On the topological complexity of surfaces and other manifolds

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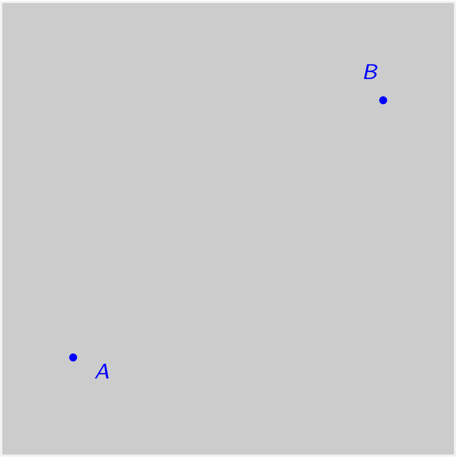
joint work with Daniel C. Cohen

08/03/2024

Motivation- Motion planning problem of a mechanical system.



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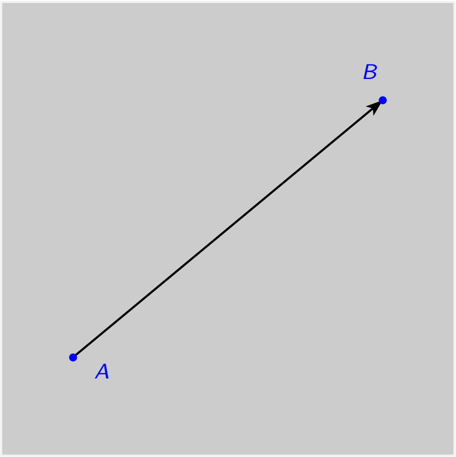

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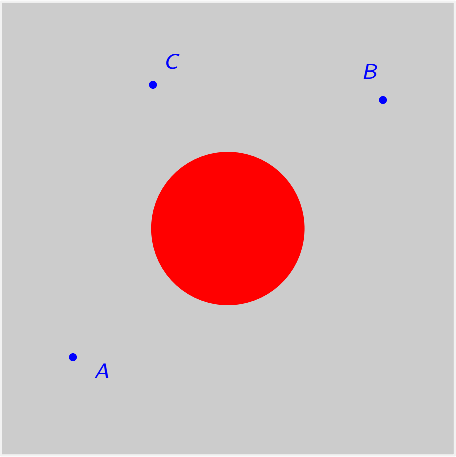

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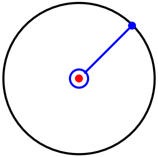

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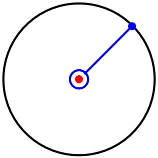
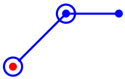


in the plane $\mathbb{R}^2$: X =circle $\qquad X = S^1$

in the space $\mathbb{R}^3$: X =sphere $\qquad X = S^2$

- System=articulated arm with two axis and fixed origin



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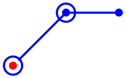

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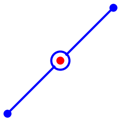
- System= bar revolving about its center (in $\mathbb{R}^3$)



$X = \mathbb{RP}^2$ = projective plane = {lines of $\mathbb{R}^3$ through $\vec{0}$}

Motion planner

Let X be a nice topological space, say a manifold, a CW-complex.

$$\begin{array}{rcl} s : X \times X & \to & X^{[0,1]} = \{\gamma : [0,1] \to X \text{ continuous}\} \\ (A, B) & \mapsto & \gamma \text{ such that } \gamma(0) = A,\ \gamma(1) = B \end{array}$$

In other words, it is a section $s : X \times X \to X^{[0,1]}$ of the

evaluation map $\begin{array}{rcl} ev_{0,1} : X^{[0,1]} & \to & X \times X \\ \gamma & \mapsto & (\gamma(0), \gamma(1)) \end{array}$ $\quad ev_{0,1} \circ s = id$

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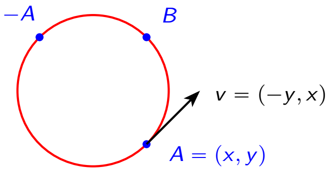
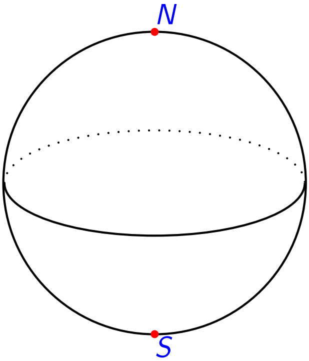
Such a section always exists when X is path-connected but is not continuous in general.

$X = S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$



$$s(A, B) = \begin{cases} \text{shortest path if } B \neq -A \\ \text{counterclockwise meridian if } B = -A \end{cases}$$

$X = S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$



$V(x, y, z) = (-y, x, 0)$

$$s(A, B) = \begin{cases} \text{shortest path if } B \neq -A \\ \text{meridian given by } V(A) \text{ if } B = -A \text{ and } A \notin \{N, S\} \\ \text{a preferred meridian if } (A, B) = (N, S) \text{ or } (S, N) \end{cases}$$

TC- Topological Complexity

TC= minimal number of continuous local sections -1

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Definition. (Farber, 2003) Suppose X is path-connected. $\mathrm{TC}(X)$ is the least n such that

$$X \times X = F_0 \cup ... \cup F_n$$

- para $i \neq j$, $F_i \cap F_j = \emptyset$,
- $F_i \subset X \times X$ is nice (ENR - Euclidian Neighborhood Retract),
- on each F_i there exists a **continuous** local section of $ev_{0,1} : X^{[0,1]} \to X \times X$

Equivalently: $\mathrm{TC}(X)$ is the least n such that

$$X \times X = U_0 \cup \ldots \cup U_n$$

where each U_i is an **open set** with a local continuous section of $ev_{0,1}$.

- TC is a homotopy invariant:

  $$\text{If } X \simeq Y \text{ then } \mathrm{TC}(X) = \mathrm{TC}(Y).$$

- $\mathrm{TC}(X) = 0$ if and only if X is contractible ($X \simeq *$).

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TC of the spheres

As seen before: $\mathrm{TC}(S^1) \leqslant 1$ and $\mathrm{TC}(S^2) \leqslant 2$.

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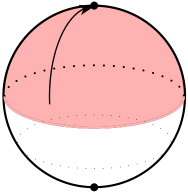
Theorem. (Grant, Lupton, Oprea)

$$\mathrm{TC}(X) = 1 \text{ iff X is homotopically equivalent to } S^{2n-1}.$$

Lusternik-Schnirelmann category

Definition. The Lusternik-Schnirelmann category of X, $\mathrm{cat}X$, is the least integer n such that X can be covered by $n+1$ open sets, each of which is contractible **in** X,

$$X = A_0 \cup \ldots \cup A_n \qquad A_0, \ldots, A_n \text{ contractible } \textbf{in } X.$$



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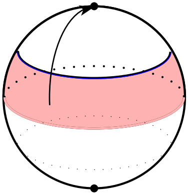

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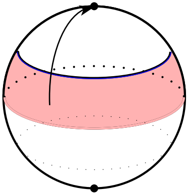

- cat is a homotopy invariant
- $\mathrm{cat}(X) = 0$ iff $X \simeq *$ and $\mathrm{cat}(S^n) = 1$ for any $n \geqslant 1$
- $\mathrm{cat}(X) \leqslant \dim(X)$

Theorem. (Farber + classical results)

$$\mathrm{cat}(X) \leqslant \mathrm{TC}(X) \leqslant \mathrm{cat}(X \times X) \leqslant 2\mathrm{cat}(X) \leqslant 2\dim(X).$$

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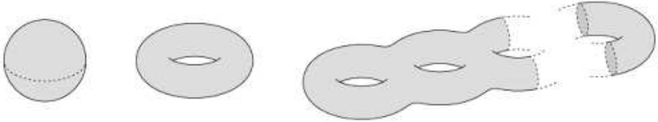
This only can happen when $\pi_1(X) \neq 0$ because

$$\pi_1(X) = 0 \Rightarrow \mathrm{cat}(X) \leqslant \frac{\dim X}{2} \quad \text{and} \quad \mathrm{TC}(X) \leqslant \dim(X).$$

Surfaces (compact, connected, without boundary)

Orientable surfaces



S^2 $\quad T = S^1 \times S^1 \quad$ torus with g holes $T_g = \underbrace{T\#T\#\cdots\#T}_{g}$.

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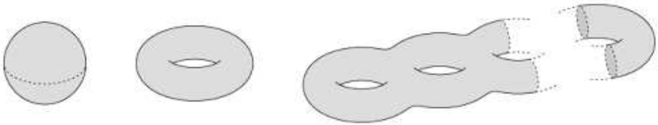

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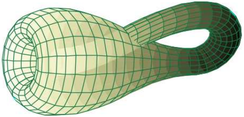
Theorem. (Farber, 2003)

- $\mathrm{TC}(S^2) = 2$
- $\mathrm{TC}(T) = 2$
- for $g \geqslant 2$, $\mathrm{TC}(T_g) = 4$.

Nonorientable surfaces: $\mathbb{RP}^2$, Klein bottle K, ...



$$K = \mathbb{RP}^2 \# \mathbb{RP}^2, \qquad N_g = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{g}$$

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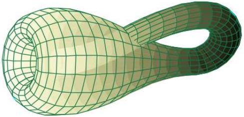

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Theorem.(Farber, Tabachnikov, Yuzvinsky, 2003) $\mathrm{TC}(\mathbb{RP}^2) = 3$

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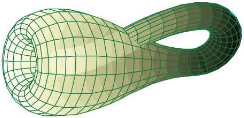

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Theorem.(Dranishnikov, 2016) For $g \geqslant 4$, $\mathrm{TC}(N_g) = 4$.

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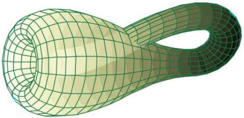

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Theorem.(Dranishnikov, 2016) For $g \geqslant 4$, $\mathrm{TC}(N_g) = 4$.

Theorem.(Cohen, V, 2017) $\mathrm{TC}(K) = 4$. For $g \geqslant 2$, $\mathrm{TC}(N_g) = 4$.

Connected sums of $\mathbb{RP}^n$

In analogy to $N_g = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ (g copies), we consider

$$\mathcal{P}_g^n := \underbrace{\mathbb{RP}^n \# \cdots \# \mathbb{RP}^n}_{g \text{ copies}}$$

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Theorem. (FTY) For $n = 1, 3$ or 7, $\mathrm{TC}(\mathbb{RP}^n) = n$.

For $n \neq 1, 3, 7$, $\mathrm{TC}(\mathbb{RP}^n)$ is the least integer k such that there exists an immersion of $\mathbb{RP}^n$ in $\mathbb{R}^k$.

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In particular, $\mathrm{TC}(\mathbb{RP}^n) \leqslant 2n - 1$.

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4. $\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b} \times \mathbb{Z}_{p^c}$
5. $\mathbb{Z}^r \times (\mathbb{Z}_2)^s$ with either n odd or n even such that $r < 2n$

Then $\mathrm{TC}(M) \leqslant 2\dim(M) - 1$.

Cohomological lower bounds

- Cup-length of $H^*(X; \Bbbk)$

$$\mathrm{cl}_{\Bbbk}(X) = \max\{n \mid a_1 \cdots a_n \neq 0, a_i \in H^{>0}(X; \Bbbk)\}$$

$$\boxed{\mathrm{cl}_{\Bbbk}(X) \leqslant \mathrm{cat}(X) \leqslant \dim(X)}$$

Ex: $\mathrm{cl}_{\Bbbk}(S^1 \times S^1) = \mathrm{cat}(S^1 \times S^1) = 2.$

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  $$\boxed{\mathrm{zcl}_{\mathbb{k}}(X) = \max\{n \mid z_1 \cdots z_n \neq 0, z_i \text{ zero divisor}\}}$$

Theorem. (Farber) $\mathrm{zcl}_{\mathbb{k}}(X) \leqslant \mathrm{TC}(X)$

Cohomological lower bounds

$$\left.\begin{array}{l}\mathrm{cat}(X)\\ \mathrm{zcl}_{\mathbb{k}}(X)\end{array}\right\} \leqslant \mathrm{TC}(X) \leqslant 2\mathrm{cat}(X) \leqslant 2\dim(X)$$

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- Therefore $\mathrm{zcl}_{\mathbb{Q}}(S^n) = 2$ when n is even.

The calculation of $\mathrm{zcl}_{\mathbb{k}}$ (together with good upper bounds) has been sufficient for determining the value of TC in many cases including the orientable surfaces, $\mathbb{RP}^2$,...

For N_g, $g \geqslant 2$, the best we can say with this approach (taking $\mathbb{k} = \mathbb{Z}_2$) is

$$3 \leqslant \mathrm{TC}(N_g) \leqslant 4$$

For proving $\mathrm{TC}(N_g) = 4$, $g \geqslant 2$

- We have used a notion of zero-divisor with twisted coefficients, in particular the Costa-Farber class:

$$\mathfrak{v} \in H^1(X \times X; I(G))$$

where $G = \pi_1(X)$, $I(G) = \{\sum n_i g_i \mid \sum n_i = 0\} \subset \mathbb{Z}[G]$ given with an action of $G \times G$.

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- $\mathrm{TC}(X) = 2n \iff \mathfrak{v}^{2n} \neq 0$.
- If $G = \mathbb{Z}_2$ then $\mathfrak{v}^{2n} = 0$ and $\mathrm{TC}(X) \leqslant 2n - 1$.

M closed manifold with $\dim M = n$ and $\pi_1(M) = G$

Proposition. If $\pi_1(M) = G$ is abelian then

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  - $\widetilde{\mathbb{Z}}$: $\mathbb{Z}$ with the action of G given by the orientation character $w : G = \pi_1(M) \to \{\pm 1\}$.
  - $[M] \in H_n(M; \widetilde{\mathbb{Z}}) \cong \mathbb{Z}$ is the (twisted) fundamental class of M.

Study of $\alpha_*(\mathsf{c} \times \mathsf{c})$ for $\mathsf{c} \in H_n(BG; \widetilde{\mathbb{Z}})$

G finitely generated abelian group with action on $\widetilde{\mathbb{Z}}$.

$H_*(BG; \widetilde{\mathbb{Z}})$ is a Pontrjagin algebra with a strictly anti-commutative product $\wedge$

$$\mathsf{c} \wedge \mathsf{d} = (-1)^{|\mathsf{c}||\mathsf{d}|} \mathsf{d} \wedge \mathsf{c} \quad \text{with } \mathsf{c} \wedge \mathsf{c} = 0 \text{ when } |\mathsf{c}| \text{ is odd.}$$

Study of $\alpha_*(c \times c)$ for $c \in H_n(BG; \widetilde{\mathbb{Z}})$

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$$c \wedge d = (-1)^{|c||d|} d \wedge c \quad \text{with } c \wedge c = 0 \text{ when } |c| \text{ is odd.}$$

Considering the morphism induced by the inversion of G

$$I : H_*(BG; \widetilde{\mathbb{Z}}) \to H_*(BG; \widetilde{\mathbb{Z}})$$

we have

$$\alpha_*(c \times c) = c \wedge I(c).$$

If $I(c) = \pm c$ and $|c|$ is odd then $\alpha_*(c \times c) = \pm c \wedge c = 0$.

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    1. $\mathbb{Z}^r$ with either n odd or n even such that $r < 2n$
    2. $\mathbb{Z}^r \times \mathbb{Z}_{p^a}$ with p prime and $r < n$
    3. $\mathbb{Z}^r \times \mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$ with p with $r \leqslant 1$
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All the conditions are sharp. In (2) $r < n$ is sharp because for $G = \mathbb{Z}^7 \times \mathbb{Z}_3$, there exists $c \in H_7(BG; \widetilde{\mathbb{Z}})$ such that $\alpha_*(c \times c) \neq 0$.

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By using surgery, we realize this class as $c = \gamma_*([M])$ where

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We get a mfld M with $\pi_1(M) = \mathbb{Z}^7 \times \mathbb{Z}_3$ and $\mathrm{TC}(M) = 2 \dim M$.