On the topological complexity of manifolds with abelian fundamental group

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joint work with

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Topological Complexity (M. Farber, 2003)

X - space of all possible positions of a mechanical system.

Motion from position x_0 to position x_1 = path in X from x_0 to x_1.

A motion planning algorithm is a section $s : X \times X \to X^{[0,1]}$ of

$$ev_{0,1} : X^{[0,1]} \to X \times X, \quad \gamma \mapsto (\gamma(0), \gamma(1))$$

That is, $s(x, y)$ is a path from x to y.

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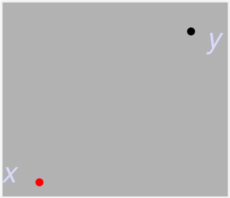
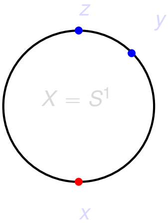
$X =$ 



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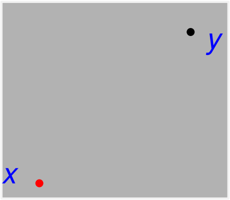

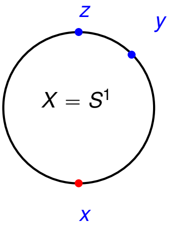

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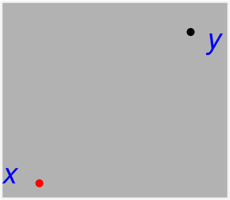

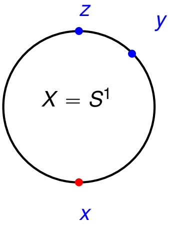

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Definition and some properties (Farber)

Let X be a path-connected space (CW-complex, manifold).

Definition. $\mathrm{TC}(X)$ is the least integer n s.t. $X \times X$ can be covered by $n+1$ open sets on each of which $ev_{0,1}$ admits a local **continuous** section.

- TC is a homotopy invariant.
- $\mathrm{TC}(X) = 0$ iff X is contractible, $\mathrm{TC}(S^{2k+1}) = 1$, $\mathrm{TC}(S^{2k}) = 2$.
- $\mathrm{TC}(X) \leqslant 2\dim(X)$.

  We say that $\mathrm{TC}(X)$ is **maximal** when $\mathrm{TC}(X) = 2\dim(X)$.

  This only can happen when $\pi_1(X) \neq 0$ because

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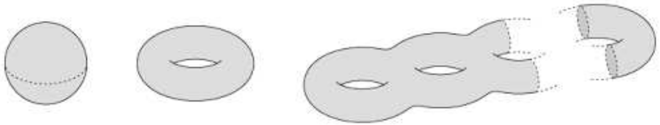
Orientable (closed) surfaces



S^2 $T = S^1 \times S^1$ torus with g holes $T_g = \underbrace{T \# T \# \cdots \# T}_{g}$.

Theorem. (Farber, 2003)

- $\mathrm{TC}(S^2) = 2$
- $\mathrm{TC}(T) = 2$
- for $g \geqslant 2$, $\mathrm{TC}(T_g) = 4$.

Important tool: $\mathrm{TC}(X) \geqslant \mathrm{zcl}_{\Bbbk}(X)$=maximal length of a nontrivial product in the kernel of the cup-product

$$\cup : H^*(X; \Bbbk) \otimes H^*(X; \Bbbk) \to H^*(X; \Bbbk)$$

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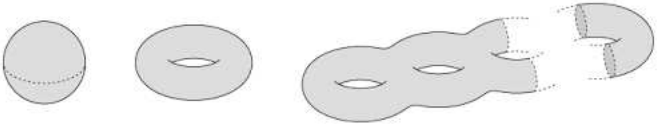

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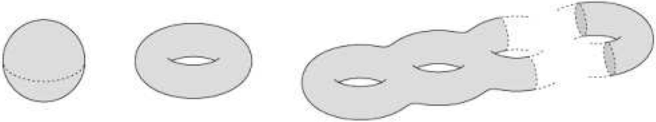

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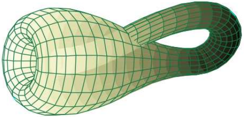
Nonorientable surfaces: $\mathbb{RP}^2$, Klein bottle K, ...



$$K = \mathbb{RP}^2 \# \mathbb{RP}^2, \qquad N_g = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{g}$$

Theorem.(Farber, Tabachnikov, Yuzvinsky, 2003) $\mathrm{TC}(\mathbb{RP}^2) = 3$

Theorem.(Dranishnikov, 2016) For $g \geqslant 4$, $\mathrm{TC}(N_g) = 4$.

Theorem.(Cohen, V, 2017) $\mathrm{TC}(K) = 4$. For $g \geqslant 2$, $\mathrm{TC}(N_g) = 4$.

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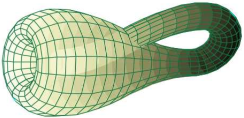

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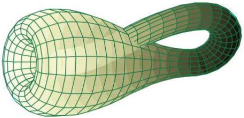

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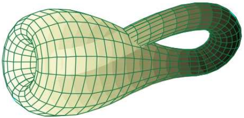

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Real projective spaces

In analogy to $N_g = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ (g copies), we consider

$$\mathcal{P}_g^n := \underbrace{\mathbb{RP}^n \# \cdots \# \mathbb{RP}^n}_{g \text{ copies}}$$

Theorem. (Cohen-V., 2018) For $n \geqslant 2$ and $g \geqslant 2$, $\mathrm{TC}(\mathcal{P}_g^n) = 2n$.

Case $g = 1, \mathcal{P}_1^n = \mathbb{RP}^n$ (Farber, Tabachnikov, Yuzvinsky - 2003)

Theorem. (FTY)

- For $n = 1, 3$ or 7, $\mathrm{TC}(\mathbb{RP}^n) = n$.
- For $n \neq 1, 3, 7$, $\mathrm{TC}(\mathbb{RP}^n)$ is the least integer k such that there exists an immersion of $\mathbb{RP}^n$ in $\mathbb{R}^k$.

In particular, $\mathrm{TC}(\mathbb{RP}^n) \leqslant 2n - 1$.

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TC of spaces with small fundamental group

Theorem. (Costa-Farber, 2010) X finite CW-complex of dim n.

1. If $\pi_1(X) = \mathbb{Z}_2$, then $\mathrm{TC}(X) < 2n$.
2. If $\pi_1(X) = \mathbb{Z}_3$ then $\mathrm{TC}(X) < 2n$ when either n is odd or n is even and the 3-adic expansion of $n/2$ contains at least one digit 2.

The condition in the even dimensional case of (2) is sharp:

For X = 6-dimensional skeleton of the lens space $L_3^7 = S^7/\mathbb{Z}_3$,

$$\pi_1(X) = \mathbb{Z}_3 \quad \text{and} \quad TC(X) = 2\dim(X)$$

Note that X is not a closed manifold.

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TC of manifolds with abelian π_1 (Cohen-V., 2021)

Theorem. Let M be a **nonorientable** manifold with **abelian** fundamental group. Then $\mathrm{TC}(M) \leqslant 2\dim(M) - 1$.

Theorem. Let M be an n-dimensional **orientable** manifold with **abelian** fundamental group $\pi_1(M)$ of one of the following forms:

1. $\mathbb{Z}^r$ with either n odd or n even such that $r < 2n$
2. $\mathbb{Z}^r \times \mathbb{Z}_{p^a}$ with p prime and $r < n$
3. $\mathbb{Z}^r \times \mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$ and $r \leqslant 1$
4. $\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b} \times \mathbb{Z}_{p^c}$
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M closed manifold with $\dim M = n$ and $\pi_1(M) = G$

Building on works of Costa-Farber and Dranishnikov we prove:

Proposition. If $\pi_1(M) = G$ is abelian then

$$\mathrm{TC}(M) < 2n \Leftrightarrow \alpha_*(\mathbf{m} \times \mathbf{m}) = 0 \quad \text{in } H_{2n}(BG; \widetilde{\mathbb{Z}}) = H_{2n}(G; \widetilde{\mathbb{Z}}).$$

- α_* is induced by $\quad \alpha : G \times G \to G, \quad \alpha(a, b) = ab^{-1}$.
- $\mathbf{m} = \gamma_*([M]) \in H_n(BG; \widetilde{\mathbb{Z}})$ where:
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Study of $\alpha_*(\mathbf{c} \times \mathbf{c})$ for $\mathbf{c} \in H_n(BG; \widetilde{\mathbb{Z}})$

G finitely generated abelian group with action on $\widetilde{\mathbb{Z}}$.

$H_*(BG; \widetilde{\mathbb{Z}})$ is a Pontrjagin algebra with a strictly anti-commutative product $\wedge$

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Considering the morphism induced by the inversion of G

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All the conditions are sharp.

In (2), the condition $r < n$ is sharp because for $G = \mathbb{Z}^7 \times \mathbb{Z}_3$, there exists $\mathbf{c} \in H_7(BG; \widetilde{\mathbb{Z}})$ such that $\alpha_*(\mathbf{c} \times \mathbf{c}) \neq 0$.

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Realization – $G = \mathbb{Z}^7 \times \mathbb{Z}_3$ – $BG = (S^1)^7 \times L_3^\infty$

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- $\phi_*([N]) = \mathbf{c}$ but $\pi_1(N) = \mathbb{Z}^7 * \mathbb{Z}_3$ is not abelian.
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