An Introduction to the Geometry of Symmetric Spaces

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Lie groups

For any Lie group G we have a canonical involution:

$$\mathfrak{s}_e : G \to G, \quad \text{written} \quad g \mapsto g^{-1}.$$

1. $\mathfrak{s}_e(e) = e$.
2. $\mathfrak{s}_e \circ \mathfrak{s}_e = \mathrm{Id}_G$.

Symmetries $(\mathfrak{s}_a)_{a\in G}$ in a Lie group G

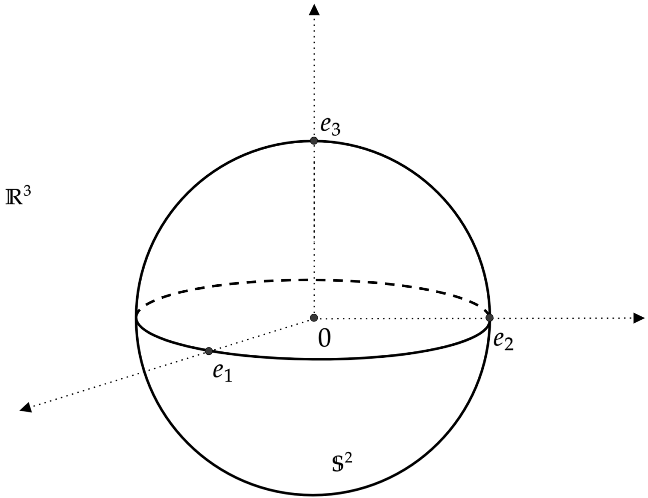
For any fixed point a in G, there exists a smooth involution $\mathfrak{s}_a : G \to G$ such that a is an **isolated fixed** point for it.
It is defined by:

$$\begin{array}{ccc} G & \xrightarrow{\mathfrak{s}_e} & G \\ {\scriptstyle l_{a^{-1}}}\uparrow & & \downarrow{\scriptstyle l_a} \\ G & \xrightarrow{\mathfrak{s}_a} & G, \end{array}$$

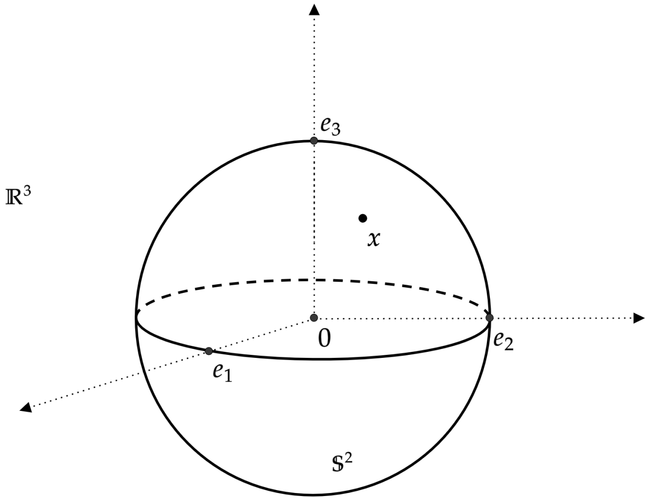
where $l_g(x) := gx$ is the left translation by the element $g \in G$.
More precisely, the map $\mathfrak{s}_a$ is given by:

$$\mathfrak{s}_a(b) := ab^{-1}a.$$

What about the unit sphere $\mathbb{S}^2$?



The unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$



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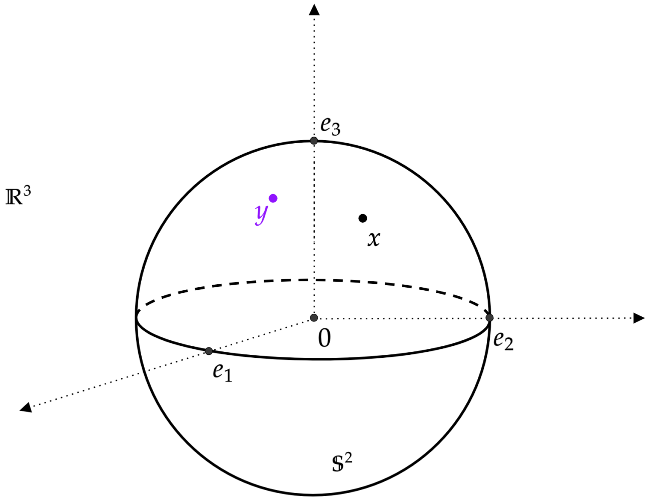

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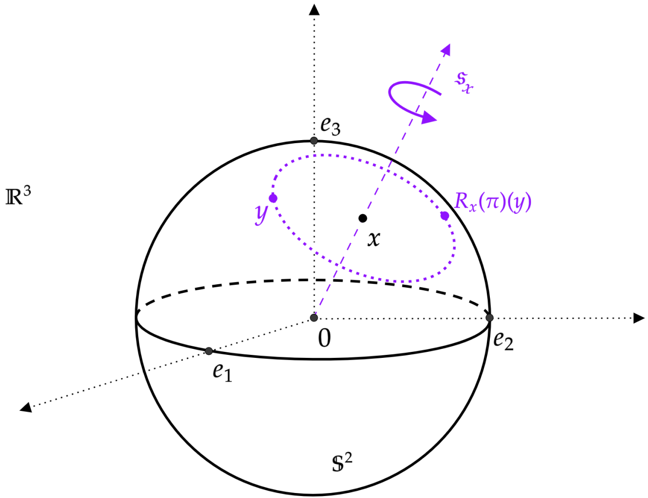

Symmetries on the Poincaré Half-Plane

Consider $\mathbb{H} := \{z = x + iy \in \mathbb{C} /\ y > 0\}$ with the metric $ds^2 := \frac{1}{y^2}(dx^2 + dy^2)$. The group $\mathrm{SL}(2, \mathbb{R})$ acts transitively and isometrically on $\mathbb{H}$ via:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}.$$

Moreover, a symmetry (an isometry of $\mathbb{H}$) at i is given by:

$$\mathfrak{s}_i(z) := -\frac{1}{z} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot z.$$

Hence, under conjugaison by elements in $\mathrm{SL}(2, \mathbb{R})$ we get symmetries: $\mathfrak{s}_a : \mathbb{H} \to \mathbb{H}$ for any $a \in \mathbb{H}$.

What is a Symmetric Space?

Definition

A *symmetric space* is a connected smooth manifold M with a smooth family of involutions $\{\mathfrak{s}_x\}_{x \in M}$, in the sense that

$$\begin{array}{ccc} M \times M & \longrightarrow & M \\ (x, y) & \longmapsto & \mathfrak{s}_x(y), \end{array}$$

is smooth, and which satisfies the following properties:

1. $\mathfrak{s}_x(x) = x, \quad \forall\, x \in M;$
2. $\mathfrak{s}_x \circ \mathfrak{s}_y \circ \mathfrak{s}_x = \mathfrak{s}_{\mathfrak{s}_x(y)}, \quad \forall\, x, y \in M;$
3. For each $x \in M$, there exists a neighborhood $U_x \subseteq M$ of x such that x is the only fixed point of $\mathfrak{s}_x$ in U_x.

Example: Symmetric Pairs

Definition

A *symmetric pair* is a triple (G, H, σ) such that:

1. G is a connected Lie group and H a closed subgroup;
2. $\sigma : G \to G$ is an involutive automorphism of G satisfying the following condition

$$\mathrm{Fix}^\circ(\sigma) \subseteq H \subseteq \mathrm{Fix}(\sigma),$$

where $\mathrm{Fix}(\sigma) := \{g \in G \mid \sigma(g) = g\}$.

Examples:

- $(\mathrm{GL}^+(n, \mathbb{R}), \mathrm{SO}(n), \sigma)$, where $\sigma(A) := (A^{-1})^T$.
- $(G \times G, \Delta G, \sigma)$, where $\sigma(a, b) := (b, a)$.

From Symmetric Pairs to Symmetric Spaces

Theorem

Let (G, H, σ) be a symmetric pair, then $M := G/H$ is a symmetric space.

sketch. Let $\pi : G \to M, g \mapsto \overline{g}$ be the projection map.
Define:

$$\mathfrak{s}_{\overline{e}}(\overline{b}) := \overline{\sigma(b)}, \qquad \mathfrak{s}_{\overline{a}} := \lambda_a \circ \mathfrak{s}_{\overline{e}} \circ \lambda_{a^{-1}},$$

where $\lambda_a : M \to M, \quad \overline{b} \mapsto \overline{ab}$.

...

Example: The sphere S^2 as a homogeneous space of a symmetric pair

Consider S^2, $G := Iso(S^2)^\circ = SO(3)$ and $s_\circ : S^2 \to S^2$ the geodesic symmetry given by

$$s_\circ(x, y, z) = (-x, -y, z).$$

Then an involutive automorphism $\sigma : G \to G$ is given by

$$\sigma(g) = s_\circ g s_\circ^{-1}$$

and satisfy

$H := Fix(\sigma)^\circ = \{ \begin{pmatrix} A & \\ & 1 \end{pmatrix} \; / \; A \in SO(2)\} \cong SO(2)$. We have the canonical isomorphism $\varphi : G/H \cong S^2$ given by $\varphi([g]) := ge_3$.

Proposition

Let (G, H, σ) be a symmetric pair, then $M := G/H$ is a reductive homogeneous G-space. More precisely we have:

1. *The Lie algebra of H is* $\mathfrak{h} := \{u \in \mathfrak{g} \ / \ \sigma'(u) = u\}$.
2. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, *where* $\mathfrak{m} := \{u \in \mathfrak{g} \ / \ \sigma'(u) = -u\}$.
3. $\mathrm{Ad}(H)(\mathfrak{m}) \subseteq \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$.

Definition

Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be an involutive automorphism of a Lie algebra $\mathfrak{g}$.

- The pair $(\mathfrak{g}, \tau)$ is called an *involutive* Lie algebra.
- The canonical decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h} := \ker(\tau - \mathrm{Id}_{\mathfrak{g}})$ and $\mathfrak{m} := \ker(\tau + \mathrm{Id}_{\mathfrak{g}})$.

The following relations hold:
$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}.$

Conversely, we have

Proposition

Any involutive automorphism $\tau : \mathfrak{g} \to \mathfrak{g}$ *of a Lie algebra* $\mathfrak{g}$ *gives rise to a symmetric pair* $(\widetilde{G}, \widetilde{H}, \widetilde{\sigma})$, *where*

- $\widetilde{G}$ *is a simply connected Lie group having* $\mathfrak{g}$ *as Lie algebra;*
- $\widetilde{H} := \langle \exp_{\widetilde{G}}(\mathfrak{h}) \rangle$, *with* $\mathfrak{h} := \ker(\tau - \mathrm{Id}_{\mathfrak{g}})$;
- $\widetilde{\sigma} \in \mathrm{Aut}(\widetilde{G})$ *such that* $\widetilde{\sigma}' = \tau$.

The Canonical decomposition of $\mathfrak{g}$ for the previous Examples

- For $(\mathrm{GL}^+(n, \mathbb{R}), \mathrm{SO}(n), \sigma)$, we have

$$\mathfrak{h} = \mathfrak{so}(n), \quad \text{and} \quad \mathfrak{m} = \mathrm{Sym}(n, \mathbb{R}).$$

- For $(G \times G, \Delta G, \sigma)$, we have

$$\mathfrak{h} = \{(u, u) \mid u \in \mathfrak{g}\}, \quad \text{and} \quad \mathfrak{m} = \{(u, -u) \mid u \in \mathfrak{g}\}.$$

Automorphisms of a Symmetric Space

An *automorphism* of a symmetric space M is a diffeomorphism $\Phi : M \xrightarrow{\simeq} M$ such that:

$$\Phi \circ \mathfrak{s}_x \circ \Phi^{-1} := \mathfrak{s}_{\Phi(x)}, \qquad \forall\, x \in M.$$

We will denote by $\mathrm{Aut}(M)$ the group of all automorphisms of M.

Let (M, μ) be a symmetric space. Define the *group of displacements* of M by:

$$G(M) := \langle \mathfrak{s}_x \circ \mathfrak{s}_y \,;\ x, y \in M \rangle.$$

If $\Phi \in \mathrm{Aut}(M)$, then

$$\Phi \circ \left(\mathfrak{s}_x \circ \mathfrak{s}_y\right) \circ \Phi^{-1} = \mathfrak{s}_{\Phi(x)} \circ \mathfrak{s}_{\Phi(y)}.$$

Hence $G(M)$ is a normal subgroup of $\mathrm{Aut}(M)$.

Let $o \in M$ be a fixed point, then for each $x, y \in M$ we have

$$\mathfrak{s}_x \circ \mathfrak{s}_y = \mathfrak{s}_x \circ \mathfrak{s}_o \circ \mathfrak{s}_o \circ \mathfrak{s}_y = (\mathfrak{s}_x \circ \mathfrak{s}_o) \circ (\mathfrak{s}_y \circ \mathfrak{s}_o)^{-1}.$$

Thus

$$G(M) = \langle \mathfrak{s}_x \circ \mathfrak{s}_o\, ;\ x \in M \rangle.$$

From Symmetric Spaces to Symmetric Pairs

Theorem

Let (M,μ) be a symmetric space and $o \in M$. Then:

1. *$G(M)$ is a connected Lie group.*
2. *$G(M)$ acts transitively on M.*
3. *$(G(M), H_o, \sigma_o)$ is a symmetric pair, where H_o denotes the isotropy group of o, and σ_o given by:*

$$\sigma_o : G(M) \to G(M), \quad F \mapsto \mathfrak{s}_o \circ F \circ \mathfrak{s}_o.$$

   Moreover, M is isomorphic to $G(M)/H_o$.

4. *The canonical decomposition of $\mathfrak{g}(M)$ corresponding to σ_o is $\mathfrak{g}(M) = \mathfrak{m} \oplus [\mathfrak{m}, \mathfrak{m}]$,, where $\mathfrak{m} := \ker(\sigma_o' + \mathrm{Id}_{\mathfrak{g}})$.*

For a full proof one can see Loos, Ottmar. *Symmetric spaces: General theory.* Vol. 1. WA Benjamin, 1969.

Affine Symmetric Spaces

Definition

An *affine symmetric space* is a connected smooth manifold M endowed with a connection ∇ which satisfies the following:

- For each $x \in M$, there exists an affine map $\mathfrak{s}_x : M \to M$ such that:

$$\mathfrak{s}_x(\gamma(t)) = \gamma(-t),$$

  where $\gamma : (-\varepsilon, \varepsilon) \to M$ is a geodesic of ∇ with $\gamma(0) = x$.

The affine map $\mathfrak{s}_x : M \to M$ is called the *geodesic symmetry* about x.

$\mathfrak{s}_x : M \to M$, $\mathfrak{s}_x(\gamma(t)) = \gamma(-t)$.

Clearly a geodesic symmetry $\mathfrak{s}_x$ is different from Id_M and admits x as an isolated fixed point.
Moreover, if $\gamma : (-\varepsilon, \varepsilon) \to M$ is a geodesic of ∇ with $\gamma(0) = x$ and $u_x = \dot{\gamma}(0)$, then

$$\begin{aligned} T_x\mathfrak{s}_x(u_x) &= \frac{d}{dt}_{|t=0} \mathfrak{s}_x(\gamma(t)) \\ &= \frac{d}{dt}_{|t=0} \gamma(-t) \\ &= -u_x. \end{aligned}$$

Thus

$$T_x\mathfrak{s}_x = -\mathrm{Id}_{T_xM}.$$

Furthermore, using the following Lemma

Lemma

Let M be a connected smooth manifold, and ∇ a connection on it. If $F_1, F_2 : M \to M$ are two affine maps such that:

$$F_1(x_0) = F_2(x_0), \quad \text{and} \quad T_{x_0}F_1 = T_{x_0}F_2.$$

Then $F_1 = F_2$.

We deduce that

- $\mathfrak{s}_x \circ \mathfrak{s}_x = \mathrm{Id}_M, \quad \forall\, x \in M;$
- $\mathfrak{s}_x \circ \mathfrak{s}_y \circ \mathfrak{s}_x = \mathfrak{s}_{\mathfrak{s}_x(y)}, \quad \forall\, x, y \in M.$

(M, ∇)

An affine symmetric space

(M, ∇) $\Longrightarrow$ $(M, \{\mathfrak{s}_x\}_{x \in M})$

An affine symmetric space — A symmetric space

Properties

Proposition

Let (M, ∇) *be an affine symmetric space, then we have*

1. ∇ *is complete.*
2. $\mathrm{Aff}(M, \nabla)$ *acts transitively on* M*, and the same is true for its identity component* $\mathrm{Aff}^0(M, \nabla)$*.*

Symmetric Pair associated to Affine Symmetric Space

If (M, ∇) is an affine symmetric space, then we can write

$$M \cong \mathrm{Aff}^0(M, \nabla)/H_{x_0},$$

where H_{x_0} denotes the isotropy group of a point $x_0 \in M$ in $\mathrm{Aff}^0(M, \nabla)$. Let $\mathfrak{s}^0$ be the geodesic symmetry about x_0, and define a homomorphism

$$\sigma^\nabla : \mathrm{Aff}^0(M, \nabla) \to \mathrm{Aff}^0(M, \nabla), \quad \text{written} \quad F \mapsto \mathfrak{s}^0 \circ F \circ \mathfrak{s}^0.$$

Proposition

The triple $\left(\mathrm{Aff}^0(M, \nabla), H_{x_0}, \sigma^\nabla\right)$ *is a symmetric pair.*

Correspondence: $(M, \nabla) \leftrightarrows (G, H, \sigma)$

$$(M, \nabla) \quad \Longrightarrow \quad \left(\mathrm{Aff}^0(M, \nabla), H_{x_0}, \sigma^{\nabla}\right)$$

An affine symmetric space $\qquad$ A symmetric pair

- The next step: Expression of the canonical connection ∇ associated to a symmetric pair (G, H, σ)? i.e. G-invariant connection on G/H for which $\overline{\sigma} : G/H \to G/H$ is an affine map.

Reductive homogeneous G-spaces

A homogeneous G-space G/H is called *reductive* if there exists a vector subspace $\mathfrak{m} \subset \mathfrak{g}$ such that:

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}, \quad \text{and} \quad \mathrm{Ad}(H)(\mathfrak{m}) \subseteq \mathfrak{m},$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively.

Nomizu Theorem

Theorem

Let $M := G/H$ be a reductive homogeneous G-space with a fixed reductive decomposition, i.e

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}, \quad \text{and} \quad \mathrm{Ad}(H)(\mathfrak{m}) \subseteq \mathfrak{m}.$$

Then there exists a one-to-one correspondence between the set of G-invariant connections on M and the set of bilinear maps $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ which are $\mathrm{Ad}(H)$-invariant, i.e

$$\mathrm{Ad}_h \alpha(u, v) = \alpha\left(\mathrm{Ad}_h u, \mathrm{Ad}_h v\right),$$

for $u, v \in \mathfrak{m}$ and $h \in H$.

Let $M := G/H$ be a reductive homogeneous G-space with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. For each $u \in \mathfrak{g}$, we define a vector field $u^* \in \mathfrak{X}(M)$, called the *fundamental vector field* associated to u by:

$$u^*_{\overline{a}} := \frac{d}{dt}_{|t=0} \overline{\exp_G(tu)a}, \qquad \forall\, \overline{a} \in M.$$

Moreover, we have a linear isomorphism between $\mathfrak{m}$ and $T_{\overline{e}}M$, given by:

$$\begin{array}{cccc} \mathrm{I}_{\overline{e}} & : & \mathfrak{m} & \xrightarrow{\cong} & T_{\overline{e}}M \\ & & u & \longmapsto & u^*_{\overline{e}}. \end{array}$$

If ∇ is a G-invariant connection on M, then its associated bilinear map $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is defined as follows[1]:

$$\alpha(u, v) := \mathrm{I}_{\overline{e}}^{-1}\Big((\nabla_{u^*} v^*)_{\overline{e}} \Big) + [u, v]_{\mathfrak{m}}.$$

[1]For $w \in \mathfrak{g}$, we denote by $w_{\mathfrak{m}}$ the projection of w on $\mathfrak{m}$.

Further, the torsion T^{∇} of the G-invariant connection ∇ gives rise to a bilinear map $T^{\alpha} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ written as

$$T^{\alpha}(u, v) := \alpha(u, v) - \alpha(v, u) - [u, v]_{\mathfrak{m}}.$$

Hence

Corollary

Let ∇ be a G-invariant connection on M and α its associated bilinear map. Then ∇ is torsion-free if and only if for any $u, v \in \mathfrak{m}$

$$\alpha(u, v) = \frac{\alpha(u, v) + \alpha(v, u)}{2} + \frac{1}{2}[u, v]_{\mathfrak{m}},$$

i.e. the bilinear map $\alpha_{\mathrm{sym}}(u, v) := \alpha(u, v) - \frac{1}{2}[u, v]_{\mathfrak{m}}$ is symmetric.

Particular G-invariant connections on M

- The natural connection ∇^0 given by:

$$\alpha^0(u,v) = \frac{1}{2}[u,v]_{\mathfrak{m}}, \qquad \forall\, u,v \in \mathfrak{m}.$$

  It is **torsion-free**.

- The canonical connection ∇^c given by:

$$\alpha^c(u,v) = 0, \qquad \forall\, u,v \in \mathfrak{m}.$$

  It is invariant under parallelism i.e the torsion and the curvature tensors of ∇^c are both parallel.

Remark. $\nabla^c = \nabla^0$ if and only if $[\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{h}$.

Nomizu's Theorem allows us to transfer geometric conditions to algebra, or algebraic conditions to geometry.

Proposition

Let $M := G/H$ be a reductive homogeneous G-space with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and ∇ a G-invariant connection on M with $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ its associated bilinear map. For each $u \in \mathfrak{m}$, we have

$$\alpha(u,u) = 0 \quad \Leftrightarrow \quad t \mapsto \overline{\exp_G(tu)} \text{ is a geodesic of } \nabla.$$

Proof. Let $u \in \mathfrak{m}$ and $\gamma : \mathbb{R} \to M$, $t \mapsto \overline{\exp_G(tu)}$. Since $\dot{\gamma}(t) = u^*_{\gamma(t)}$, then a direct computation yields

$$\nabla_{\dot{\gamma}}\dot{\gamma}(t) = \left(\lambda_{\exp_G(tu)}\right)_* \alpha(u,u)^*_{\bar{e}}. \quad \blacksquare$$

Notice that if ∇ is a G-invariant connection on M whose geodesics through $\overline{e}$ are exactly the curves $t \mapsto \overline{\exp_G(tu)}$ for any $u \in \mathfrak{m}$, then the geodesics through another point $\overline{a}$ of M are exactly the curves $t \mapsto \overline{\exp_G(t\mathrm{Ad}_a u)a}$, with $u \in \mathfrak{m}$.

Corollary

On a reductive homogeneous G-space $M := G/H$ with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, the natural connection ∇^0 is the only G-invariant torsion-free connection whose geodesics are exactly the curves $t \mapsto \overline{\exp_G(t\mathrm{Ad}_a u)a}$, with $u \in \mathfrak{m}$ and $\overline{a} \in M$.

Example. A connected Lie group G, viewed as a reductive homogeneous $(G \times G)$-space, endowed with its natural bi-invariant connection!

From Symmetric Pairs to Affine Symmetric Spaces

Theorem

Let (G, H, σ) be a symmetric pair, then $(G/H, \nabla^0)$ is an affine symmetric space.

Proof. Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ be the canonical decomposition of $\mathfrak{g}$ and ∇^0 the natural torsion-free G-invariant connection on M associated to the bilinear map $\alpha^0 \equiv 0$. Consider the following smooth map on M

$$\mathfrak{s}^0 : M \to M, \qquad \overline{a} \mapsto \overline{\sigma(a)}.$$

This is well defined because $H \subseteq \mathrm{Fix}(\sigma)$, and satisfies

$$\mathfrak{s}^0 \circ \mathfrak{s}^0 = \mathrm{Id}_M \,.$$

Proof. $\mathfrak{s}^0 \in \mathrm{Aff}(M, \nabla^0)$

Define a connection ∇ on M by:

$$\nabla_X Y := \mathfrak{s}^0_* \left(\nabla^0_{\mathfrak{s}^0_* X} \mathfrak{s}^0_* Y \right), \qquad \forall\, X, Y \in \mathfrak{X}(M).$$

Let us show that $\nabla = \nabla^0$. First, for each $a \in G$, we have the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \mathfrak{s}^0\ } & M \\ \lambda_a \big\downarrow & & \big\downarrow \lambda_{\sigma(a)} \\ M & \xrightarrow{\ \mathfrak{s}^0\ } & M \end{array}.$$

Thus ∇ is G-invariant. Let α be its associated bilinear map.

Proof. $\mathfrak{s}^0 \in \mathrm{Aff}(M, \nabla^0)$

For each $u \in \mathfrak{m}$ and $a \in G$ we have

$$\begin{aligned}\left(\mathfrak{s}^0_* u^*\right)_{\overline{a}} &= \frac{d}{dt}_{|t=0} \mathfrak{s}^0 \left(\overline{\exp_G(tu)\sigma(a)}\right) \\ &= \frac{d}{dt}_{|t=0} \overline{\exp_G(-tu)a} \\ &= -u^*_{\overline{a}}.\end{aligned}$$

Thus

$$\mathfrak{s}^0_* u^* = -u^*, \qquad \forall\, u \in \mathfrak{m}.$$

Proof. $\mathfrak{s}^0 \in \mathrm{Aff}(M, \nabla^0)$

Hence for $u, v \in \mathfrak{m}$ we have

$$\begin{aligned} \alpha(u,v) &= \mathrm{I}_{\overline{e}}^{-1}\Big((\nabla_{u^*} v^*)_{\overline{e}} \Big) \\ &= \mathrm{I}_{\overline{e}}^{-1}\Big(s^0_* \left(\nabla^0_{u^*} v^*\right)_{\overline{e}} \Big) \\ &= -\mathrm{I}_{\overline{e}}^{-1}\Big(\alpha^0(u,v)^*_{\overline{e}} \Big) \\ &= 0, \end{aligned}$$

which implies that $\nabla = \nabla^0$ and therefore $\mathfrak{s}^0 \in \mathrm{Aff}(M, \nabla^0)$.

Proof. $\mathfrak{s}^0$ is a geodesic symmetry about $\overline{e}$

Now it only remains to check that $\mathfrak{s}^0$ is a geodesic symmetry about $\overline{e}$. Let $t \mapsto \overline{\exp_G(tu)}$ be a geodesic through $\overline{e}$ with $u \in \mathfrak{m}$, then

$$\begin{aligned} \mathfrak{s}^0\left(\overline{\exp_G(tu)}\right) &= \overline{\sigma\left(\exp_G(tu)\right)} \\ &= \overline{\exp_G(-tu)}. \end{aligned}$$

Thus $\mathfrak{s}^0$ is a geodesic symmetry about $\overline{e}$.

Finally, for any $\overline{a} \in M$ we define the geodesic symmetry about $\overline{a}$ as follow

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{s}^0} & M \\ \uparrow{\scriptstyle \lambda_{a^{-1}}} & & \downarrow{\scriptstyle \lambda_a} \\ M & \xrightarrow{\mathfrak{s}_{\overline{a}}} & M. \end{array}$$

One can check easily that $\mathfrak{s}_{\overline{a}}$ satisfies all the conditions required for a geodesic symmetry. ∎

$$(G, H, \sigma) \quad \Longrightarrow \quad \left(G/H, \nabla^0\right)$$

A symmetric pair $\Longrightarrow$ An affine symmetric space

Invariant Pseudo-Riemannian Metrics on a Reducitve Homogeneous G-space

Theorem

Let $M := G/H$ be a reductive homogeneous G-space with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. There is a natural one-to-one correspondence between the set of G-invariant pseudo-Riemannian metrics on M and the set of $\mathrm{Ad}(H)$-invariant non-degenerate symmetric bilinear forms on $\mathfrak{m}$.

For the sake of simplicity, we shall use the same notation $\langle \cdot\,, \cdot \rangle$ to denote both the G-invariant pseudo-Riemannian metric on M, and its associated $\mathrm{Ad}(H)$-invariant non-degenerate symmetric bilinear form on $\mathfrak{m}$.

Proposition

Let $M := G/H$ be a reductive homogeneous G-space with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, and let $\langle \cdot\,, \cdot \rangle$ be a G-invariant pseudo-Riemannian metric on M. The Levi-Civita connection ∇^{LC} of $\langle \cdot\,, \cdot \rangle$ is G-invariant and its associated bilinear map $\alpha^{\mathrm{LC}} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is given by:

$$\alpha^{\mathrm{LC}}(u, v) := \frac{1}{2}[u, v]_{\mathfrak{m}} + \alpha^{\mathrm{LC}}_{\mathrm{sym}}(u, v),$$

where $\alpha^{\mathrm{LC}}_{\mathrm{sym}} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is the symmetric bilinear map defined by:

$$\langle \alpha^{\mathrm{LC}}_{\mathrm{sym}}(u, v), w \rangle = \frac{1}{2}\Big\{ \langle [w, u]_{\mathfrak{m}}, v \rangle + \langle u, [w, v]_{\mathfrak{m}} \rangle \Big\},$$

for all $u, v, w \in \mathfrak{m}$.

Proof. A direct computation using Koszul's formula shows that ∇^{LC} is G-invariant. Moreover, for $u, v, w \in \mathfrak{m}$ we have

$$\begin{aligned}
\langle \alpha^{\mathrm{LC}}(u,v), w\rangle &= \langle \nabla^{\mathrm{LC}}_{u^*} v^*, w^*\rangle_{\overline{e}} + \langle [u,v]^*, w^*\rangle_{\overline{e}} \\
&= \frac{1}{2}\Big\{ \langle [u,v]^*, w^*\rangle_{\overline{e}} + \langle [w,u]^*, v^*\rangle_{\overline{e}} + \langle u^*, [w,v]^*\rangle_{\overline{e}} \Big\} \\
&= \frac{1}{2}\Big\{ \langle [u,v]_{\mathfrak{m}}, w\rangle + \langle [w,u]_{\mathfrak{m}}, v\rangle + \langle u, [w,v]_{\mathfrak{m}}\rangle \Big\} \\
&= \langle \frac{1}{2}[u,v]_{\mathfrak{m}} + \alpha^{\mathrm{LC}}_{\mathrm{sym}}(u,v), w\rangle,
\end{aligned}$$

where

$$\langle \alpha^{\mathrm{LC}}_{\mathrm{sym}}(u,v), w\rangle := \frac{1}{2}\Big\{ \langle [w,u]_{\mathfrak{m}}, v\rangle + \langle u, [w,v]_{\mathfrak{m}}\rangle \Big\}. \quad \blacksquare$$

Corollary

With the notations of the previous proposition, The Levi-Civita connection ∇^{LC} *of* $\langle\cdot\,,\cdot\rangle$ *coincides with the natural connection* ∇^0 *associated to the decomposition* $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ *if and only if*

$$\langle [u,v]_{\mathfrak{m}}, w\rangle + \langle v, [u,w]_{\mathfrak{m}}\rangle = 0, \qquad \forall\, u,v,w \in \mathfrak{m}.$$

Corollary

Let (G,H,σ) *be a symmetric pair. A* G*-invariant pseudo-Riemannian metric on* G/H*, if there exists any, induces the canonical connection.*

Semi-simple Lie Algebras

Definition

Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra.

- $\mathfrak{g}$ is *simple* if it is nonabelian and does not contain any ideal distinct from $\{0\}$ and $\mathfrak{g}$.
- $\mathfrak{g}$ is *semi-simple* if does not contain any nonzero solvable ideal. ($\mathfrak{a}$ is solvable i.e. there exists n s.t. $\mathcal{D}^n(\mathfrak{a}) = \{0\}$).

Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra. Then the following statements are equivalent:

1. $\mathfrak{g}$ is semi-simple.
2. $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$, where the $\mathfrak{g}_i$'s are ideals of $\mathfrak{g}$ which are simple (as Lie algebras).
3. $\mathfrak{g}$ has no nonzero abelian ideal.
4. The Killing form $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ of $\mathfrak{g}$ is non-degenerate.

Cartan involution

Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be an automorphism with $\tau^2 = \mathrm{Id}_{\mathfrak{g}}$. Then, the bilinear form

$$B^\tau(u, v) := -B_{\mathfrak{g}}(u, \tau(v)),$$

is symmetric, where $B_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$. τ is called a *Cartan involution* if B^τ is an inner product on $\mathfrak{g}$.

Proposition

$\theta(A) := -A^t$ is an involution of $M_n(\mathbb{R})$. If $\mathfrak{g} \subset M_n(\mathbb{R})$ is a subalgebra such that

$$\theta(\mathfrak{g}) \subset \mathfrak{g}, \quad \text{and} \quad Z(\mathfrak{g}) = \{0\},$$

then, $\tau := \theta_{|\mathfrak{g}}$ is a Cartan involution of $\mathfrak{g}$.

It is the case, for example, of the subalgebras $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{so}(p, q)$.

Proof. We have to show that for any $X \in \mathfrak{g}$, s.t. $X \neq 0$

$$B^{\tau}(X,X) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_{X^t}) > 0 \quad ?$$

Consider the canonical inner product on $\mathfrak{g}$:

$$\langle X, Y\rangle := \mathrm{tr}(X^t Y),$$

this induces an inner product on $\mathrm{End}(\mathfrak{g})$:

$$\langle\langle f_1, f_2\rangle\rangle := \mathrm{tr}(f_1^T \circ f_2),$$

where $f_1^T : \mathfrak{g} \to \mathfrak{g}$ is the transpose defined through $\langle \cdot\,, \cdot\rangle$.
A small computation shows that $\mathrm{ad}_{X^t} = (\mathrm{ad}_X)^T$. ■

Theorem

Let (G, H, σ) be a symmetric pair such that G is semi-simple. Then the canonical connection on G/H is induced by a G-invariant pseudo-Riemannian metric. If moreover σ' is a Cartan involution, then the canonical connection on G/H is induced by a G-invariant Riemannian metric.

Proof. Define an $\mathrm{Ad}(H)$-invariant symmetric bilinear form on $\mathfrak{m}$ by:

$$\langle \cdot\,,\cdot \rangle : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}, \quad \text{written} \quad \langle u, v \rangle := -B_{\mathfrak{g}}(u, v),$$

where $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is the Killing form of $\mathfrak{g}$. Furthermore, since $\mathfrak{g}$ is semi-simple and $B_{\mathfrak{g}}(\mathfrak{h}, \mathfrak{m}) = 0$, we deduce that $\langle \cdot\,,\cdot \rangle$ is non-degenerate. ■

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