Sectional category, Relative category and Topological complexity

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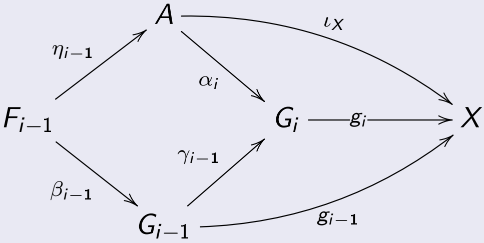
March 18, 2023

Definition

For any map $\iota_X \colon A \to X$, the *Ganea construction* of ι_X is the following sequence of homotopy commutative diagrams ($i > 0$) :



where the outside square is a homotopy pullback, the inside square is a homotopy pushout and the map $g_i \colon G_i \to X$ is the whisker map induced by this homotopy pushout. The induction starts with $g_0 = \iota_X \colon A \to X$.

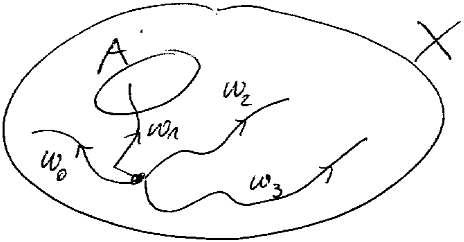
Geometric construction

If $\iota_X \colon A \to X$ is a closed cofibration (in which case $\iota_X(A) \cong A$),

$$G_i(\iota_X) \simeq \{\ (\omega_0, \omega_1, \ldots, \omega_i) \in (X^{[0,1]})^{i+1} \text{ such that } \omega_j(0) = \omega_0(0) \text{ for all } j \text{ and } \omega_k(1) \in A \text{ for at least one } k\ \}$$

and

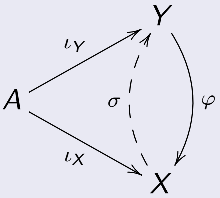
$$g_i \colon G_i \to X \colon (\omega_0, \omega_1, \ldots, \omega_i) \mapsto \omega_0(0).$$



Domination

Definition

Consider the following diagram



such that $\varphi \circ \iota_Y \simeq \iota_X$.
1) If there is a homotopy section σ of φ, i.e. $\varphi \circ \sigma \simeq \mathrm{id}_X$, we say that ι_X is *(simply) dominated by* ι_Y *along* φ.
2) If there is a homotopy section σ of φ such that $\sigma \circ \iota_X \simeq \iota_Y$, we say that ι_X is *relatively dominated by* ι_Y *along* φ.

We omit 'along φ' if the context is clear enough.

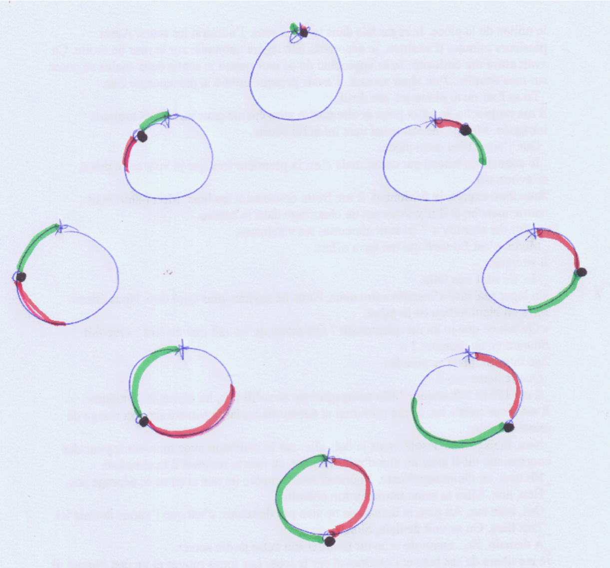
Definition

Let $\iota_X\colon A \to X$ be any map.
1) The *sectional category* of ι_X is the least integer n such that the map $\iota_X\colon A \to X$ is dominated by $\alpha_n\colon A \to G_n(\iota_X)$ along $g_n\colon G_n(\iota_X) \to X$.
2) The *relative category* of ι_X is the least integer n such that the map ι_X is relatively dominated by α_n along g_n.

We denote the sectional category by $\mathrm{secat}\,(\iota_X)$, and the relative category by $\mathrm{relcat}\,(\iota_X)$. If A is the zero object $*$, we write $\mathrm{cat}\,(X) = \mathrm{secat}\,(\iota_X) = \mathrm{relcat}\,(\iota_X)$.

Example

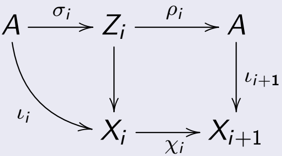
Section of $G_1(S^1) \to S^1$.



An equivalent characterisation

Proposition

We have $\mathrm{secat}\,(\iota_X) \leqslant n$ *(respectively:* $\mathrm{relcat}\,(\iota_X) \leqslant n$*) if and only there exists a sequence of homotopy commutative diagrams :*



such that $\iota_0 = \mathrm{id}_A$, $\rho_i \circ \sigma_i \simeq \mathrm{id}_A$, *the square is a homotopy pushout and* ι_X *is simply (respectively: relatively) dominated by* ι_n.

Lemma

Assume we have a homotopy commutative diagram :

$$\begin{array}{ccc} B & \xrightarrow{\kappa_Y} & Y \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ A & \xrightarrow[\iota_X]{} & X \end{array}$$

If ϕ has a homotopy section, then $\operatorname{secat}(\iota_X) \leqslant \operatorname{secat}(\kappa_Y)$.

In particular (with $B = *$ and $X = Y$), for any map $\iota_A \colon A \to X$, we have $\operatorname{secat}(\iota_X) \leqslant \operatorname{cat}(X)$

A homotopy pullback doesn't increase categories

Lemma

Assume we have a homotopy pullback :

$$\begin{array}{ccc} B & \xrightarrow{\kappa_Y} & Y \\ \downarrow & & \downarrow \\ A & \xrightarrow[\iota_X]{} & X \end{array}$$

Then $\operatorname{secat}(\kappa_Y) \leqslant \operatorname{secat}(\iota_X)$ *and* $\operatorname{relcat}(\kappa_Y) \leqslant \operatorname{relcat}(\iota_X)$.

A homotopy pushout cannot increase relative category

Lemma

Assume we have a homotopy pushout :

$$\begin{array}{ccc} A & \xrightarrow{\iota_X} & X \\ \downarrow & & \downarrow \\ B & \xrightarrow[\kappa_Y]{} & Y \end{array}$$

Then $\mathrm{relcat}\,(\kappa_Y) \leqslant \mathrm{relcat}\,(\iota_X)$.

In particular (with $B = *$), for any map $\iota_A\colon A \to X$, if C is the homotopy cofibre of ι_X, we have $\mathrm{cat}\,(C) \leqslant \mathrm{relcat}\,(\iota_X)$

Sectional and relative categories differ at most by one

Theorem

For any map $\iota_X : A \to X$*, we have :*

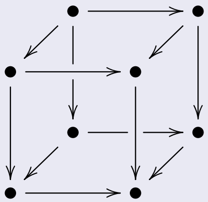
$$\mathrm{secat}\,(\iota_X) \leqslant \mathrm{relcat}\,(\iota_X) \leqslant \mathrm{secat}\,(\iota_X) + 1.$$

Cube axiom

The deepest properties of the sectional and relative categories rely on this 'axiom' which is satisfied in usual model categories.

Axiom (Cube axiom)

For any homotopy commutative diagram :



if the bottom face is a homotopy pushout and the four vertical faces are homotopy pullbacks, then the top face is a homotopy pushout.

A particular case of inequality

The following corollary shows that the sectional and relative categories of a map differ whenever the category of its homotopy cofibre is greater than the category of its target :

Corollary

For any map $\iota_X\colon A \to X$ *with homotopy cofibre* C *such that* $\operatorname{cat}(X) < \operatorname{cat}(C)$, *we have* $\operatorname{secat}(\iota_X) = \operatorname{cat}(X)$ *and* $\operatorname{relcat}(\iota_X) = \operatorname{cat}(C) = \operatorname{cat}(X) + 1$.

Example

The homotopy cofibre of the Hopf fibration $h\colon S^3 \to S^2$ is $\mathbb{C}P^2$ and we have $\operatorname{cat}(S^2) = 1 < \operatorname{cat}(\mathbb{C}P^2) = 2$. Thus $\operatorname{secat}(h) = 1$ and $\operatorname{relcat}(h) = 2$.

A particular case of equality

Proposition

Let $i\colon F \to E$ be the homotopy fibre of $f\colon E \to B$. If f has a homotopy section then

$$\mathrm{cat}\,(E/F) = \mathrm{relcat}\,(i) = \mathrm{cat}\,(B) = \mathrm{secat}\,(i)$$

where E/F is the homotopy cofibre of i.

Example

The map $\mathrm{in}_1 = (\mathrm{id}_A, 0)\colon A \to A \times B$ is the (homotopy) fibre of $\mathrm{pr}_2\colon A \times B \to B$, thus $\mathrm{cat}\,((A \times B)/A) = \mathrm{secat}\,(\mathrm{in}_1) = \mathrm{relcat}\,(\mathrm{in}_1) = \mathrm{cat}\,(B)$.

Another particular case of equality

Theorem

Let be given a CW-complex A and a $(q-1)$-connected map $\iota_X\colon A \to X$. If $\dim A < (\mathrm{secat}\,\iota_X + 1)q - 1$ then $\mathrm{secat}\,\iota_X = \mathrm{relcat}\,\iota_X$.

Example

Let $\iota\colon S^r \to S^m$ with $r \geqslant m$. If $r < 2m - 1$, then $\mathrm{relcat}\,(\iota) = \mathrm{secat}\,(\iota)$; this is 1 except for the identity for which it is 0. In particular this means that $\alpha_1 : S^r \to S^r \bowtie_{S^m} S^r$ factorizes through ι up to homotopy.

Example

Let h be any of the Hopf maps $S^3 \to S^2$, $S^7 \to S^4$ and $S^{15} \to S^8$. Since they have a target of category 1 and a homotopy cofibre of category 2, we have $\mathrm{secat}\, h = 1$ while $\mathrm{relcat}\, h = 2$. This is a conterexample wich illustrates that the inequality in the hypothesis of the previous theorem is sharp, because in the three cases we have exactly $\dim A = (\mathrm{secat}\, h + 1)q - 1$.

Sectional category of g_i

Proposition

For any map $\iota_X\colon A \to X$*, we have :*

$$\operatorname{secat} g_i(\iota_X) = \lfloor \frac{\operatorname{secat} \iota_X}{i+1} \rfloor$$

where $\lfloor x \rfloor$ *means the greatest integer less than or equal to* x*.*

Example

If i_X a map with $\operatorname{secat}(\iota_X) = 7$, the successive values of $\operatorname{secat}(g_i)$ for $0 \leqslant i \leqslant 7$ are

$$7 \quad 3 \quad 2 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0.$$

Sectionnal and relative categories of α_i

Proposition

Let $\iota_X \colon A \to X$ be any map. Consider the map $\alpha_i \colon A \to G_i(\iota_X)$ of the Ganea construction. We have :

$$\min\{i, \mathrm{secat}\,(\iota_X)\} \leqslant \mathrm{secat}\,(\alpha_i) \leqslant \mathrm{relcat}\,(\alpha_i) = \min\{i, \mathrm{relcat}\,(\iota_X)\}.$$

Note that the first inequality can be strict, for instance for $\iota_* \colon A \to *$ and $\alpha_1 \colon A \to A \bowtie A$ (join of A with itself), which is a null map, i.e. it factors through the zero object up to homotopy.

Conjectures

Conjecture

For any map $\iota_X \colon A \to X$, *any* $i \geqslant 0$, *we have*

$$\mathrm{secat}\,(\alpha_i) = \mathrm{relcat}\,(\alpha_i) = \min\{i, \mathrm{relcat}\,(\iota_X)\}.$$

Another more tricky conjecture is :

Conjecture

For any map $\iota_X \colon A \to X$, *if* ι_X *has a homotopy retraction, then we have* $\mathrm{secat}\,(\iota_X) = \mathrm{relcat}\,(\iota_X)$.

Complexity of a space

Definition

Let X be any object.
We define the *complexity* of X to be the sectional category of the diagonal map $\Delta\colon X \to X \times X$.
Analogously, we define the *relative complexity* of X to be the relative category of the diagonal.

We use the following notations : $\mathrm{compl}\,(X) = \mathrm{secat}\,(\Delta)$ and $\mathrm{relcompl}\,(X) = \mathrm{relcat}\,(\Delta)$.
Actually complexity is the (normalized) topological complexity of Farber, and relative complexity is the monoidal complexity of Iwase and Sakai. A positive answer to our second conjecture would imply that $\mathrm{compl}\,(X) = \mathrm{relcompl}\,(X)$.

Complexity of a map

Definition

For any map $\iota_X \colon A \to X$, we define the *complexity* of ι_X (respectively: *relative complexity*) as the sectional category (respectively: relative category) of $\delta_1(\iota_X) \colon A \to X \times A$. where $\delta_1(\iota_X)$ is the whisker map of ι_X and id_A.

We write $\mathrm{compl}\,(\iota_X) = \mathrm{secat}\,(\delta_1(\iota_X))$.
In particular $\mathrm{compl}\,(\mathrm{id}_X) = \mathrm{compl}\,(X)$, since $\delta_1(\mathrm{id}_X) \simeq \Delta$.

Inequalities for complexity

Proposition

For any object X and any map $\iota_X : A \to X$,

$$\mathrm{cat}\,(X) \leqslant \mathrm{compl}\,(\iota_X) \leqslant \mathrm{compl}\,(X) \leqslant \mathrm{cat}\,(X \times X).$$

Example

M. Farber has shown that the complexity of a sphere is 1 if the dimension is odd and 2 if the dimension is even.

Example

Consider the Hopf fibration $S^7 \to S^4$ and factor by the action of S^1 on S^7 to get $\iota\colon \mathbb{C}P^3 \to S^4$. The map $\delta_1 : \mathbb{C}P^3 \to S^4 \times \mathbb{C}P^3$ induces $\delta_1^*\colon H^*(S^4) \otimes H^*(\mathbb{C}P^3) \to H^*(\mathbb{C}P^3)$. We can find an element a of $H^*(S^4) \otimes H^*(\mathbb{C}P^3)$ such that $a \in \ker \delta_1^*$ and $a^2 \neq 0$, so by a theorem of A.S. Schwarz, $\mathrm{compl}\,(\iota) \geqslant 2$. On the other hand by the previous proposition $\mathrm{compl}\,(\iota) \leqslant \mathrm{cat}\,(S^4) = 2$. So $\mathrm{compl}\,(\iota) = 2$.

A case of equality

Corollary

Let be given any map $\iota_X\colon A \to X$ *between CW-complexes,* A *connected and* X $(q-1)$*-connected. If* $\dim A < (\operatorname{compl}(\iota_X)+1)q - 1$, *then*

$$\operatorname{cat}((A \times X)/A) \leqslant \operatorname{relcompl}(\iota_X) = \operatorname{compl}(\iota_X) \leqslant \operatorname{cat}(A \times X)$$

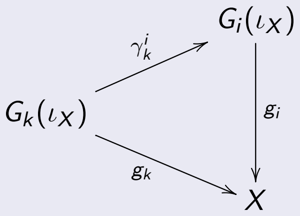
where $(A \times X)/A$ *is the homotopy cofibre of* (id_A, ι_X).

Example

Consider the Hopf fibration $S^7 \to S^4$ and factor by the action of S^1 on S^7 to get $\iota\colon \mathbb{C}P^3 \to S^4$. We have $\dim \mathbb{C}P^3 = 6 < 3.4 - 1 = (\operatorname{compl}(\iota)+1).q - 1$, so $\operatorname{relcompl}(\iota) = \operatorname{compl}(\iota) = 2$.

Definition

Let $\iota_X : A \to X$ be any map. Consider the homotopy commutative diagram :



where $\gamma_k^i \simeq \gamma_{i-1} \circ \gamma_{i-2} \circ \cdots \circ \gamma_{k+1} \circ \gamma_k$ $(k < i)$ and $\gamma_k^k = \mathrm{id}_{G_k}$. The *relative category of order* k of ι_X is the least integer $n \geqslant k$ such that the map $g_k : G_k(\iota_X) \to X$ is relativement dominated by $\gamma_k^n : G_k(\iota_X) \to G_n(\iota_X)$ along $g_n : G_n(\iota_X) \to X$.

We denote this integer by $\mathrm{relcat}_k\,(\iota_X)$. When $A \simeq *$, we write $\mathrm{cat}_k\, X = \mathrm{relcat}_k\,(\iota_X)$.

Theorem

For any map $\iota_X : A \to X$, any k, we have :

$$k \leqslant \mathrm{relcat}_k(\iota_X) \leqslant \mathrm{relcat}_{k+1}(\iota_X) \leqslant \mathrm{relcat}_k(\iota_X) + 1.$$

Remarks

Remark

$\mathrm{relcat}_k\,(\iota_X) = k$ *if and only if* $g_k(\iota_X)$ *is a homotopy equivalence.*

Remark

The inequalities of the previous theorem imply that if $\mathrm{relcat}\,(\iota_X) = n$ *there can be at most* n *integers* k *such that* $\mathrm{relcat}_k\,(\iota_X) = \mathrm{relcat}_{k+1}\,(\iota_X)$.

The last remark suggests the following definition :

Definition

For any map $\iota_X : A \to X$, define $\mathrm{scratch}\,(\iota_X)$ as the number of integers k such that $\mathrm{relcat}_k\,(\iota_X) = \mathrm{relcat}_{k+1}\,(\iota_X)$.

If $A \simeq *$, we write $\mathrm{scratch}\,(X) = \mathrm{scratch}\,(\iota_X)$.

Examples

Example

We have $\mathrm{cat}_k\, S^n = k + 1$ for all k. So $\mathrm{scratch}\,(S^n) = 0$.

Example

Let X be the Eilenberg-Mac Lane space $K(\mathbb{Q}, 1)$. It is known that $\mathrm{cat}\,(X) = 2$. Because $G_1(X) \simeq \Sigma\Omega X$ has the homotopy type of a wedge of circles, we have $\mathrm{cat}_1\,(X) = 2$. And we have $\mathrm{cat}_k\,(X) = k + 1$ for $k \geqslant 1$. So $\mathrm{scratch}\,(X) = 1$.

References

http ://math.univ-lille1.fr/∼doeraene/