Lecture on the Hilali conjecture

The First International Conference on Algebraic Topology and its Applications in Robotics In Honor of Professor Mohamed Rachid HILALI

Saloua CHOUINGOU

University Moulay Ismail, Meknès, Morocco

March, 17-18, 2023

Plan

Rational Homoyopy Theory

Rational homotopy theory is the study of rational homotopy types of spaces and of the properties of spaces and maps that are invariant under rational homotopy equivalence.

Topological Version

Conjecture H, 1990

Let X be a simply connected CW-complex of elliptic space, then

$$\dim \pi_*(X) \otimes \mathbb{Q} \leq \dim H^*(X, \mathbb{Q}).$$

Elliptic Space

Let X be a simply-connected space. We say that X is rationally elliptic if the dimensions of cohomology and homotopy are both finite, i.e.,

$$\dim \pi_*(X) \otimes \mathbb{Q} < \infty \text{ and } \dim H^*(X;\mathbb{Q}) < \infty.$$

For example :

- The spheres $\mathbb{S}^n$;
- The Eilenberg-Mac Lane space $K(\mathbb{Q}, 2n+1)$.

A Sullivan algebra is a cochain commutative graded algebra of the form $(\wedge V, d)$ where d is a differential, that is, d is a linear map $d : V \longrightarrow V$ such that

$$d(xy) = (dx)y + (-1)^{\deg x} x(dy); x, y \in V$$

.

The Sullivan algebra $(\wedge V, d)$ is called minimal if

$$dV \subset \wedge^{\geq 2} V$$

.

The Sullivan model

Sullivan minimal models have been used to describe the rational homotopy of a space.

The Sullivan model

Sullivan minimal models have been used to describe the rational homotopy of a space.

D. Sullivan, 1978

If $(\wedge V, d)$ is the Sullivan minimal model of X, there are isomorphism's :

$$H^*(X, \mathbb{Q}) = H^*(\wedge V, d)$$

$$\pi_*(X) \otimes \mathbb{Q} \simeq Hom_{\mathbb{Q}}(V^*, \mathbb{Q}).$$

The Sullivan model

Example

1. The minimal Sullivan model of $\mathbb{C}P^n$ is of the form :

$$(\wedge(a,b),d), \quad da = 0, db = a^{n+1}, \quad \deg a = 2.$$

Algebraic Version

In terms of Sullivan minimal models, the Hilali conjecture can be rewritten as follows :

Conjecture H, 1990

Let $(\Lambda V, d)$ is a 1-connected elliptic model, then we have :

$$\dim V \leq \dim H^*(\Lambda V, d)$$

M.R. Hilali, 1990

The Conjecture H holds for pure models.

$(\Lambda V, d)$ is a pure Sullivan model if

$$V = V^{even} \oplus V^{odd} \text{ with } dV^{\text{even}} = 0 \text{ and } dV^{\text{odd}} \subset \Lambda V^{\geq 2} V^{\text{even}}$$

M.R.Hilali and M.I.Mamouni, 2006

The conjecture H is solved in the following cases :

1. H-spaces ;
2. Hyperelliptic spaces under the condition that :

$$\chi_c \geq \frac{1}{2}\left(1 + \sqrt{-12\chi_\pi - 15}\right)$$

hyperelliptic spaces $(\Lambda V, d)$:

$$\begin{cases} dV^{\text{even}} & = 0 \\ dV^{\text{odd}} & \subset \Lambda^+ V^{\text{even}} \otimes \Lambda V^{\text{odd}} \end{cases}$$

The cohomological Euler-Poincaré charachteristic :

$$\chi_c := \sum_{k \geq 0} \dim H^k(\wedge V, d)$$

M.R.Hilali and M.I.Mamouni, 2007

The conjecture H is solved in the following cases :

1. 1-connected and hyperelliptic spaces with the additional condition that $-2 \leq rk_0(X) + \chi_\pi \leq 0$;
2. 1-connected and elliptic spaces with the additional condition that $0 \leq fd(X) - rk_0(X) \leq 6$;
3. 1-connected and elliptic spaces with the additional condition that $fd(x) \leq 10$;
4. Symplectic manifolds ;
5. Cosymplectic manifolds ;

• $rk_0(X)$: The toral rank of a space X is the maximum of n such that the toric $\mathbb{T}^n$ acts almost freely on X ;
• $fd(X)$: The formal dimension is the largest integer k such that $H^k(X, \mathbb{Q}) \neq 0$.

M.R.Hilali and M.I.Mamouni, 2008

The conjecture H is solved in the following cases :

1. Formal spaces ;
2. Nilmanifolds ;
3. Coformal spaces with odd degree generators only ;
4. Elliptic minimal model $(\Lambda V, d)$ has a differential, homogeneous of length at least 3 ;
5. Elliptic minimal model with an homogeneous-length differential and whose rational Hurewicz homorphism is non-zero in some odd degree.

O. Nakamura, T. Yamaguchi, 2011

1. The conjecture H is true if X is is a simply connected elliptic space with formal dimension less than or equal to 16, i.e.,

$$fd(x) \leq 16$$

J.Fernández de Bobadilla, J.Fresán, V.Muñoz and A.Murillo, 2014

The conjecture H holds for hyperelliptic spaces.

The conjecture H holds for 2-stage spaces.

$(\Lambda W, d)$ is 2-stage algebras, if $W = U \oplus V$ with :
(1) $\mathrm{d}(U) = 0$;
(2) $\mathrm{d}(V) \subseteq \Lambda U$.

M.R.Hilali, M.I.Mamouni and H.Yamoul, 2015

The Hilali conjecture holds for the configuration spaces of a rationally elliptic and simply connected topological space when it already holds for the space itself.

B.B.El Krafi, M.R. Hilali, M.I.Mamouni, 2016

The Hilali conjecture holds for any coformal space X whose rational homotopy Lie algebra $\mathbb{L}$ is of nilpotency 1 or 2.

H.Aaya, M.A.Hilali, M.R.Hilali and T.Jawad, 2019

The Hilai conjecture holds for a sullivan minimal model $(\Lambda V, d)$, where $V = \mathbb{Q}(a_1, \cdots, a_n)$ with $|a_i|$ are odd integers under the condition that for all $i, 1 \leq i \leq n$,

$$A_i = \Lambda(a_1, \cdots, a_i), da_i = P_i \in A_i, \text{ such that } [P_i^2] = 0 \text{ in } H^*(A_{i-1}).$$

S. Cattalani and A.Milivojevic, 2019

The Hilali conjecture holds in formal dimension ≤ 20

S.Yokura, 2019

For any such space X there exists a positive integer n_0 such that for any $n \geqq n_0$ the strict inequality $\dim\left(\pi_*\left(X^n\right) \otimes \mathbb{Q}\right) < \dim H_*\left(X^n; \mathbb{Q}\right)$ holds.

Y.Rami, 2019

The Hilali conjecture holds for sullivan minimal model $(\Lambda V, d)$ such that $(\Lambda V, d_k)$ is elliptic with $V = V^{\text{odd}}$ and $k \geq 3$.

S.Chouingou and M.R.Hilali, 2019

The Hilali conjecture is true for every space X simply connected elliptic such that $\dim H^*(X;\mathbb{Q}) \leq 10$

The relative Hilali conjecture (topologic version)

Let $f : X \to Y$ be a continuous map between two elliptic spaces, then

$$\dim \text{ Ker } \pi_*(f)_{\mathbb{Q}} \leq \dim \text{ Ker } H_*(f;\mathbb{Q}) + 1. \qquad \text{(RH)}$$

The relative Hilali conjecture (topologic version)

Let $f : X \to Y$ be a continuous map between two elliptic spaces, then

$$\dim \text{ Ker } \pi_*(f)_{\mathbb{Q}} \leq \dim \text{ Ker } H_*(f;\mathbb{Q}) + 1. \qquad \text{(RH)}$$

Where

$$\begin{aligned} \text{Ker } \pi_*(f)_{\mathbb{Q}} \quad &: \quad = \oplus_{i\geq 1}\text{Ker } (\pi_i(f)_{\mathbb{Q}} : \pi_i(X)_{\mathbb{Q}} \to \pi_i(Y)_{\mathbb{Q}}) \text{ and} \\ \text{Ker } H_*(f;\mathbb{Q}) \quad &: \quad = \oplus_{i\geq 0}\text{Ker } (H_i(f;\mathbb{Q}) : H_i(X;\mathbb{Q}) \to H_i(Y;\mathbb{Q})). \end{aligned}$$

References

- Amann M., A note on the Hilali conjecture. Forum Mathematicum. 29(2), 251- 258, 2017.
- J. F. De Bobadilla, J. Fresan, V. Mu~noz and A. Murillo, The Hilali conjecture for hyper elliptic spaces. Mathematics without boundaries : Surveys in Pure Mathematics, P. Pardalos and Th.M. Rassias, eds. (2014) 21-36.
- S. Cattalani and A.Milivojevic, Verifiying the Hilali conjecture in formal dimension ≤ 20, 2019.
- S. Chouingou and M.R. Hilali. Classification of Rational Homotopy Type for 10-Cohomological Dimension Elliptic Spaces. Mathematica, 62 (85)-1 :16-32, 2020.
- El Krafi B.B., Hilali M.R. and Mamouni M.I., On Hilali's conjecture related to Halperin's. JHRS 3, 493-501, 2016. Texts in Mathematics. 205, Springer-Verlag, New York, 2001.

References

- Hilali M.R., Action du tore T^n sur les espaces simplement connexes. PhD thesis, Université catholique de Louvain, 1980.
- Hilali M.R. and Mamouni M.I., A conjectured lower bound for the cohomological dimension of elliptic spaces. J. Homotopy Relat. Struct. 3(1), 379-384, 2008.
- Hilali M.R., Mamouni M.I. and Yamoul H., On the Hilali Conjecture for Confi- guration Spaces of Closed Manifolds. Afr. Diaspora J. Math. (N.S.) 18, no. 1, 1-11, 2015.
- Nakamura O. and Yamaguchi T., Lower bounds of Betti numbers of elliptic spaces with certain formal dimensions. Kochi J. Math. 6, 9-28, 2011.
- Y. Rami. Gaps in the Milnor-Moore spectral sequence and the Hilali conjecture. Ann. Math. Qu ´ebec. https ://doi.org/10.1007/s40316-018-0107-4 :2018.

Thank you
for your
attention