On the rational topological complexity of coformal elliptic spaces

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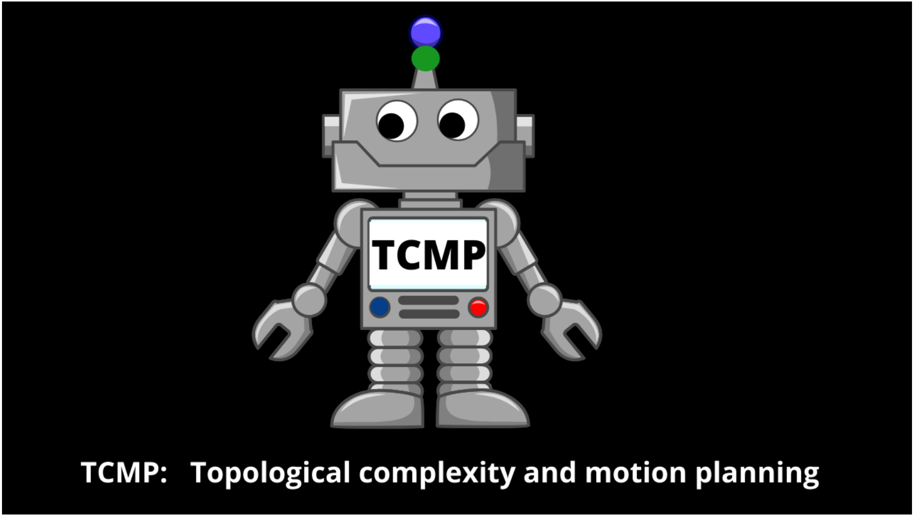
TCMP
TCMP: Topological complexity and motion planning

Introduction

Let S be a topological space.

Definition

The topological complexity of S denoted by $\mathrm{TC}(S)$ is the least integer n such that there exists a family of open subsets $U_0, \cdots, U_n$ covering $S \times S$ and local continuous sections of the evaluation map
$e_{0,1} : S^{[0,1]} \longrightarrow S \times S, \quad \gamma \longmapsto (\gamma(0), \gamma(1))$.

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The Lusternik–Schnirelmann category of S, $\mathrm{cat}(S)$, is the least integer n such that there exists a family of open subsets $U_0, \cdots, U_n$ covering S, each of which is contractible in S.

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The Lusternik–Schnirelmann category of S, $\mathrm{cat}(S)$, is the least integer n such that there exists a family of open subsets $U_0, \cdots, U_n$ covering S, each of which is contractible in S.

- TC and cat are homotopy invariants and satisfy the inequality

$$\mathrm{cat}(S) \leq \mathrm{TC}(S) \leq 2\mathrm{cat}(S).$$

- In particular for spheres S^n we have

$$\begin{cases} \mathrm{TC}(S^n) = \mathrm{cat}(S^n) = 1 \text{ if } n \text{ is odd} \\ \mathrm{TC}(S^n) = 2\mathrm{cat}(S^n) = 2 \text{ if } n \text{ is even.} \end{cases}$$

Actually $\mathrm{TC}(S^n) = \mathrm{zcl}_{\mathbb{Q}}(S^n)$.

- $\mathrm{zcl}_{\mathbb{Q}}(S)$ is the maximal length of a non-trivial product in the kernel of the multiplication $H^*(S;\mathbb{Q}) \otimes H^*(S;\mathbb{Q}) \to H^*(S;\mathbb{Q})$.

- For S^n we have $H^*(S^n;\mathbb{Q}) = \frac{\mathbb{Q}[x]}{(x^2)}$ with $|x| = n$ and

$$(x \otimes 1 - 1 \otimes x)^2 = \begin{cases} 0 \text{ if } |x| \text{ is odd} \\ \neq 0 \text{ if } |x| \text{ is even .} \end{cases}$$

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From now on S is a simply-connected CW-complex of finite type.
The rational topological complexity and LS-category are defined by

$$\mathrm{TC}_0(S) := \mathrm{TC}(S_0), \quad \mathrm{cat}_0(S) := \mathrm{cat}(S_0)$$

where S_0 is the rationalization of S. This means

- S_0 is a rational space-$\pi_*(S_0)$ or equivalently $H^*(S_0;\mathbb{Z})$ is a rational vector space.
- There exists $\rho_S : S \to S_0$ which induces $\pi_*(S) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_*(S_0)$ and $H^*(S_0) \xrightarrow{\cong} H^*(S;\mathbb{Q})$.

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- $\mathrm{zcl}_{\mathbb{Q}}(S) \leq \mathrm{TC}_0(S) \leq \mathrm{TC}(S)$
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- The general goal is to study TC_0 in terms of cat_0 for elliptic spaces.
- S is said (rationally) **elliptic** if $H^*(S;\mathbb{Q}) < \infty$ and $\pi_*(S)\otimes\mathbb{Q} < \infty$.
- The main tools in this context are Sullivan models.
- A minimal Sullivan model $(\Lambda V, d)$ of S is a cochain algebra which is free as a commutative graded algebra and satisfies

$$d(V)\subset \Lambda^{\geq 2}V, \quad V\cong \pi_*(S)\otimes\mathbb{Q}, \quad H^*(\Lambda V, d) = H^*(S;\mathbb{Q}).$$

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Examples of minimal Sullivan models

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- For the homogeneous space $S = \frac{SU(6)}{SU(3)\times SU(3)}$, $(\Lambda V, d) = (\Lambda(x_1, x_2, y_1, y_2, z, d)$ with $|x_1| = 4$, $|x_2| = 6$, $dx_1 = 0$, $dx_2 = 0$, $dy_1 = x_1^2$, $dy_2 = x_2^2$ and $dz = x_1x_2$.

degree	4	6	13	15	19
$H^*(S;\mathbb{Q})$	$[x_1]$	$[x_2]$	$[x_1z - x_2y_1]$	$[x_2z - x_1y_2]$	$[x_1x_2z - x_1^2y_2]$

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- The minimal Sullivan model $(\Lambda V, d)$ is said **pure** if

$$dV^{even} = 0 \quad \text{and} \quad dV^{odd} \subset \Lambda V^{even}.$$

- The space S is said **formal** if there is a quasi-isomorphism

$$(\Lambda V, d) \xrightarrow{\simeq} (H^*(S;\mathbb{Q}), 0).$$

- Note that S^n is a formal space while $\frac{SU(6)}{SU(3)\times SU(3)}$ is not.

Theorem (Lechuga-Murillo)

For any formal space, we have

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Definition: for S elliptic the **homotopy characteristic** is defined by

$$\chi_\pi(S) = \dim \pi_{even}(S) \otimes \mathbb{Q} - \dim \pi_{odd}(S) \otimes \mathbb{Q}.$$

- $\chi_\pi(S) \leq 0$ and if $(\Lambda V, d)$ is a minimal model of S then
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- If $\chi_\pi(S) = 0$, S is automatically formal and called F_0-space.

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If S is a pure formal elliptic space then $\mathrm{TC}_0(S) = 2\mathrm{cat}_0(S) + \chi_\pi(S)$.

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Theorem (Jessup-Murillo-Parent, 2012)

Let $(\Lambda V, d)$ be a minimal Sullivan model of S. If the projection

$$\rho_m : (\Lambda V \otimes \Lambda V, d) \to \left(\frac{\Lambda V \otimes \Lambda V}{(\ker \mu_{\Lambda V})^{m+1}}, \bar{d} \right),$$

where $\mu_{\Lambda V} : \Lambda V \otimes \Lambda V \to \Lambda V$ is the multiplication, admits a homotopy retraction, then $\mathrm{TC}_0(S) \leq m$.

Theorem (Carrasquel, 2017)

$$\mathrm{TC}_0(S) \leq m \Leftrightarrow \rho_m \text{ admits a homotopy retraction.}$$

Note that $\ker \mu_{\Lambda V}$ is the ideal of $\Lambda V \otimes \Lambda V$ generated by the elements $\underbrace{v \otimes 1}_{v} - \underbrace{1 \otimes v}_{v'}$ where $v \in V$.

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- If ρ_m admits a homotopy retraction then $H(\rho_m)$ is injective.

- Denoting by $\mathrm{HTC}(S)$ the least integer m for which $H(\rho_m)$ is injective, we have

$$\mathrm{zcl}_{\mathbb{Q}}(S) \leq \mathrm{HTC}(S) \leq \mathrm{TC}_0(S).$$

- If $(\Lambda V, d)$ is a Sullivan model, we may use the notations $\mathrm{TC}(\Lambda V)$, $\mathrm{HTC}(\Lambda V)$, $\mathrm{cat}(\Lambda V)$ instead of $\mathrm{TC}_0(S)$, $\mathrm{HTC}(S)$, $\mathrm{cat}_0(S)$.

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Example

We consider the homogeneous space $S = \frac{SU(6)}{SU(3)\times SU(3)}$ whose minimal Sullivan model is

$$(\Lambda V, d) = (\Lambda(x_1, x_2, y_1, y_2, z), d)$$

where $|x_1| = 4$, $|x_2| = 6$, $dx_1 = dx_2 = 0$, $dy_1 = x_1^2$, $dy_2 = x_2^2$, $dz = x_1 x_2$. We will see that $\mathrm{TC}_0(S) \geq 5$.

We first construct

$$\Omega := (x_1 - x_1')(x_2 - x_2')(z - z') - \frac{1}{2}(y_2 - y_2')(x_1 - x_1')^2 - \frac{1}{2}(y_1 - y_1')(x_2 - x_2')^2$$

which satisfies $d\Omega = 0$, $\Omega \in (\ker \mu_{\Lambda V})^3$ and $[\Omega] \neq 0$.

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We consider the quasi-isomorphism

$$\varphi : (\Lambda(x_1, x_2, y_1, y_2, z), d) \to (A, \bar{d}) = \left(\frac{\Lambda(x_1, x_2)}{(x_1^2, x_2^2)} \otimes \Lambda(z), \bar{d} \right)$$

degree	4	6	13	15	19
$H^*(A)$	$[x_1]$	$[x_2]$	$[x_1 z]$	$[x_2 z]$	$[x_1 x_2 z]$

We next construct the cocycles $\beta_1, \beta_2 \in \ker \mu_{\Lambda V}$ such that $[\Omega \beta_1 \beta_2] \neq 0$.

Explicitly $\beta_1 = (x_2 - x_2'), \quad \beta_2 = (x_1 z - x_2 y_1 - x_1' z' + x_2' y_1')$. We have

$$[(\varphi \otimes \varphi)(\Omega \beta_1 \beta_2)] = -4[x_1 x_2 z \cdot x_1' x_2' z'] \neq 0.$$

Since $\Omega \beta_1 \beta_2 \in (\ker \mu_{\Lambda V})^5$ and $[\Omega \beta_1 \beta_2] \neq 0$ it follows that $\mathrm{TC}_0(S) \geq 5$.

- Here $\mathrm{cat}_0(S) = \dim \pi_{odd}(S) \otimes \mathbb{Q} = 3$ (d is quadratic).

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Since $\Omega\beta_1\beta_2 \in (\ker \mu_{\Lambda V})^5$ and $[\Omega\beta_1\beta_2] \neq 0$ it follows that $\mathrm{TC}_0(S) \geq 5$.

- Here $\mathrm{cat}_0(S) = \dim \pi_{odd}(S) \otimes \mathbb{Q} = 3$ (d is quadratic).

We consider the quasi-isomorphism

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degree	4	6	13	15	19
$H^*(A)$	$[x_1]$	$[x_2]$	$[x_1 z]$	$[x_2 z]$	$[x_1 x_2 z]$

We next construct the cocycles $\beta_1, \beta_2 \in \ker \mu_{\Lambda V}$ such that $[\Omega \beta_1 \beta_2] \neq 0$.

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A lower bound of TC_0 for coformal pure elliptic spaces

Definition: $(\Lambda V, d)$ is **coformal** if $dV \subset \Lambda^2 V$.

Theorem (Félix-Halperin)

When S is elliptic and admits a coformal Sullivan model, we have

$$\mathrm{cat}_0(S) = \dim \pi_{odd}(S) \otimes \mathbb{Q} = \dim(V^{odd}).$$

Theorem (Lower bound)

Let $(\Lambda V, d)$ be a pure elliptic coformal model. For every basis $\mathcal{B}$ of V^{even}, we have

$$\mathrm{cat}(\Lambda V) + L(\Lambda V, \mathcal{B}) \leq \mathrm{TC}(\Lambda V)$$

or equivalently

$$\dim(V^{odd}) + L(\Lambda V, \mathcal{B}) \leq \mathrm{TC}(\Lambda V)$$

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$L(\Lambda V, \mathcal{B})$

Suppose that $(\Lambda V, d) = (\Lambda X \otimes \Lambda Y, d)$ with $X = V^{even}$, $Y = V^{odd}$
$dX = 0$, $dY \subset \Lambda X$ and let $\mathcal{B} = \{x_1, \cdots, x_n\}$ be a basis of X.

We consider the extension $(\Lambda W_{\mathcal{B}}, d)$ such that $\Lambda W_{\mathcal{B}} = \Lambda(V \oplus U)$ with U is a vector space generated by u_i satisfying $du_i = x_i^2$ for all $i = 1, \cdots, n$.

As d is a quadratic differential, it induces a bigraded differential

$$d_{p,q} : \Lambda^p X \otimes \Lambda^q (Y \oplus U) \to \Lambda^{p+2} X \otimes \Lambda^{q-1}(Y \oplus U)$$

as well as a bigradation on $H(\Lambda W_{\mathcal{B}})$.

$$H_{p,q}(\Lambda W_{\mathcal{B}}) = \frac{\ker(d_{p,q})}{Im(d_{p-2,q+1})}.$$

We define

$$L(\Lambda V, \mathcal{B}) := max\{r : \exists \alpha_1, \cdots, \alpha_r \in H_{odd,*}(\Lambda W_{\mathcal{B}}) \text{ with } \alpha_1 \cdots \alpha_r \neq 0\}.$$

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For $\frac{SU(6)}{SU(3)\times SU(3)}$ we have

$$\Lambda V = \Lambda(\overset{0}{\underset{}{\overset{\uparrow}{x_1}}}, \overset{0}{\overset{\uparrow}{x_2}}, \overset{x_1^2}{\overset{\uparrow}{y_1}}, \overset{x_2^2}{\overset{\uparrow}{y_2}}, \overset{x_1x_2}{\overset{\uparrow}{z}}) \hookrightarrow \Lambda W_{\mathcal{B}} = \Lambda(x_1, x_2, y_1, y_2, z, \overset{x_1^2}{\overset{\uparrow}{u_1}}, \overset{x_2^2}{\overset{\uparrow}{u_2}})$$

The elements

$$\alpha_1 = x_2 \quad \text{and} \quad \alpha_2 = x_1 z - x_2 y_1$$

are cocycles with classes in $H_{1,*}(\Lambda W_{\mathcal{B}})$.

We have $[\alpha_1][\alpha_2] \neq 0$ in $H(\Lambda W_{\mathcal{B}})$. Therefore $L(\Lambda V, \mathcal{B}) \geq 2$.

Actually we can check that $L(\Lambda V, \mathcal{B}) = 2$.

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An upper bound of TC_0 for certain pure elliptic spaces

Theorem

Let $(\Lambda V, d)$ be a Sullivan model and $(\Lambda V \otimes \Lambda U, d)$ be an extension of $(\Lambda V, d)$ with U is a vector space concentrated in odd degrees such that $dU \subset \Lambda V$, then

$$\mathrm{TC}(\Lambda V \otimes \Lambda U) \leq \mathrm{TC}(\Lambda V) + \dim U.$$

Theorem (Upper bound)

Let $(\Lambda V, d)$ be a pure elliptic minimal Sullivan model. If there exist

- *an extension $(\Lambda Z, d) \hookrightarrow (\Lambda V, d)$ where $Z^{even} = V^{even}$ and $(\Lambda Z, d)$ is an F_0-model*
- *an integer k such that $dV \subset \Lambda^k V$*

then

$$\mathrm{TC}(\Lambda V) \leq 2\mathrm{cat}(\Lambda V) + \chi_\pi(\Lambda V).$$

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$$\mathrm{TC}(\Lambda V) \leq 2\mathrm{cat}(\Lambda V) + \chi_\pi(\Lambda V).$$

Corollary

Let $(\Lambda V, d)$ be a pure coformal elliptic Sullivan model. Assume that there exists an extension $(\Lambda Z, d) \hookrightarrow (\Lambda V, d)$ where $(\Lambda Z, d)$ is an F_0-model satisfying $Z^{even} = V^{even}$. Then

$$\mathrm{TC}(\Lambda V) \leq 2\mathrm{cat}(\Lambda V) + \chi_\pi(\Lambda V) = \dim V.$$

Example

For $\frac{SU(6)}{SU(3)\times SU(3)}$ we have $(\Lambda Z, d) \hookrightarrow (\Lambda V, d) = (\Lambda(x_1, x_2, y_1, y_2, z), d)$ where

- $(\Lambda Z, d) = (\Lambda(x_1, x_2, y_1, y_2), d)$ is an F_0-model.
- $\chi_\pi(\Lambda V) = -1, \quad \mathrm{cat}(\Lambda V) = 3 = \dim V^{odd}$.
- Conclusion: $\mathrm{TC}(\Lambda V) \leq 5$ and finally $\mathrm{TC}_0(S) = 5$.

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Special family

Theorem

For any model of the form $(\Lambda V, d) = (\Lambda(x_i, u_i, y), d)$ *with* $dx_i = 0$, $du_i = x_i^2$ *for all* $i = 1, \cdots, n$ *and* $dy = \sum_{i,j} \lambda_{i,j} x_i x_j$. *We have*

$$\mathrm{TC}(\Lambda V) = 2\mathrm{cat}(\Lambda V) + \chi_\pi(\Lambda V) = \dim V.$$

Note that by the upper bound theorem and by considering the extension $\Lambda(x_i, u_i) \hookrightarrow \Lambda(x_i, u_i, y)$ we have

$$\mathrm{TC}(\Lambda V) \leq 2\mathrm{cat}(\Lambda V) + \chi_\pi(\Lambda V) = 2n + 1.$$

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- We consider the quasi-isomorphism

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- There exists a cocycle $\Omega \in (\ker \mu_{\Lambda V})^{n+1}$ with

$$(\varphi \otimes \varphi)(\Omega) = (-1)^n \prod_{i=1}^{n} (x_i - x_i')(y - y').$$

- Construction of an explicit cocycle $\beta \in (\ker \mu_{\Lambda V})^n$ such that $(\varphi \otimes \varphi)(\Omega \cdot \beta)$ is the top class of $A \otimes A$.

- In conclusion $\mathrm{TC}(\Lambda V) \geq 2n + 1$ and $\mathrm{TC}(\Lambda V) = 2n + 1 = \dim V$.

Work in progress

- Investigate the calculation and properties of the cuplength $L(\Lambda V, \mathcal{B})$.
- Extend the obtained results to other classes of elliptic spaces.

References

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