Higher topological complexity and the Eilenberg-Moore functor Ext

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Plan

Topological complexity of a space X, $TC(X)$, is introduced by M. Farber which is seen as the *sectional category*, $secat(\Delta_X)$, of the diagonal map $\Delta_X : X \to X \times X$. If $f : X \to Y$ is a continuous map, $secat(f)$ is the smallest integer m for which there are $m+1$ local homotopy sections $s_i : U_i \to Y$ for f whose sources form an open covering of X. Later, Y. Rudyak generalizes Farber's concept and introduces that of higher topological complexity, $TC_n(X)$ ($n \geq 2$), which turns out to be the sectional category $secat(\Delta_X^n)$ of the n-diagonal $\Delta_X^n : X \to X^n$.

Determining $TC_n(X)$ is as difficult as it is in the case of Lusternik-Scnirelmann category $cat_{LS}(X) = secat(* \hookrightarrow X)$, however thanks to rational homotopy theory methods, it is possible to establish better approximation of this latter, namely $cat_{LS}(X_0) \leq cat_{LS}(X)$ with X_0 denoting the rationalization X_0 of X.

Then, J. Carrasquel used the characterization à la Félix-Halperin to give an explicit definition of the higher topological complexity $TC_n(X_0)$ that turns out to lower $TC_n(X)$ (cf. Definition 4.1 below), and he also introduced higher (rational) *homology (resp. module) topological complexity* $HTC_n(X)$, (resp. $MTC_n(X)$) and showed that they interpolate $zcl_n(X) := nilkerH^*(\Delta_X^n)$ and $TC_n(X_0)$.

We are interested in the study of the invariant $HTC_n(X)$ when X is a *Gorenstein space*. Gorenstein spaces were introduced to provide new characterizations of spaces that satisfy Poincaré duality . They give so much interest to the invariant

$$\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K})),$$

where $\mathbb{K}$ is a field.

Inspired by Carrasquel's characterizations we introduce *Ext*-versions of the topological complexities mentioned earlier.

First we shall recall some definitions and properties used throughout this work.

A differential graded algebra (A, d) (dga for short) is a graded algebra A together with a linear map $d : A \to A$ of degree $|d| = +1$ that is a derivation : $d(ab) = d(a)b + (-1)^{|a|}ad(b)$, and satisfying $d \circ d = 0$.

A morphism of dga $f : (A, d) \to (B, d)$ is a linear map of degree zero satisfying $f(aa') = f(a)f(a')$, and the compatibility with the differential d : $f(da) = d(f(a))$.

A dga algebra A is said to be augmented if it is endowed with a morphism $\varepsilon : A \to \mathbb{K}$ of graded algebras.

A (left) graded (A, d) module is a graded module M equipped with a linear map $A \otimes M \to M$, $a \otimes m \mapsto am$ of degree zero such that $a(bm) = (ab)m$ and $1m = m$, and a differential d satisfying $d(am) = (da)m + (-1)^{|a|}a(dm), \quad m \in M, \ a \in A$.

A morphism of (left) graded modules over a dga (A, d) is a morphism $f : (M, d) \to (N, d)$ compatible with the differential : $d \circ f = f \circ d$.

A left (A, d)-module (M, d) is said semi-free if it is the union of an increasing sequence $M(0) \subset M(1) \subset M(2) \cdots \subset M(n) \subset \cdots$ of sub (A, d)-modules such that $M(0)$ and each $M(i)/M(i-1)$ is A-free on a basis of cycles.

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A semi-free resolution of an (A, d)-module (M, d) is an (A, d)-semi-free module (P, d) together with a quasi-isomorphism $m : (P, d) \xrightarrow{\simeq} (M, d)$ of (A, d)-modules.

Particularly, let $(P, d) \xrightarrow{\simeq} (\mathbb{Q}, 0)$ be an (A, d)-semi-free resolution of $(\mathbb{Q}, 0)$. This defines the graded (A, d)-module

$$Hom_A((P, d), (A, d)) = \bigoplus_{p\geq 0} Hom_A^{p,*}((P, d), (A, d)) = \bigoplus_{p\geq 0}\bigoplus_{i\geq 0} Hom_A(P^i, A^{i+p}),$$

which, endowed with the differential

$$D(f) = d \circ f - (-1)^p f \circ d; \quad f \in Hom_A^{p,*}((P, d), (A, d)),$$

yields the Eilenberg-Moore Ext functor :

$$\mathcal{E}xt_{(A,d)}(\mathbb{K}, (A, d)) = H^*(Hom_A((P, d), (A, d)), D).$$

This is an invariant up to homotopy of differential graded algebras, and if $(A, d) \xrightarrow{\simeq} (B, d)$ is a quasi-isomorphism of differential graded algebras, then $\mathcal{E}xt_{(A,d)}(\mathbb{K}, (A, d))$ is identified with $\mathcal{E}xt_{(B,d)}(\mathbb{K}, (B, d))$

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Let $(P,d) \xrightarrow{\simeq} (\mathbb{K},0)$ be a minimal (A,d)-semifree resolution of $(\mathbb{K},0)$. Consider the chain map

$$\mathrm{Hom}_{(A,d)}((P,d),(A,d)) \longrightarrow (A,d)$$

given by $f \mapsto f(z)$, where $z \in P$ is a cocycle representing 1 in $\mathbb{K}$. Passing to homology, we obtain the natural map

$$ev_{(A,d)} : \mathcal{E}xt_{(A,d)}(\mathbb{K},(A,d)) \longrightarrow H^*(A,d),$$

called the evaluation map of (A,d). The definition of $ev_{(A,d)}$ is independent of the choice of (P,d) and z. The evaluation map of X over $\mathbb{K}$ is by definition the evaluation map of $(C^*(X,\mathbb{K}),d)$.

A *Poincaré duality algebra* over $\mathbb{K}$ is a graded algebra $H = \{H^k\}_{0\le k\le N}$ such that $H^N = \mathbb{K}\alpha$ and the pairing $<\beta,\gamma>\alpha = \beta\gamma,\ \beta \in H^k,\ \gamma \in H^{N-k}$ defines an isomorphism $H^k \xrightarrow{\cong} Hom_{\mathbb{K}}(H^{N-k},\mathbb{K})$, $0 \le k \le N$. A *Poincaré space* at $\mathbb{K}$ is a space whose cohomology with coefficients in $\mathbb{K}$ is a Poincaré duality algebra.

A differential graded algebra (A,d) is called Gorenstein if the vector space $\mathcal{E}xt_{(A,d)}(\mathbb{K},(A,d))$ has dimension one. A space X is called Gorenstein over $\mathbb{K}$ if the cochain algebra $C^*(X;\mathbb{K})$ is a Gorenstein algebra.

For a simply connected finite CW complex, $C^*(X;\mathbb{K})$ is Gorenstein if and only if $H^*(X;\mathbb{K})$ is a Poincare duality algebra.

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Introduction and preliminaries

$\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K}))$ and the evaluation map

$\mathcal{E}xt$-Homology topological complexity

Application : Adams-Hilton model

Let X be a simply-connected finite type CW-complex and (P, d) a semi-free resolution of $\mathbb{Q}$ and $(\Lambda V, d) \xrightarrow{\simeq} C^*(X, \mathbb{Q})$ its minimal Sullivan model and .

Recall that a Sullivan algebra is a free cdga $(\Lambda V, d)$, where

$$(\Lambda V, d) = \text{Exterior}(V^{odd}) \otimes \text{Symmetric}(V^{even})$$

generated by the graded $\mathbb{K}$-vector space $V = \oplus_{i \geq 0} V_i$ which has a well ordered basis $\{x_\alpha\}_{\alpha \in I}$ such that $dx_\alpha \in \Lambda V_{<\alpha}$ $(\Lambda V_{<\alpha} = span\{v_{\gamma, \gamma<\alpha}\})$.

Let A and $\mathcal{A}$ denote respectively $Hom_A((P, d), (A, d))$ and $\mathcal{E}xt_{\Lambda V}(\mathbb{Q}, \Lambda V)$. Let $f, g : P \to S$ in A,

$$\begin{aligned} f \cdot g := \mu_{\mathcal{E}xt}(f \otimes g) : P \otimes_{\Lambda V} P &\longrightarrow \Lambda V \\ x \otimes y &\longmapsto (-1)^{|g||x|} f(x) g(y). \end{aligned}$$

- $*$ $f \cdot g$ is a ΛV-morphism.
- $*$ $P \otimes_{\Lambda V} P$ is an $(\Lambda V, d)$-semifree resolution of $\mathbb{Q}$.

We obtain a well-defined map of vector spaces (multiplication)

$$\begin{aligned} \mu_{\mathcal{E}xt} : \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{A} &\longrightarrow \mathcal{A} \\ [f] \otimes [g] &\longmapsto [f \cdot g] \end{aligned}$$

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$*$ $\mu_{\mathcal{E}xt}$ has a unit element.

Indeed, consider $\tilde{\varepsilon} : \Lambda V \otimes \Lambda sV \to \Lambda V$ the composition $i \circ (\varepsilon \otimes \varepsilon_{\Lambda sV})$ where $\varepsilon : \Lambda V \to \mathbb{Q}$ is the augmentation, and the map

$$\begin{array}{rccc} \theta = Id_{\Lambda V \otimes \Lambda sV} \otimes \varepsilon_{\Lambda sV} : & \Lambda V \otimes \Lambda sV \otimes \Lambda sV & \longrightarrow & \Lambda V \otimes \Lambda sV \\ & 1 \otimes sv \otimes 1 & \longmapsto & 1 \otimes sv; \\ & 1 \otimes sv \otimes sw & \longmapsto & 0; \\ & 1 \otimes 1 \otimes sv & \longmapsto & 0 \end{array}$$

makes the following diagram commutative :

$$\begin{tikzcd} \mathbb{Q} & & \Lambda V \otimes \Lambda sV \arrow[ll, "\simeq"'] \arrow[d, "f"] \\ \Lambda V \otimes \Lambda sV \otimes \Lambda sV \arrow[u, "\simeq"] \arrow[urr, "\theta"] \arrow[rr, "f \cdot \tilde{\varepsilon}"'] & & \Lambda V. \end{tikzcd}$$

Passing to cohomology, we get $[f] \cdot [\tilde{\varepsilon}] = [f]$ and similarly $[\tilde{\varepsilon}] \cdot [f] = [f]$. Henceforth, the class $[\tilde{\varepsilon}]$ defines a unit element for $\mu_{\mathcal{A}}$.

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  Let τ be the flip map $\tau : P \otimes_S P \to P \otimes_S P\, ; x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$.
  The diagram

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We respectively conclude that A is a homotopy commutative differential graded algebra with unit and $\mathcal{A}$ is a graded commutative $\mathbb{Q}$-algebra with unit.

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Finally, it is clear that the following diagram, where *cev* is the chain evaluation map of $(\Lambda V, d)$, is commutative :

$$\begin{tikzcd} \mathrm{A} \otimes \mathrm{A} \arrow[r] \arrow[d, "cev \otimes cev"] & \mathrm{A} \arrow[d, "cev"] \\ (\Lambda V, d) \otimes (\Lambda V, d) \arrow[r, "\mu_{\Lambda V}"] & (\Lambda V, d) \end{tikzcd}$$

Thus, passing to cohomology, we deduce that the evaluation map is a morphism of graded algebra.

Theorem 1.

The $\mathbb{Q}$-vector space $\mathcal{E}xt_{(S,d)}(\mathbb{Q}, (S, d))$, endowed with $\mu_{\mathcal{E}xt}$, is a graded commutative algebra with unit. And the evaluation map $\mathcal{E}xt_{(\Lambda V,d)}(\mathbb{Q}, (\Lambda V, d)) \to H^*(X; \mathbb{Q})$ is a morphism of algebras.

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Application : Adams-Hilton model

Let (A, d) be any cdga model for a space X, and $\theta : (\Lambda V, d) \xrightarrow{\simeq} (A, d)$ its minimal Sullivan model. The cdga morphism

$$\mu_n^\theta := (\mathrm{Id}_A, \theta, \dots, \theta) : (A, d) \otimes (\Lambda V, d)^{\otimes n-1} \to (A, d)$$

is a special model, called an s-model, for the path fibration $\pi_n : X^I \to X^n$.

definition 1.

(a) : $\mathrm{TC}_n(X_0)$ *is the least m such that the projection*

$$\rho_m : \left(A \otimes (\Lambda V)^{\otimes n-1}, d\right) \to \left(\frac{A \otimes (\Lambda V)^{\otimes n-1}}{\left(\ker \mu_n^\theta\right)^{m+1}}, \overline{d}\right)$$

admits an algebra retraction.

(b) : $\mathrm{mTC}_n(\mathrm{X}, \mathbb{Q})$ *is the least m such that* ρ_m *admits a retraction as* $\left(A \otimes (\Lambda V)^{\otimes n-1}, d\right)$*-module.*

(c) : $\mathrm{HTC}_n(X, \mathbb{Q})$ *is the least m such that* $H(\rho_m)$ *is injective.*

(d) : *nil* $\ker H^*(\Delta_X^n, \mathbb{Q})$ *is the longest non trivial product of elements of* $\ker H^*(\Delta_X^n), \mathbb{Q})$.

Let (A, d) be any cdga model for a space X, and $\theta : (\Lambda V, d) \xrightarrow{\simeq} (A, d)$ its minimal Sullivan model. The cdga morphism

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is a special model, called an s-model, for the path fibration $\pi_n : X^I \to X^n$.

definition 1.

(a) : $\mathrm{TC}_n(X_0)$ *is the least* m *such that the projection*

$$\rho_m : \left(A \otimes (\Lambda V)^{\otimes n-1}, d\right) \to \left(\frac{A \otimes (\Lambda V)^{\otimes n-1}}{\left(\ker \mu_n^\theta\right)^{m+1}}, \overline{d}\right)$$

admits an algebra retraction.

(b) : $\mathrm{mTC}_n(\mathrm{X}, \mathbb{Q})$ *is the least* m *such that* ρ_m *admits a retraction as* $\left(A \otimes (\Lambda V)^{\otimes n-1}, d\right)$*-module.*

(c) : $\mathrm{HTC}_n(X, \mathbb{Q})$ *is the least* m *such that* $H(\rho_m)$ *is injective.*

(d) : *nil* $\ker H^*(\Delta_X^n, \mathbb{Q})$ *is the longest non trivial product of elements of* $\ker H^*(\Delta_X^n), \mathbb{Q})$.

We have the following :

$$nil\ker H^*(\Delta_X^n,\mathbb{Q}) \leq \mathrm{HTC}_n(X,\mathbb{Q}) \leq \mathrm{mTC}_n(\mathrm{X},\mathbb{Q}) \leq \mathrm{TC}_n(X_0) \leq \mathrm{TC}_n(X).$$

In a similar way, we put $\mu_{\mathrm{A},n} : \mathrm{A}^{\otimes n} \to \mathrm{A}$ and $\mu_{\mathcal{A},n} : \mathcal{A}^{\otimes n} \to \mathcal{A}$ where $\mathrm{A} = Hom_{\Lambda V}((P,d),(\Lambda V,d))$ and $\mathcal{A} := H(\mathrm{A}) = \mathcal{E}xt_{(\Lambda V,d)}((P,d),(\Lambda V,d))$. We introduce $\mathcal{E}xt$-version of the previous invariants :

definition 2.

(a) : $\mathrm{TC}_n^{\mathcal{E}xt}(X,\mathbb{Q})$ *is the least m such that the projection*

$$\Gamma_m : (\mathrm{A}^{\otimes n}, d) \to \left(\frac{\mathrm{A}^{\otimes n}}{(\ker(\mu_{\mathrm{A},n}))^{m+1}}, \overline{d}\right)$$

admits a homotopy retraction.

(b) : $\mathrm{mTC}^{\mathcal{E}xt}{}_n(\mathrm{X},\mathbb{Q})$ *is the least m such that Γ_m admits a homotopy retraction as (A,d)-module.*

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$$\textit{nil} \ker(\mu_{\mathcal{A},n}, \mathbb{Q}) \leq \mathrm{HTC}_n^{\mathcal{E}xt}(X, \mathbb{Q}) \leq \mathrm{mTC}_n^{\mathcal{E}xt}(X, \mathbb{Q}) \leq \mathrm{TC}_n^{\mathcal{E}xt}(X_0).$$

The projections

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induce two short exact sequences linked by chain evaluation maps

$$\begin{tikzcd} 0 \arrow[r] & (\ker(\mu_{\mathrm{A},n}))^{m+1} \arrow[r] \arrow[d] & \mathrm{A}^{\otimes n} \arrow[r, "\Gamma_m"] \arrow[d] & \frac{\mathrm{A}^{\otimes n}}{(\ker(\mu_{\mathrm{A},n}))^{m+1}} \arrow[r] \arrow[d, "\theta"] & 0 \\ 0 \arrow[r] & (ker\,\mu_n)^{m+1} \arrow[r] & (\Lambda V)^{\otimes n} \arrow[r, "\rho_m"] & \frac{(\Lambda V)^{\otimes n}}{(\ker \mu_n)^{m+1}} \arrow[r] & 0. \end{tikzcd}$$

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Passing to cohomology, we get two (long) exact sequences that allows the following

Theorem 2.

Let X be a 1-connected finite type CW-complex. If X is a Gorenstein space over $\mathbb{Q}$ and $ev_{C^*(X,\mathbb{Q})} \neq 0$, then $HTC_n^{\mathcal{E}xt}(X,\mathbb{Q}) \leq HTC_n(X,\mathbb{Q})$ for any integer $n \geq 2$. Furthermore, if $(\Lambda V, d)$ is a Sullivan minimal model of X and $m = HTC_n^{\mathcal{E}xt}(X,\mathbb{Q})$, then

$$HTC_n(X,\mathbb{Q}) = HTC_n^{\mathcal{E}xt}(X,\mathbb{Q}) =: m \Leftrightarrow \begin{cases} f(1)^{\otimes n} \notin (\ker \mu_n)^{m+1}, \\ \forall z \in (\ker \mu_n)^{m+1},\ dz = 0 \\ \qquad\qquad \Rightarrow \exists z' \in (\Lambda V)^{\otimes n} \mid z = dz' \end{cases}$$

Corollary.

For each of the following conditions :

(a) X is rationally elliptic,

(b) $H_{>N}(X,\mathbb{Z}) = 0$, for some N, and X is a Gorenstein space over $\mathbb{Q}$,

(c) X is a finite 1-connected CW-complex and its Spivak fiber F_X has finite dimensional cohomology,

we have $HTC_n^{\mathcal{E}xt}(X) \leq HTC_n(X)$, for any integer $n \geq 2$.

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Remark.

An equivalent definition of $HTC_n(X, \mathbb{Q})$ when X is a Poincaré duality space reads as follows : It is the smallest integer $m \geq 0$ such that some cocycle ω representing the fundamental class of $(\Lambda V, d)^{\otimes n}$, can be written as a product of m elements of $ker(\mu_n)$ (not necessarily cocycles). Similarly, for any Gorenstein space X, $HTC_n^{\mathcal{E}xt}(X, \mathbb{Q})$ is the smallest integer m such that some cocycle representing the fundamental class of $\mathrm{A}^{\otimes n}$, namely $\Omega = [f]^{\otimes n}$ where $[f]$ designates the generating element of $\mathcal{A}^N$, can be written as a product of length m of elements in $ker(\mu_{\mathrm{A},n})$. Therefore, in order to determine $HTC_n(X, \mathbb{Q})$ we may, using the precedent theorem, calculate $m = HTC_n^{\mathcal{E}xt}(X, \mathbb{Q})$ which is quite simpler since $\mathcal{A}^*$ is one dimensional, and afterwards deal with the obstruction to have the equality.

Introduction and preliminaries

$\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K}))$ and the evaluation map

$\mathcal{E}xt$-Homology topological complexity

Application : Adams-Hilton model

The Adams-Hilton model of an arbitrary CW-complex X over an arbitrary field $\mathbb{K}$ is a chain algebra quasi-morphism $\theta_X : (TV, d) \xrightarrow{\simeq} C_*(\Omega X; \mathbb{K})$ i.e. $H_*(\theta_X)$ is an isomorphism of graded algebras. Here V satisfies $H_{i-1}(V, d_1) \cong H_i(X; \mathbb{K})$ and $d_1 : V \to V$ is the linear part of d. (TV, d) is called a *free model* of X.

Let $\mathbb{K}$ be a field with odd characteristic (thus containing $\frac{1}{2}$) and X is a q-connected ($q \geq 1$) finite CW-complex such that $\dim X \leq q \cdot char(\mathbb{K})$ i.e. $X \in CW_q(\mathbb{K})$ it has a minimal Sullivan model $(\Lambda W, d)$. Therefore, using properties of *Ext* established earlier, we obtain

$$\mathcal{E}xt_{(\Lambda W,d)}(\mathbb{K}, (\Lambda W, d)) \cong \mathcal{E}xt_{(TV,d)}(\mathbb{K}, (TV, d)),$$

thus yields the following

Proposition.

Let $X \in CW_q(\mathbb{K})$. Then, the graded vector spaces $\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K}))$ and $\mathcal{E}xt_{C_*(\Omega X;\mathbb{K})}(\mathbb{K}, C_*(\Omega X;\mathbb{K}))$ have isomorphic graded commutative algebra structures with unit.

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$\mathcal{E}xt_{(TV,d)}(\mathbb{K},(TV,d))$ is, as in the rational case, obtained in terms of the acyclic closure of $\mathbb{K}$ of the form $(TV \otimes (\mathbb{K} \oplus sV), \delta)$, where the differential δ satisfies $\delta s + sd = id$, d being the differential of TV. That is, for any element $z \otimes sv$ of $TV \otimes (\mathbb{K} \oplus sV)$, we have

$$\delta(z \otimes sv) = dz \otimes sv + (-1)^{|z|} zv \otimes 1 - (-1)^{|z|} z \otimes sdv.$$

Notice that any element f in $Hom^p_{(TV,d)}((TV \otimes (\mathbb{K} \oplus sV), \delta), (TV, d))$ is entirely determined by its image of $1 \otimes (\mathbb{K} \oplus sV)$ since $TV \otimes (\mathbb{K} \oplus sV)$ is a left (TV, d)-module acting on the first factor. Thus we have

$$\begin{aligned}(D(g))(1 \otimes sv) &= d \circ f(1 \otimes sv) - (-1)^p f \circ \delta(1 \otimes sv)\\ &= df(1 \otimes sv) - (-1)^{p(|v|+1)} vf(1) + (-1)^p f(1 \otimes sdv).\end{aligned}$$

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A straightforward argument allows the following

(a) $g \in Im(D) \Leftrightarrow g(1 \otimes sv) = df(1 \otimes sv) - (-1)^{p(|v|+1)} vf(1) + (-1)^p f(1 \otimes sdv)$ for some f.

(b) $f \in Ker(D) \Leftrightarrow df(1 \otimes sv) = (-1)^{p(|v|+1)} vf(1) - (-1)^p f(1 \otimes sdv)$.

Using the standard convention $\mathcal{A}^{-p} = \mathcal{A}_p$, for all $q \in \mathbb{Z}$ and the equations obtained, we calculate Ext for a space $X = S^q \cup_\varphi e^{q+1}$, $q \geq 2$, the space where the cell e^{q+1} is attached by a map of degree r. The Adams-Hilton model of X has the form (TV, d) where V is a $\mathbb{K}$-vector space generated by a and a' with $deg(a) = q-1$, $deg(a') = q$, $da = 0$ and $da' = -ra$.

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Notice that we have two cases since we have
$H_*(X,\mathbb{Z}) = H_0(X,\mathbb{Z}) \oplus H_q(X,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z}$

(i) If $char(\mathbb{K}) = 0$ or co-prime with r, we have $H^*(X,\mathbb{K}) = H^0(X,\mathbb{K}) \cong \mathbb{K}$. In this case, $H^*(X,\mathbb{K})$ has formal dimension $fd(X) = 0$, thus, it is a Poincaré duality space. Moreover, since it has finite dimensional cohomology, it is also a Gorenstein space. Explicit calculations conclude that $\mathcal{E}xt^0_{(TV,d)}(\mathbb{Q},(TV,d)) = \mathbb{Q}$ and for $i \neq 0$, $\mathcal{E}xt^i_{(TV,d)}(\mathbb{Q},(TV,d)) = 0$.

(ii) If $char(\mathbb{K})$ divides r then, $H^*(X,\mathbb{K}) = H^0(X,\mathbb{K}) \oplus H^q(X,\mathbb{K}) \oplus H^{q+1}(X,\mathbb{K}) \cong \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$. Thus, since $q \geq 2$, X is neither a Poincaré duality space nor a Gorenstein space. In this case, $fd(X) = q+1$, so that $\mathcal{E}xt^k_{(TV,d)}(\mathbb{K},(TV,d)) = 0, \forall k > q+1$.

Following the same computation process as in the first case we recover the previous inequality, however we might have $\mathcal{E}xt^{-i}_{(TV,d)}(\mathbb{K},(TV,d)) = 0$ for finitely many $i \geq -(q+1)$ (e.g. for $q = 7$ we have $\mathcal{E}xt^7_{(TV,d)}(\mathbb{K},(TV,d)) = 0$).

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