# Classification of compact complex manifolds

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# Cohomological point of view

$X$ compact complex manifold, $\dim_{\mathbb{C}} X = n$.

1. **De Rham cohomology** (depends only on the differential structure): $\forall k = 0, \cdots, 2n$

$$H^k_{DR}(X,\mathbb{C}) = \frac{\ker\{d : C^\infty_k(X,\mathbb{C}) \longrightarrow C^\infty_{k+1}(X,\mathbb{C})\}}{\operatorname{Im}\{d : C^\infty_{k-1}(X,\mathbb{C}) \longrightarrow C^\infty_k(X,\mathbb{C})\}}$$

with $\Delta = dd^* + d^*d$ is the Laplacian associated to the De Rham cohomology which is self-adjoint and elliptic.

2. For every constant $h \in \mathbb{R} \setminus \{0\}$, let

$$d_h := h\partial + \bar{\partial} : C^\infty_k(X,\mathbb{C}) \longrightarrow C^\infty_{k+1}(X,\mathbb{C}), \quad k \in \{1, \cdots, 2n\}$$

the linear maps:

$$\theta_h : \Lambda^k T^*X \longrightarrow \Lambda^k T^*X, \quad u = \sum_{p+q=k} u^{p,q} \longmapsto \theta_h u := \sum_{p+q=k} h^p u^{p,q},$$

are isomorphisms for $h \neq 0$ and the operators $d$ and $d_h$ are related by

$$d_h = \theta_h d \theta_h^{-1}.$$

Then $d_h^2 = 0$ inducing the $d_h$-**cohomology**

$$H_{d_h}(X, \mathbb{C}) = \frac{\ker d_h}{\operatorname{Im}\ d_h}$$

When a Hermitian metric $\omega$ has been fixed on $X$, the formal adjoint $d_h^\star$ of $d_h$ w.r.t. $\omega$ induces together with $d_h$ a Laplace-type operator in the usual way:

$$\Delta_h := d_h\, d_h^\star + d_h^\star d_h : C_k^\infty(X,\, \mathbb{C}) \longrightarrow C_k^\infty(X,\, \mathbb{C}),$$

for every $k \in \{0, \ldots, 2n\}$. This **h-Laplacian** is elliptic (cf. [Pop17]).

1. $X$ is an **h-$\partial\bar{\partial}$-manifold** if for every $k \in \{0, 1, \ldots, 2n\}$ and every $k$-form $u \in \ker d_h \cap \ker d_{-h^{-1}}$, the following exactness conditions are equivalent:

$$\begin{aligned} u \in \operatorname{Im} d_h \quad & \iff \quad u \in \operatorname{Im} d_{-h^{-1}} \iff u \in \operatorname{Im} d \\ & \iff \quad u \in \operatorname{Im}(d_h\, d_{-h^{-1}}) = \operatorname{Im}(\partial\bar{\partial}). \end{aligned}$$

**Proposition [B22]**

Let $h \in \mathbb{R} \setminus \{0\}$ be an arbitrary constant. Let $X$ be a compact complex $h$-$\partial\bar{\partial}$-manifold with $\dim_{\mathbb{C}} X = n$.

1. Every $d_h$-cohomology class contains a $d$-closed representative.
2. Let $k \in \{0, \cdots, 2n\}$. The following map

$$\begin{aligned} F: \quad H^k_{d_h}(X, \mathbb{C}) &\longrightarrow H^k_{DR}(X, \mathbb{C}) \\ [\alpha]_h &\longmapsto \{\alpha\} \end{aligned}$$

   is well defined. Moreover, $F$ is an isomorphism.

1. **Aeppli cohomology** is defined, for any $p,\ q \in \{0, 1, \cdots, n\}$, by:

$$H_A^{p,q}(X, \mathbb{C}) = \frac{\ker \partial\bar{\partial}}{(\text{Im } \partial + \text{Im } \bar{\partial})}$$

One defines the operator

$$\Delta_A := \partial\partial^* + \bar{\partial}\bar{\partial}^* + (\partial\bar{\partial})^*(\partial\bar{\partial}) + (\partial\bar{\partial})(\partial\bar{\partial})^* + (\partial\bar{\partial}^*)(\partial\bar{\partial}^*)^* + (\partial\bar{\partial}^*)^*(\partial\bar{\partial}^*)$$

The $4^{\text{th}}$ order Aeppli Laplacian is self-adjoint and elliptic. One obtains

$$C^\infty_{p,q}(X, \mathbb{C}) = \ker \Delta_A \oplus (\text{ Im } \partial + \text{ Im } \bar{\partial}) \oplus \text{ Im } (\partial\bar{\partial})^*$$

$$\ker \Delta_A = \ker \partial^* \cap \ker \bar{\partial}^* \cap \ker (\partial\bar{\partial})$$

Hodge isomorphism: $H_A^{p,q}(X, \mathbb{C}) \simeq \ker \Delta_A = \mathcal{H}_A^{p,q}(X, \mathbb{C})$.

1. $h$-**Aeppli cohomology** is defined, for any $k = 0, \cdots, 2n$, as

$$H^k_{h,A}(X, \mathbb{C}) = \frac{\ker d_h d_{h-1}}{(\mathrm{Im}\ d_h + \mathrm{Im}\ d_{h-1})}$$

where all the vector spaces involved are subspaces of the space $C^\infty_k(X, \mathbb{C})$ of smooth $k$-forms on $X$.

**[BP18]**

$$H^k_{h,A}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_A(X, \mathbb{C})$$

## Proposition

For every $h \in R \setminus \{0\}$. Let $X$ be a compact complex $h$-$\partial\bar{\partial}$-manifold with $\dim_{\mathbb{C}} X = n$.

1. Every $d_h d_{-h^{-1}}$-cohomology class contains a $d_h$-closed representative.
2. The following map

$$G: \quad H^k_{h,A}(X,\mathbb{C}) \longrightarrow H^k_{d_h}(X,\mathbb{C})$$
$$[\Omega]_{h,A} \longmapsto [\Omega]_{d_h}$$

is well defined. Furthermore $G$ is an isomorphism.

# Metrical point of view

Let $\Omega$ be a $C^\infty$ **strictly weakly positive** $(p,p)-$**form** on $X$. $\Omega$ is called

$$\exists \alpha^{i,2p-i} \in C^\infty_{i,2p-i}(X,\mathbb{C}) \text{ for } i \in \{0,\cdots,p-1\} \quad \Longrightarrow \quad \partial\bar{\partial}\Omega = 0$$

$$d\left(\sum_{i=0}^{p-1} \alpha^{i,2p-i} + \Omega + \sum_{i=0}^{p-1} \overline{\alpha^{i,2p-i}}\right) = 0 \qquad (p - \text{SKT } [B19])$$

$$(p - \text{HS } [B19])$$

For every $h \in \mathbb{R} \setminus \{0\}$,

1. $\omega$ is called $h$-**strongly Gauduchon (**$h$-**sG) metric** if there exists $\Omega^{n-2,n} \in C^\infty_{n-2,n}(X, \mathbb{C})$ such that

$$d_h\left(\frac{1}{h}\Omega^{n-2,n} + \omega^{n-1} + h\overline{\Omega^{n-2,n}}\right) = 0.$$

2. $\Omega$ is called $hp$-**Hermitian symplectic (**$hp$-**HS) form** if there exist $\Omega^{i,2p-i} \in C^\infty_{i,2p-i}(X, \mathbb{C})$ and $\Omega^{2p-i,i} \in C^\infty_{2p-i,i}(X, \mathbb{C})$ with $i = 0, \cdots, p-1$ such that

$$d_h\left(\sum_{i=0}^{p-1}\Omega^{i,2p-i} + \Omega + \sum_{i=0}^{p-1}\Omega^{2p-i,i}\right) = 0.$$

3. $X$ is said to be $h$-**sG (resp.** $hp$-**HS) manifold** if there exists an $h$-**sG metric (resp.** $hp$-**HS form)** on $X$.

$$h - \mathrm{sG} \iff \mathrm{sG}$$

$$\forall u \in \ker d_h \cap \ker d_{-h^{-1}}; \quad \Longrightarrow \quad h - \mathrm{sG}$$

$$u \in \mathrm{Im}\ d_{-h^{-1}} \Longrightarrow u \in \mathrm{Im}\ d_h d_{-h^{-1}}$$

$$\Longrightarrow \quad h - \mathrm{Gauduchon}$$

$$X \text{ is } hp-\mathrm{HS} \; + p = n-1 \Longrightarrow X \text{is either sG or balanced}$$

On the $h$-$\partial\bar{\partial}$-manifold, one has:

$$hp-\mathrm{HS} \iff p-\mathrm{SKT}\ .$$

# Application

Let $\mathcal{X}$ be a complex manifold and let $\Delta$ be an open ball containing the origin in $\mathbb{C}^m$ for some $m \in \mathbb{N}^*$.

A *holomorphic family of compact complex manifolds* is a proper holomorphic submersion $\pi : \mathcal{X} \longrightarrow \Delta$.

By a result of Ehresmann ([Voi07], Theorem 9.3), all the fibres $X_t := \pi^{-1}(t)$, for all $t \in \Delta$, are $C^\infty$-diffeomorphic to a fixed $C^\infty$ manifold $X$. Therefore, the holomorphic family $(X_t)_{t\in\Delta}$ of compact complex manifolds can be viewed as a single $C^\infty$ manifold $X$ endowed with a $C^\infty$ family of complex structures $(J_t)_{t\in\Delta}$.

# Main result

**Theorem**

*For every* $h \in \mathbb{R} \setminus \{0\}$ *an arbitrary constant. Let* $\pi : \mathcal{X} \longmapsto \Delta$ *be a holomorphic family of compact complex manifolds of dimension* $n$ *and* $p \in \{0, \cdots, n\}$*. If* $X_0$ *is a p-SKT h-*$\partial\bar{\partial}$*-manifold, then* $X_t$ *is a p-SKT h-*$\partial\bar{\partial}$*-manifold for every* $t \in \Delta$*, after possibly shrinking* $\Delta$ *about* $0$*.*

**Theorem**

*Let $(X_t)_{t\in\Delta}$ be a holomorphic family of compact complex manifolds. If $X_0$ is an hp-HS h-$\partial\bar{\partial}$-manifold for some $h \in \mathbb{R}\setminus\{0\}$, then $X_t$ is an hp-HS h-$\partial\bar{\partial}$-manifold for every $t \in \Delta$ close enough to 0.*

**Proof** Suppose that $X_0$ is an *hp*-HS *h*-$\partial\bar{\partial}$-manifold. There exist $\Omega^{i,2p-i} \in C^{\infty}_{i,2p-i}(X,\mathbb{C})$ and $\Omega^{2p-i,i} \in C^{\infty}_{2p-i,i}(X,\mathbb{C})$ for $i = 0,\cdots,p-1$ such that

$$d_h\tilde{\Omega} = d_h\left(\sum_{i=0}^{p-1}\Omega^{i,2p-i} + \Omega + \sum_{i=0}^{p-1}\Omega^{2p-i,i}\right) = 0.$$

$\tilde{\Omega}$ is a $d_h$-closed $2p$-form on $X_0$. Then,

$$d\tilde{\Omega} = -dd_h u$$

where $u$ is a $(2p-1)$-form. The splitting of $u$ into pure-type forms reads:

$$u = \sum_{i=0}^{p-1} u^{i,2p-i-1} + \sum_{i=0}^{p-1} u^{2p-i-1,i}. \tag{1}$$

Applying $\partial\bar{\partial}$ to the equation (1) implies that

$$\begin{aligned}\Psi : \quad &= \sum_{i=0}^{p-1} \left(\Omega^{i,2p-i} + (1-h)\,\bar{\partial}u^{i,2p-i-1}\right) \\ &+\Omega + \textstyle\sum_{i=0}^{p-1} \left(\Omega^{2p-i,i} - (1-h)\,\partial u^{2p-i-1,i}\right)\end{aligned}$$

is a $d$-closed $(2p)$-form such that $\Omega$ is its component of type $(p,p)$.

$(\Omega_t)_{t\in\Delta}$: the smooth family of component of $\Psi$ of type $(p,p)$ for the complex structure $J_t$ of $X_t$.

The forms $\Omega_t$ vary in a $C^\infty$ way with $t\in\Delta$ for $t$ close to 0. Additionally, $\Omega>0$ implies that $\Omega_t>0$ for $t$ close to 0 (cf. [B19]). Note by

$$\Psi_t=\sum_{i=0}^{p-1}\Omega_t^{i,2p-i}+\Omega_t+\sum_{i=0}^{p-1}\Omega_t^{2p-i,i}$$

the $d$-closed $2p$-form on $X_t$ with $\Omega_t$ is its component of type $(p,p)$.

This implies that $\Omega_t$ is a $\partial_t\bar{\partial}_t$-closed $(p,p)$-form on $X_t$. The $h$-$\partial\bar{\partial}$-property is deformation-open (cf. [BP18]). Consequently, $X_t$ is an $hp$-HS $h$-$\partial\bar{\partial}$-manifold for any $t$ close to 0.

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Thank you $\ddot{\smile}$