On Quandle Colorings of Knots and Links

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Outline

Definition of a knot

Definition

A knot is the image of a circle S^1 in S^3 (or $\mathbb{R}^3$) by an embedding

$$i \;:\; S^1 \;\hookrightarrow\; S^3 \; (or \; \mathbb{R}^3)$$

denoted by $K = i(S^1)$.

What are Knots and Links?

Examples of Knots

Two of the simplest examples of knots are:

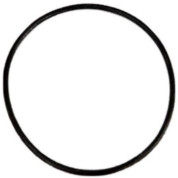

Figure: The trivial knot

Figure: The trefoil knot

Definition

A link is the image of n disjoint copies $S^1 \sqcup \cdots \sqcup S^1$ of a circle S^1 in S^3 (or $\mathbb{R}^3$) by an embedding

$$i \quad : \quad S^1 \sqcup \cdots \sqcup S^1 \quad \hookrightarrow \quad S^3 \; (or \; \mathbb{R}^3)$$

denoted by $L = K_1 \sqcup \cdots \sqcup K_n$ where K_j, $1 \leq j \leq n$ is the image of the j th copy of S^1.

- A Knot is a link with one component $(n = 1)$.

Some of the simplest examples of links are:

Figure: The trivial link with two components

What are Knots and Links?

Examples of Links

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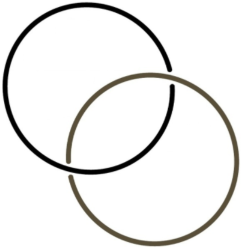


Figure: The hopf link

Knot equivalence

The main problem in knot theory is the following: When are two Knots (Links) the same?

Knot equivalence

Reidemeister moves

An approach to knot equivalence will be in term of three operations on the knot diagrams called Reidemeister moves.

- A Reidemeister move is one of the three following ways to change a projection of the knot.

Knot equivalence

Reidemeister moves

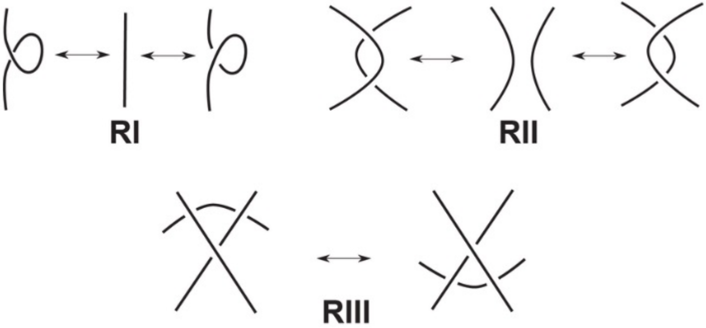


Figure: Reidemeister moves of type I, II and III.

Knot equivalence

Definition

Two diagrams D_1 and D_2 are equivalent if and only if they are related by a sequence of Reidemeister moves and planar isotopies.

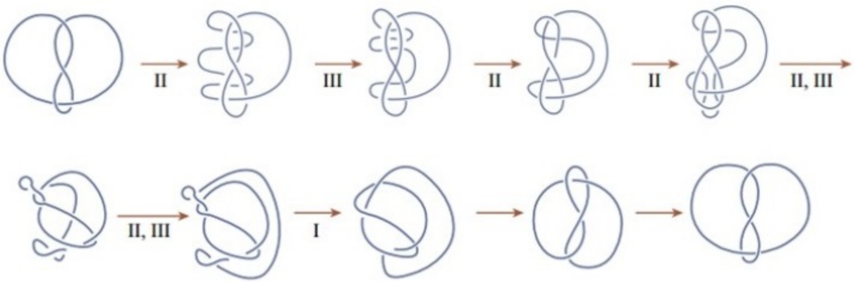
Theorem

Two knots or links are equivalent if and only if their diagrams are equivalent.

Knot equivalence

Example

Reidemeister moves on a knot.



Knot invariants

We can use knot invariants to help us tell whether or not two knot diagrams represent the same knot.

Definition

A knot invariant is a function that assigns a quantity or a mathematical expression to a knot, which is preserved under knot equivalence. In other words, if two knots are equivalent, then they must be assigned the same quantity or expression. However, the converse is not necessarily true; if two knots are assigned the same invariant, it does not necessarily mean that they are equivalent.

Quandles were introduced in order to give an algebraic interpretation of the Reidemeister moves.

Definition

A *quandle*, is a non-empty set Q equipped with a binary operation

$$\begin{array}{cccc} * : & Q \times Q & \longrightarrow & Q \\ & (x, y) & \longmapsto & x * y \end{array}$$

satisfying the following three axioms:

1. For all $x \in Q$, $x * x = x$.
2. For all $y \in Q$, the map $R_y : Q \to Q$ defined by $R_y(x) = x * y$, $x \in Q$, is bijective.
3. For all $x, y, z \in Q$, $(x * y) * z = (x * z) * (y * z)$ (right self-distributivity).

We write $x *^{-1} y$ for $R_y^{-1}(x)$.

Example

Alexander quandle
An Alexander quandle M is a $\mathbb{Z}[t, t^{-1}]$-module endowed with the following binary operation:

$$x * y = tx + (1 - t)y \text{ for all } x, y \in M.$$

Note that we have $x *^{-1} y = t^{-1}x + (1 - t^{-1})y$.

Example

Linear Alexander quandle

Given positive integers $m < n$ such that $\gcd(m, n) = 1$, we let the underlying set to be $\mathbb{Z}_n$ with the binary operation:

$$x * y = mx + (1 - m)y \pmod{n}, \text{ for all } x, y \in \mathbb{Z}_n.$$

Note that we have $x *^{-1} y = m^{-1}x + (1 - m^{-1})y \pmod{n}$.

Coloring links by quandles

Let $(Q, *)$ be a quandle and D a diagram of an oriented link L. A coloring of D by Q is a map $\mathcal{C}$ from the set of arcs of D denoted by $\mathcal{A}$ to Q,

$$\mathcal{C}\colon \mathcal{A} \to Q$$

such that at each crossing of D, if the over-arc α_1 is colored by $\mathcal{C}(\alpha_1) = y$ and the incoming under-arc is colored by $\mathcal{C}(\alpha_2) = x$ then the outcoming under-arc is colored by $\mathcal{C}(\alpha_3) = x * y$ or $\mathcal{C}(\alpha_3) = x *^{-1} y$ according to the rule depicted in the following Figure

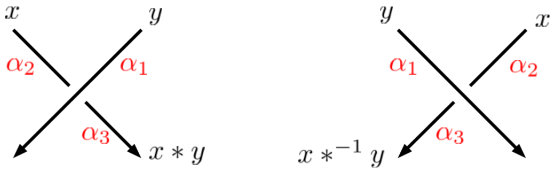


Figure: Coloring conditions.

Coloring links by Alexander quandles

If Q is an Alexander quandle, by collecting the coloring conditions at all crossings of D, we get a homogeneous system of linear equations over $\mathbb{Z}[t, t^{-1}]$. The matrix associated to this system of equations is called the Alexander matrix denoted A.

- The rows of A correspond to the crossings of D and the columns correspond to the arcs of D.

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- Each row has only three non-zero entries which are t, $1-t$ and -1. So on the one hand $(\lambda, \ldots, \lambda)$ is a solution for any $\lambda \in \mathbb{Z}[t, t^{-1}]$ (trivial solutions), and on the other hand the determinant of the Alexander matrix is zero.

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- A non-trivial solution of the initial homogeneous system corresponds to a non-trivial solution of the system of equations determined by the original matrix with one row and one column deleted.
- The determinant of this last submatrix is known to be the Alexander polynomial of the considered link denoted by $\Delta_L(t)$. Therefore, the existence of non-trivial solutions corresponds to working on the quotient of $\mathbb{Z}[t, t^{-1}]$ by $\Delta_L(t)$, which is a Laurent polynomial on the variable t determined up to $\pm t^n$, for any integer n.

The reduced Alexander polynomial $\Delta_K^0(t)$

let K be a knot, the reduced Alexander polynomial of K is

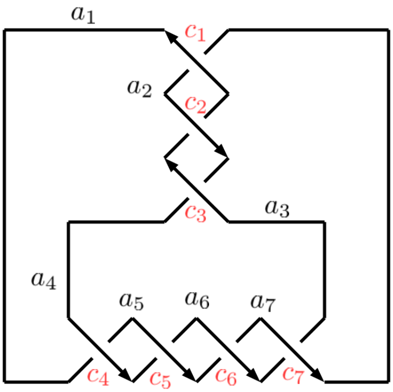
$$\Delta_K^0(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_N t^N,$$

where N is even, $c_{N-r} = c_r$, $c_{\frac{N}{2}}$ is odd and $c_0 > 0$.

Coloring knots by Alexander quandles

Example

We consider the diagram D of the knot 7_3 whose arcs are labeled as shown in the following figure



$$c_1 : a_2 = ta_7 + (1-t)a_1$$

$$c_2 : a_1 = ta_3 + (1-t)a_2$$

$$c_3 : a_4 = ta_2 + (1-t)a_3$$

$$c_4 : a_5 = ta_1 + (1-t)a_4$$

$$c_5 : a_6 = ta_4 + (1-t)a_5$$

$$c_6 : a_7 = ta_5 + (1-t)a_6$$

$$c_7 : a_3 = ta_6 + (1-t)a_7$$

Coloring knots by Alexander quandle

Example: The Alexander matrix

we get the following Alexander matrix A:

$$A = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{matrix} \\ \begin{pmatrix} 1-t & -1 & 0 & 0 & 0 & 0 & t \\ -1 & 1-t & t & 0 & 0 & 0 & 0 \\ 0 & t & 1-t & -1 & 0 & 0 & 0 \\ t & 0 & 0 & 1-t & -1 & 0 & 0 \\ 0 & 0 & 0 & t & 1-t & -1 & 0 \\ 0 & 0 & 0 & 0 & t & 1-t & -1 \\ 0 & 0 & -1 & 0 & 0 & t & 1-t \end{pmatrix} \end{array}$$

- The determinant of the minor matrix M is $\det(M) = -2t + 3t^2 - 3t^3 + 3t^4 - 2t^5$.

Coloring knots by Alexander quandle

Example: The Alexander matrix

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- The determinant of the minor matrix M is $\det(M) = -2t + 3t^2 - 3t^3 + 3t^4 - 2t^5$.
- The reduced Alexander polynomial of the knot 7_3 is $\Delta^0_{7_3}(t) = 2 - 3t + 3t^2 - 3t^3 + 2t^4$.

Coloring links by linear Alexander quandle

if m and n are coprime integers, a coloring of a diagram D of a link L by the quandle $\Lambda_{n,m}$ is a map from the set $\mathcal{A}$ of arcs of L to $\Lambda_{n,m}$ satisfying the coloring conditions, such non-trivial coloring exists if n divides $\Delta_L^0(m)$. That coloring is called an (n, m)-coloring.

Minimum number of colors

Kauffman and Lopes [2] studied the number of distinct colors appeared in a non-trivially (p, m)-colored knot diagram.

Definition

Let L be a link admitting non-trivial (p, m)-colorings. Let D be a diagram of L and let $n_{D,p,m}$ be the minimum number of colors it takes to equip D with a non-trivial (p, m)-coloring. We let

$$mincol_{p,m}(L) = min\{n_{D,p,m} | D \text{ is a diagram of } L\}$$

and refer to it as the minimum number of colors for non-trivial (p, m)-colorings of L.

A lower bound for the minimum number of link colorings by linear Alexander quandles

Theorem [*Kauffman and Lopes*]

Theorem (Kauffman and Lopes)

Let K be a knot i.e., a 1-component link. Let p be an odd prime. Let m be an integer such that K admits non-trivial (p, m)-colorings $\pmod{p}$. If $m \neq 2$ (or $m = 2$ but $\Delta^0_K(m) \neq 0$) then

$$2 + \lfloor \log_M p \rfloor \leq mincol_{p,m}(K),$$

where $M = max\{|m|, |m-1|\}$.

An improvement of the lower bound for the minimum number of knot colorings by linear Alexander quandles

Lemma

Let K be a knot and $\Delta_K^0(t) = \sum_{i=0}^{k} c_i t^i$, its reduced Alexander polynomial. Let m be an integer, $m > 1$, and $p = \Delta_K^0(m)$. If $m > \max_{0 \leq i \leq k}\{|c_i|\} + 1$, then $2 + \lfloor \log_m p \rfloor$ is either $k+1$ or $k+2$.

Remark 1

Remark

The proof shows that the condition $m > \max_{0 \leq i \leq k} \{|c_i|\} + 1$ *is needed in the only one case where the non-null coefficients do not alternate and the two penultimate non-null coefficients have negative signs. Otherwise the weaker condition* $m > \max_{0 \leq i \leq k} \{|c_i|\}$ *suffices.*

Theorem 1

Theorem

Let K be a knot. Let $\Delta_K^0(t) = \sum_{i=0}^{k} c_i t^i$ be the reduced Alexander polynomial of K. Let m be an integer, such that $m > \max_{0 \leq i \leq k} \{|c_i|\} + 1$ and $p = \Delta_K^0(m)$ is an odd prime integer.

1. *If $c_k = 1$ and the penultimate non-zero coefficient is negative, then*
$$k + 1 \leq mincol_{p,m}(K).$$
2. *If $c_k > 1$ or the penultimate non-zero coefficient is positive, then*
$$k + 2 \leq mincol_{p,m}(K).$$

Applications to L-space knots

The non-zero coefficients of the reduced Alexander polynomial of an L-space knot K are all ± 1, and they alternate in sign [4].

Corollary

Let K be an L-space knot. If m is an integer such that $m > 1$ and $p = \Delta_K^0(m)$ is an odd prime, then

$$k + 1 \leq mincol_{p,m}(K).$$

Applications to L-space knots

Torus knots

A torus link $T_{p,q}$ is a special kind of links that lies on a surface of an unknotted torus in $\mathbb{R}^3$. It is created by traveling p times vertically and q times horizontally around the torus.

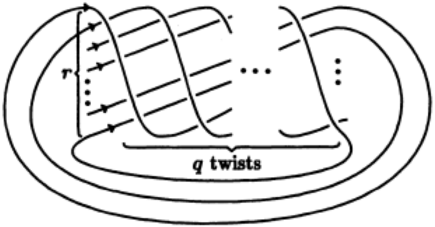

Figure: Torus link $T_{p,q}$.

- If p and q are coprime then $T_{p,q}$ is a knot. A torus link arises if p and q are not coprime in which case the number of components is $gcd(p, q)$.

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- Torus knots are L-space knots [4].
- If $T_{a,b}$ is a torus knot, then the non-zero coefficients of the reduced Alexander polynomial $\Delta^0_{T_{a,b}}(t)$ are all ± 1, and they alternate in sign [4].

Proposition

Let $T_{a,b}$ be a torus knot. Let m be an integer such that, $m > 1$ and $p = \Delta^0_{T_{a,b}}(m)$ is an odd prime, then

$$c(T_{a,b}) - (a-2) \leq mincol_{p,m}(T_{a,b}) \leq c(T_{a,b}),$$

where $c(T_{a,b})$ is the crossing number of the torus knot $T_{a,b}$.
In particular

$$mincol_{p,m}(T_{2,b}) = c(T_{2,b}).$$

The crossing number is the minimum number of crossing in any diagram of a knot.

Applications to L-space knots

Pretzel knots $P(-2, 3, 2l+1)$

A pretzel link $P(p_1, p_2, p_3, \dots, p_n)$ is defined by an n-tuple $(p_1, p_2, p_3, \dots, p_n)$, $n \geq 3$, such that each p_i is nonzero integer. The absolute value of p_i is the number of half twists and the sign of p_i is either positive or negative as seen in the following figure

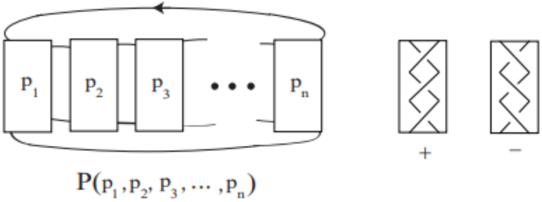

Figure: Pretzel link $P(p_1, p_2, p_3, \dots, p_n)$.

The pretzel link $P(p_1, p_2, p_3, \dots, p_n)$ is a knot if and only if both n and all the p_i are odd or exactly one of the p_i is even.

Applications to L-space knots

Pretzel knots $P(-2, 3, 2l + 1)$

Lidman and Moore showed that $P(-2, 3, 2l + 1)$, $l \geq 0$, are the only L-space Pretzel knots [3]. In what follows, we prove that if we color a diagram of the Pretzel knot $P(-2, 3, 2l + 1)$, $l \geq 0$, by the linear Alexander quandle $\mathbb{Z}_p[t]/(t-2)$ where p is an odd prime, then the lower bound in Theorem 1 is reached.

Proposition

Let $K = P(-2, 3, a)$ be a pretzel knot where a is an odd positive integer. Let $m > 1$ be an integer such that $p = \Delta_K^0(m)$ is an odd prime. If $\Delta_K^0(m) \neq 0$ then

$$a + 4 \leq mincol_{p,m}(K).$$

In particular, if $m = 2$ then

$$a + 4 = mincol_{p,m}(K).$$

A lower bound for the minimum number of link colorings by linear Alexander quandles

Theorem

Theorem

Let L be a link whose reduced Alexander polynomial $\Delta_L^0(t) \neq 0$. Let m be an integer and p a prime factor of $\Delta_L^0(m)$ such that L admits non-trivial (p, m)-colorings. Suppose that $\Delta_L^0(m) \neq 0$ then

$$2 + \lfloor \log_M p \rfloor \leq mincol_{p,m}(L),$$

where $M = max\{|m|, |m-1|\}$.

References

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[4] P. Ozsváth, A. Stipsicz and Z. Szabó, *Concordance homomorphisms from knot Floer homology*, Advances in mathematics, Vol. 315 (2017), 366-426.
[5] P. Ozsváth, Z. Szabó, *On knot Floer homology and lens space surgeries*, Topology, Vol. 44, No. 6, (2005), pp. 1281-1300.

Thank you for your attention