The Hilali Conjecture and The Milnor-Moore spectral Sequence

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First International Conference On Algebraic Topology and its Application in Robotics.
Meknès: 17-18 Mars 2023.

Plan

1. Hilali Conjecture : H conjecture
2. Milnor-Moore spectral sequence

Let X be a simply connected topological space such that each $H_i(X,\mathbb{Q})$ is finite dimensional.
There is a free commutative differential graded algebra $(\Lambda V, d)$ and a sequence of quasi-isomorphisms :

$$(\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X) \xrightarrow{\simeq} D(X) \xleftarrow{\simeq} C^*(X,\mathbb{Q}) \qquad (1)$$

where :

(i) $V = \oplus_{i\geq 2} V^i$ is a graded $\mathbb{Q}$-vector space and each V^i is finite dimensional.

(ii) $\Lambda V = Symetric(V^{even}) \otimes Exterior(V^{odd})$.

(iii) $d : \Lambda V \to \Lambda V$ is a derivation i.e.
$d(ab) = (da)b + (-1)^{|a||b|}a(db)$, and $d \circ d = 0$.
Thus, d is a differential determined by $d_{|V}$.

(iv) $d : V \to \Lambda^{\geq 2} V$ i.e. d is decomposable.

(v) V is naturally isomorphic with $Hom_{\mathbb{Z}}(\pi_*(X),\mathbb{Q})$ and
$H^*(X,\mathbb{Q}) \cong H^*(\Lambda V, d)$.

Here $A_{PL}(X)$ is the *commutative* cochain algebra of *polynomial differential forms on* X *with rational coefficients*.

Definition

$(\Lambda V, d)$ is called a *minimal Sullivan model* of X.

H conjecture : Topological version

$$\dim H^*(X, \mathbb{Q}) \geq \dim \pi_*(X) \otimes \mathbb{Q}.$$

H conjecture : Algebraic version

$$\dim H^*(\Lambda V, d) \geq \dim V.$$

Bar constructions

Let (A, d_A) be an augmented differential graded algebra over $\mathbb{Q}$, with unity $\eta : \mathbb{Q} \hookrightarrow A$ and augmentation $\varepsilon_A : A \to \mathbb{Q}$.
We Assume that (A, d) is 1-connected ($A^0 = \mathbb{Q}$ and $A^1 = 0$) and of finite type ($\dim A^i < \infty$, $i \geq 2$).
Let $\bar{A} = \ker(\varepsilon)$ and $W = s\bar{A}$ its suspension defined by $(s\bar{A})^i = \bar{A}^{i+1}$.

- The *tensor co-algebra* on W :

$T'(W) = \oplus_{k \geq 0} T^k(W)$ endowed with :
- the diagonal $\Delta : T(W) \to T(W) \otimes T(W)$ given by :

$$\Delta([a_1|\dots|a_k]) = [a_1|\dots|a_k] \otimes 1 \quad + \sum_{i=1}^{i=k-1} [a_1|\dots|a_i|a_{i+1}|\dots|a_k] \\ +1 \otimes [a_1|\dots|a_k]$$

- co-unity $\varepsilon : T'(W) \to \mathbb{Q}$ and co-augmentation $\mathbb{Q} \hookrightarrow T'(W)$.

Bar construction

The bar construction of an augmented dga (A, d_A) is the co-augmented tensor co-algebra $BA = T'(W)$ with the co-derivation $d = d_0 + d_1$ given by :

$$d_0([sa_1|\dots|sa_k]) = -\sum_{i=1}^{i=k}(-1)^{n_i}[a_1|\dots|sa_{i-1}|sd_A a_i|a_{i+1}|\dots|a_k]$$

$$\begin{cases} d_1([sa]) = 0 \\ d_1([sa_1|\dots|sa_k]) = \sum_{i=2}^{i=k}(-1)^{n_i}[a_1|\dots|sa_{i-1}|sa_{i-1}a_i|a_{i+1}|\dots|a_k]. \end{cases}$$

Here $n_i = \sum_{j<i} deg(sa_j)$.
$d^2 = 0$: $d_0^2 = 0$ since $d_A^2 = 0$, $d_0 d_1 + d_1 d_0 = 0$ since d_A is a derivation and $d_1^2 = 0$ because A is associative.

Cobar constructions

Let (C, d_C) be a co-augmented differential graded co-algebra with co-multiplication $\Delta : C \to C \otimes C$, co-unity $\varepsilon : C \to \mathbb{Q}$ and co-augmentation $\eta : \mathbb{Q} \to C$.
If $\bar{C} = \ker(\varepsilon)$, then

$$\overline{\Delta}(c) = \Delta(c) - c \otimes 1 - 1 \otimes c \in \overline{\Delta}(c) \otimes \overline{\Delta}(c).$$

This defines $\overline{\Delta} : \bar{C} \to \bar{C} \otimes \bar{C}$
Denote by $s^{-1}\bar{C}$ the de-suspension of $\bar{C}$ given by $(s^{-1}\bar{C})^i = \bar{C}^{i-1}$.

Cobar construction

The cobar construction of (C, d_C) is the augmented tensor algebra $\Omega C = T(s^{-1}\bar{C})$ endowed with the differential given by the derivation $d = d_0 + d_1$ where :

$$d_0(s^{-1}x) = -s^{-1}(d_C x),\ x \in \bar{C}$$

$$d_1(s^{-1}x) = \sum_i (-1)^{deg(x_i)} s^{-1}x_i \otimes s^{-1}y_i,\ x \in \bar{C},\ \bar{\Delta}(x) = \sum_i x_i \otimes y_i.$$

Milnor-Moore spectral sequence

If (A, d_A) is the augmented cochain algebra $C^*(X, \mathbb{Q}) =: C^*(X)$, the two constructions give then an augmented differential graded algebra $\Omega BC^*(X) = T(V)$ with $V = s^{-1}\overline{BC^*(X)}$.
A standard filtration on $\Omega BC^*(X)$ is given by :

$$F^p := F^p(\Omega BC^*(X)) = T^{\geq p}V, \;\; p \geq 0. \tag{2}$$

Definition

The filtration (2) induce the spectral sequence :

$$E_2^{p,q} = Ext^{p,q}_{H_*(\Omega X, \mathbb{Q})}(\mathbb{Q}, \mathbb{Q}) \Rightarrow H^{p+q}(X, \mathbb{Q}) \tag{3}$$

called the (cohomology) *Milnor-Moore spectral of X*.

Remark

The constructions above show clearly that any morphism $\varphi : (A, d_A) \to (B, d_B)$ of augmented dga yields a spectral sequence homomorphism. Moreover, this is an isomorphism between the first terms provided that φ is a quasi-isomorphism.

In particular, let $(\Lambda V, d)$ be a minimal Sullivan model of X. The quasi-isomorphisms in (1) enable us to replace $C^*(X)$ by $(\Lambda V, d)$. In this case the filtration is :

$$F^p := F^p(\Lambda V) = \Lambda^{\geq p} V, \;\; p \geq 0.$$

It satisfies the following properties :

1. $(F^p)_{p\geq 0}$ is a bounded and decreasing sequence.
2. $dF^p \subseteq F^p$.
3. $E_0^{p,q} = (F^p/F^{p+1})^{p+q} \cong (\Lambda^p V)^{p+q}$ and $\delta_0 = 0 : E_0^{p,q+1} \to E_0^{p,q}$ since $d = d_2 + \dots$.
4. $\delta_1 : E_1^{p,q}(\cong E_0^{p,q}) \to E_1^{p+1,q}(\cong E_0^{p+1,q})$, i.e. $\delta_1 : (\Lambda^p V)^{p+q} \to (\Lambda^{p+1} V)^{p+q+1}$ is exactly the quadratic part d_2 of the differential d. Hence, $E_1^{*,*} \cong (\Lambda V, d_2)$.
5. $E_2^{p,q} \cong H^{p,q}(E_2^{*,*}, \delta_1) \cong H^{p,q}(\Lambda V, d_2)$. That is $E_2^{*,*} \cong H^*(\Lambda V, d_2)$.

Consequently, we obtain the convergent spectral sequence :

$$E_2^{p,q} = H^{p,q}(\Lambda V, d_2) \Rightarrow H^{p+q}(\Lambda V, d). \tag{4}$$

that is isomorphic from the second term with the Milnor-Moore spectral sequence.

Remark

If the differential in $(\Lambda V, d)$ has the form $d = d_k + d_{k+1} + \ldots$ then, $E_1^{*,*} \cong E_2^{*,*} \cong \ldots \cong E_{k-1}^{*,*} \cong (\Lambda V, d_k)$ so, (4) becomes :

$$E_k^{p,q} = H^{p,q}(\Lambda V, d_k) \Rightarrow H^{p+q}(\Lambda V, d). \qquad (5)$$

Recall that X is said *rationally elliptic* if both $H^*(X, \mathbb{Q})$ and $\pi_*(X) \otimes \mathbb{Q}$ or equivalently $H^*(\Lambda V, d)$ and V are finite dimensional. It results from the convergence of (5) that, if $(\Lambda V, d_k)$ is elliptic then so is $(\Lambda V, d)$.

Elliptic spaces satisfy Poincaré duality, that is, in terms of $(\Lambda V, d)$:

(a) $\exists \alpha \in H^*(\Lambda V, d)$ such that $H^N(\Lambda V, d) = \mathbb{Q}\alpha$ (some integer N) and $H^{>N}(\Lambda V, d) = 0$.

(b) The pairing $<,>: H^i(\Lambda V, d) \times H^{N-i}(\Lambda V, d) \to H^N(\Lambda V, d)$ such that $a.b =< a, b > \alpha$ is a non-degenerate bilinear map for every $i = 1, 2, \dots, N-1$.

$N = sup\{p \mid H^p(\Lambda V, d_k) \neq 0\}$ is called the *formal dimension* of X (or $(\Lambda V, d)$) and $\alpha =: [\omega]$ its fundamental class.
Recall that the *Toomer invariant* is originally defined in terms of (3) by

$$e_0(X) = sup\{p \mid E_\infty^{p,q} \neq 0\}$$

or ∞ *if such maximum doesn't exist.*

An equivalent definition is later established in [FH] :
$e_0(X) = e(\Lambda V, d)$ *is the smallest integer n such that the projection* $p_n : \Lambda V \to \Lambda V/\Lambda^{>n}V$ *induces an injection in cohomology or* ∞ *if there is no such integer.*
Similarly if $x \in H^*(\Lambda V, d)$, we put $e_0(x)$ for the smallest integer n such that $H^*(p_n)(x) \neq 0$. In particular, we have

$$e(\Lambda V, d) = e_0(\alpha).$$

We say that $H^*(\Lambda V, d)$ has an e_0-gap if it contain an element x where $e_0(x) = l$ and no element y with $e_0(y) = l - 1$.

Our main theorem is [4] :

Theorem

Any elliptic space whose minimal Sullivan model $(\Lambda V, d)$ with $d = d_k + \dots$ $(k \geq 2)$ is such that $(\Lambda V, d_k)$ is elliptic has no e_0-gaps in its cohomology.

It results that $\dim H_p^{n_p}(\Lambda V, d) \geq 1$ for every $p = 0, \dots, e(\Lambda V, d)$, so that $\dim H^*(\Lambda V, d) \geq e(\Lambda V, d)$. Moreover, $e(\Lambda V, d) = \dim V^{odd} + (k-2)\dim V^{even}$ ([?] or [LM02]). Consequently, the Hilali conjecture follow immediately if $V = V^{odd}$ (without restriction on $(\Lambda V, d_k)$) or if $k \geq 3$.

First, suppose that $d = d_k : V \to \Lambda^k V$ is **homogeneous of degree** $k \geq 2$ and $(\Lambda V, d_k)$ is elliptic. In this case $H(\Lambda V, d_k)$ has another (lower) graduation :

$$H^+(\Lambda V, d_k) = \oplus_{p\geq 1} H_p^+(\Lambda V, d_k).$$

where $H_p^+(\Lambda V, d_k)$ denotes *the subspace of cohomology classes represented by homogeneous cocycles of length* p.

G. Lupton established (in particular) that

$$\dim H_p^*(\Lambda V, d_k) \neq 0, \text{ for each } p = 0, \dots, e. \quad (6)$$

Thus, the theorem is valid for $(\Lambda V, d_k)$. Moreover, $H(\Lambda V, d_k)$ is a bi-graded Poincaré algebra in the sens that :

$$H_p^i(\Lambda V, d_k) \times H_{e-p}^{N-i}(\Lambda V, d_k) \to H_e^N(\Lambda V, d_k) \cong \mathbf{Q} \quad (7)$$

for $p = 1, 2, \dots, e-1$ (see also [2]).

For each $p = 1, \dots, e-1$ let

$$n_p = min\{i \mid H^i_p(\Lambda V, d) \neq 0\}, \quad N_{e-p} = max\{i \mid H^i_{e-p}(\Lambda V, d) \neq 0\},$$

$$n_0 = N_0 = 0, \quad n_e = N_e = N.$$

Thus $n_p + N_{e-p} = N_e,\ 0 \leq p \leq e.$
It results from (7) that for any $p = 1, \dots, e$, there exist non-zero classes $[\omega_p] \in H_p^{n_p}(\Lambda V, d_k)$ and $[\omega_{e-p}] \in H_{e-p}^{N_{e-p}}(\Lambda V, d_k)$ such that

$$[\omega_p] \otimes [\omega_{e-p}] = [\omega] := [\omega_e]. \tag{8}$$

Now we return to $(\Lambda V, d)$ with $d = d_k + \dots$.
Recall that :
- The general term of the spectral sequence is given by

$$E_r^{p,q} = Z_r^{p,q} / Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q},$$

where

$$Z_r^{p,q} = \{x \in [F^p(A)]^{p+q} \mid dx \in [F^{p+r}(A)]^{p+q+1}\}$$

and

$$B_{r-1}^{p,q} = d([F^{p-r+1}(A)]^{p+q-1}) \cap F^p(A) = d(Z_{r-1}^{p-r+1,q+r-2}).$$

- The differential $\delta_k : E_k^{p,q} \to E_k^{p+k,q-k+1}$ is induced from the differential d by the formula $\delta_k[v] = [d(v)]_k$, v being any representative in $Z_k^{p,q}$ of the class $[v]_k$ in $E_k^{p,q}$.

Put $Z(E_k^{p,q}) := Ker(\delta_k)$ and $B(E_k^{p,q}) := Im(\delta_k)$. We first establish the isomorphisms :

$$I_k^{p,q} : Z_{k+1}^{p,q} + Z_{k-1}^{p+1,q-1}/Z_{k-1}^{p+1,q-1} + dZ_{k-1}^{p-k+1,q+k-2} \xrightarrow{\cong} Z(E_k^{p,q}). \quad (9)$$

and

$$J_k^{p,q} : dZ_k^{p-k,q+k-1} + Z_{k-1}^{p+1,q-1}/Z_{k-1}^{p+1,q-1} + dZ_{k-1}^{p-k+1,q+k-2} \xrightarrow{\cong} B(E_k^{p,q}). \quad (10)$$

which imply that $E_{r+1}^* \cong H(E_r^*)$ for any $r \geq k$.
In the remainder, we will identify $H^{p,q}(\Lambda V, d_k)$ and $E_k^{p,q}$. We can then take (cf. 8)

$$\omega_p \in Z_k^{p,n_p-p},\ \omega_{e-p} \in Z_k^{e-p,N_{e-p}-e+p}.$$

and denote $[\omega_p] = \bar{\omega}_p$ and $[\omega_{e-p}] = \bar{\omega}_{e-p}$.

Using [?, Theorem 2.2(c), Lemma 2.1] We show that :

a) $\delta_k[\bar{\omega}] = \bar{0}$ so that $\omega_p \in Z_{k+1}^{p,n_p-p}$ an $d(\omega_p) \in F^{p+k+1}$.

b) $\bar{\omega}_{e-p}$ can't be a δ_k-coboundary i.e. $\bar{\omega}_{e-p} \notin B(E_k^{e-p,N_{e-p}-e+p})$.

(b) $\bar{\omega}_e$ is a δ_k-cocycle that survives to E_∞^{e,N_e-e}, in particular it survives to E_{k+1}^{e,N_e-e} and $\omega_e \in Z_{k+1}^{e,N_e-e}$.

Finally, since the filtration $F^p(\Lambda V) = \Lambda^{\geq p} V$ clearly satisfies the relation $F^p(\Lambda V) \otimes F^q(\Lambda V) \subseteq F^{p+q}(\Lambda V)$, $\forall\, p, q \geq 0$ the induced spectral sequence (5) is one of graded algebras.

This implies that $\omega_p \otimes \omega_{e-p} \in Z_{k+1}^{e,N_e-e}$. Next, since $d(\omega_p) \otimes \omega_{e-p} \pm \omega_p \otimes d(\omega_{e-p}) \in F^{e+k+1}$ we deduce that : $\omega_{e-p} \in Z_{k+1}^{e-p,N_{e-p}-e+p}$ or equivalently $\delta_k(\bar{\omega}_{e-p}) = 0$, that is $\bar{\omega}_{e-p}$ is a cocycle which survieves to $E_{k+1}^{e-p,N_{e-p}-e+p}$. This in turn imply that $\bar{\omega}_p$ is a cocycle which survives to E_{k+1}^{p,n_p-p}.

Consequently, the relation $[\omega_p] \otimes [\omega_{e-p}] = [\omega] := [\omega_e]$ is valid in E_{k+1}^{e,N_e-e}.

We then finish by induction.

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Thanks for your Attention.