**The Elephant is very like a fan:
Metaphor in discourse on science and mathematics**

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The title of my talk[1] is taken from a poem of John Godfrey Saxe (1816–1887) and was suggested by Tom Archibald in an early discussion of these ideas. I have paraphrased and shortened the poem for you:

‘The Blind Men and the Elephant’

Six men of Indostan
to learning much inclined,
all of them were blind.

The First approached the Elephant,
and happening to fall
against his broad and sturdy side, he said,
‘The Elephant is very like a wall!’

The Second, feeling of the tusk,
cried, ‘To me ’tis mighty clear
an Elephant is very like a spear!’

The Third took the
squirming trunk in his hands, and spake,
‘the Elephant is very like a snake!’

The Fourth felt about the knee.
‘The Elephant is very like a tree!’

The Fifth, who touched the ear,
Said: ‘Deny the fact who can,
‘an Elephant is very like a fan!?’

The Sixth, seizing on the swinging tail,
That fell within his scope,
quoth he, ‘the Elephant is very like a rope!’

And so these men of Indostan
Disputed loud and long,
And all were in the wrong!

The term *elephant test* refers to situations in which an idea or thing “is hard to describe, but instantly recognizable when spotted” (from Wikipedia).

In science and in mathematics, researchers are called upon to describe new and usual phenomena using ordinary language. It is suggested by cognitive scientists and linguists (George Lakoff, Rafael Núñez) that the means by which we describe difficult ideas is through metaphor.

[1]Given at University of Casablanca, June 5, 2-13.

In fact, Lakoff and Núñez would go further to suggest that the source of mathematics IS metaphor. Here is how their idea works.

The basic mechanism is the *conceptual metaphor*: a mapping between domains of discourse, linking one domain of discourse, usually familiar, experiential, sometimes of the body, to another more conceptual domain that is difficult to describe.

For example, there is the metaphor

SOCIAL ORGANIZATIONS ARE PLANTS.

Source domain: the whole plant, parts of a plant, growth of a plant, removal of parts, the root of plant, flowering, fruit

$\Longrightarrow$

Target domain: the entire organization, branches, development, reduction, origins, successes, benefits.

Of course, such a mapping is meant to generate speech. Here is another example:

IDEAS ARE FOOD

Cooking $\Longrightarrow$ thinking.

'Let me stew over this.'

Swallowing $\Longrightarrow$ accepting.

'I can't swallow that claim.'

Chewing $\Longrightarrow$ considering.

'I am chewing over the proposal.'

Digesting $\Longrightarrow$ understanding.

'I can't digest all of these ideas.'

THE MIND IS A MACHINE

Terms for thinking

'We're still trying to *grind out* the solution to this equation.'

'My mind just isn't *operating* today.'

'Boy, the *wheels are turning* now!'

'I'm a *little rusty* today.'

'We've been working all day on the problem, and we are *running out of steam*.'

Notice in these examples how the descriptive actions or words are concrete, while the things being described are conceptual.

In their description of the sources of mathematics, Lakoff and Núñez propose metaphors like:

ARITHMETIC IS OBJECT COLLECTION.

Collections can have the same number $\Longrightarrow$ numbers.

Collections can be compared $\Longrightarrow$ greater/lesser.

There is a smallest collection $\Longrightarrow$ ONE.

Putting collections together $\Longrightarrow$ addition.

The purpose of this talk is not a discussion of the use of conceptual metaphors in elementary mathematics, but in the use of the language of metaphor as a tool in the history of mathematics and science.

Some Claims for Mathematics

1) Higher mathematics employs conceptual metaphors whose source domains are found in more elementary mathematics.

2) Metaphors do not provide a perfect matching of domains. Where a metaphor fails, new metaphors are needed, and here is where interesting mathematics is born.

3) It is possible to "follow the metaphors" in order to fashion a history of topics in mathematics.

Let's consider a scientific example of 'following the metaphor' that is somewhat naive, but illustrates the idea.

ATOMS ARE DISCRETE, INDIVISIBLE UNITS OF MATTER

• Matter consists of collections of atoms. Atoms consist of units of air, earth, fire or water (6th century India, 5th century Greece).

The search for what atoms might consist of led to chemistry, elements characterized by the mixture of the elements they were made from.

• Atoms of one type appear in a constant proportion to atoms of another type in a substance.

This line of study led eventually to the periodic table of Mendeleev.

• But the atom is not of one substance, but has electrons (J.J. Thomson, 1897).

This does not mean that atoms are not indivisible, but may consist of parts, unified as a whole.

• The atom is like a *plum pudding* with electrons the raisins in a pudding of positive charge (the Thomson atom).

• The nucleus is indivisible with electrons flying around it. The atom is like a solar system (the Rutherford atom, 1909).

• From the view of quantum theory, electrons inhabit atomic orbital zones, and the atom is like a nucleus surrounded by clouds of probability of the location of electrons (the Bohr atom 1913).

Modern chemistry can be explained in this manner.

• Atoms can be split (1938, Hahn, Meitner, Frisch).

• Atoms consists of particles that are discrete, indivisible units of matter.

Examples from Mathematics

POLYNOMIALS ARE INTEGERS.

The divisibility properties of polynomials are studied as if polynomials are integers. Hence we can talk about irreducible polynomials (prime numbers), greatest common divisors of polynomials (gcd of integers), and results like Bezout's formula for polynomials.

GROUPS ARE COLLECTIONS OF MATRICES.

The fact that matrices are transformations, collections of invertible matrices can be identified as groups. The invariants of matrices, such as, determinants, traces, etc. become tools with which to study elements of groups. Thus we get representation theory.

FUNCTIONS ARE REAL NUMBERS.

Just as real numbers are limits of sequences of rational numbers, so too, continuous functions (on $[01]$) are limits of polynomial functions. The theory of limits for real numbers can be made the source domain for a mapping to metric spaces, spaces of functions, etc. where the metaphor generates questions, provides potential arguments, and leads to new results.

ANALYSIS IS ARITHMETIC.

How do we manage the theory of limits? By turning questions about limits into questions about arithmetic. Thus the $\epsilon - \delta$ definition of limits is a route to secure foundations for the theory of limits through the security of arithmetic.

SPACES OF FUNCTIONS ARE VECTOR SPACES.

Real-valued functions on a fixed domain may be added, subtracted, and multiplied by a scalar. The result of the metaphor is an approach to finding results about functions that are modeled on the behavior of vectors in a vector space, including representations with respect to certain restricted forms, kernels and images for linear transformations, etc.

Case study: Solving equations

• Ancient cultures.

SOLVING EQUATIONS IS FOLLOWING AN ALGORITHM

The earliest records of solutions to algebraic problems can be found in Egyptian and Babylonian texts. These include problems of a computational sort. The notion of a solution is called the *method of false position*: An algorithm is proposed for a convenient value, then the algorithm can be expected to be carried out the same way for the relevant value. This has the appearance of generality, and of algorithmic generality. The solution fits the purpose—when faced with an analogous problem, carry out this example to obtain a solution.

• Euclid

SOLVING EQUATIONS IS CONSTRUCTING AN APPROPRIATE LENGTH

"They had Book II of the Elements, which is geometric algebra and served much the same purpose as does our symbolic algebra." (Boyer) This book was studied carefully by Newton and expanded by him. The notion of a solution of an algebraic relation among lengths is a geometric construction demonstrating the relation for arbitrary lengths.

• Diophantus of Alexandria (∼210–285 C.E.)

SOLVING EQUATIONS IS FINDING THE RATIONAL NUMBER THROUGH A PROCESS

Diophantus treated problems of a similar nature to the problems of the Egyptians and Babylonians. His main contribution was a symbolism, clumsy but systematic. Only positive rational solutions were admitted to his problems. His methods of solution were ingenious, and indicate deep understanding. However, they were presented in the same manner at the Rhind papyrus—a particular example was described, and a method for solution in this case described. The general cases were found by following the algorithm.

• Al-Khwārizmī (9th century C.E.), Omar Khayyám (c. 10501123)

SOLVING EQUATIONS IS PART OF A SYSTEMATIC FAMILY OF PROCESSES

Al-jabr wa'l muqabalah

"Al-Khwarizmi's text can be seen to be distinct not only from the Babylonian tablets, but also from Diophantus' Arithmetica. It no longer concerns a series of problems to be resolved, but an exposition which starts with primitive terms in which the combinations must give all possible prototypes for equations, which henceforward explicitly constitute the true object of study. On the other hand, the idea of an equation for its own sake appears from the beginning and, one could say, in a generic manner, insofar as it does not simply emerge in the course of solving a problem, but is specifically called on to define an infinite class of problems." (R. Rashed and A. Armstrong)

Solving an algebraic problem becomes an independent activity. Al-Khārizmī sought solutions to general quadratics in the pursuit of solutions to inheritance problems and astronomical problems.

For Khayyám, cubic equations where solved by intersecting conics. Geometric algebra used to find solutions to classes of algebraic problems.

• Del Ferro (1465–1526), Cardano (1501–1576), Tartaglia (1499–1557), Bombelli (∼1526–1572)

SOLVING EQUATIONS IS REDUCTION TO GENERAL KNOWN FORMULAS

Solution to cubic equations remained geometric although general cases could be stated without geometry. The general solutions to the cubic and the biquadratic are due to the Italian mathematicians of the 16th century.

If $x^3 + px + q = 0$, then one solution is given by x_1 where

$$x_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

• Francois Viète (1540–1603)

SOLVING EQUATIONS IS A MIX OF FORMULAS AND GEOMETRY

Introduced a "new algebra" in which algebra applied to geometry while geometrical magnitude and number were distinct (Bos). Inspired by Diophantus who had introduced the 'unkonwn.'

"Finally the analytic are, endowed, at last, with its three forms of zetetics, poristics and exegetics, claims for itself the greatest problem of all, which is TO LEAVE NO PROBLEM UNSOLVED."

Algebra is the tool. Zetetics was the art of translating a problem into algebraic equations. Poristics concerned techniques of transforming algebraic proportionalities and equations. Exegetics is the art of deriving the arithmetical or geometrical solutions from the equations supplied by zetetics, and if necessary, transformed to amenable forms by poristics. (Bos) Letters used for variables (not the first such use, but the most influential).

• René Descartes (1596–1650)

SOLVING EQUATIONS IS THE REDUCTION TO CURVES OF ALGEBRAIC RELATIONS

Although not directly influenced by Viète, Descartes put forth a vision of the solution of algebraic equations that unified the algebraic and geometric approaches. Key insights include his recognition that introduction of a unit allowed all powers of quantities to be interpreted as lengths. This leads to our notation of powers x^n. Furthermore, all problems concern quantity. Geometry then becomes the tool in algebra and in the science that is reduced to algebra. A solution to a problem is a geometric diagram leading to the length representing the solution.

• Carl-Friedrich Gauss (1777–1865)

SOLVING EQUATIONS IS ALWAYS POSSIBLE OVER THE COMPLEX NUMBERS

Many directions come together in Gauss's work. In particular, some ancient problems were revealed to find their solutions in algebra. I am thinking of the constructibility of regular polygons, related by Gauss to cyclotomic polynomials, whose solutions were found to have structure relating to the constructibility. Gauss also proved the *Fundamental Theorem of Algebra*, that every polynomial, real or complex, can be factored into linear factors over the field of complex numbers. He gave four proofs. The proofs range over geometric algebraic, and analytic insights. A solution to a polynomial equation in Gauss's hands is a complex number.

• Niels Henrik Abel (1802–1829), Evariste Galois (1811–1832)

SOLVING EQUATIONS IS KNOWING THE SYMMETRIES OF THE ROOTS

To know that all of the roots of a polynomial are complex numbers, and to know that the structure of the roots may reveal deeper connections, as Gauss had done for cyclotomic polynomials, led Galois and Abel to investigate all of the roots in as small a field as possible containing the rational numbers. By focusing on the fields, and their transformations, the symmetries of the roots emerge, the Galois group. In this way, solvability of an algebraic equation in terms of the coefficients becomes a question of the properties of a group.

Possible historical projects:

CONTINUITY

GEOMETRY

APPLIED MATHEMATICS