UNIVERSITÉ D'ANGERS
Faculté des Sciences
Dpt de Mathématiques
2 bd Lavoisier
49045 ANGERS Cedex 01

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présentée par
Jean-Claude Thomas
Professeur émérite - UMR 6093- Université d'Angers.

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Stephen Halperin Professeur-Univ. of Maryland at College Park
Adresse : Dean of the College of Computer, Mathematical and Physical sciences,
Av Building, University of Maryland, College Park
MD 20742, USA
email : shalper@umd.edu

Je collabore avec S. Halperin depuis plus de 30 ans.

Le séjour de S. Halperin à Angers se déroulera en même temps que celui de Y. Félix (Professeur UCL-Belgique). Ce dernier bénéficie d'un support de Professeur invité à l'Université d'Angers pendant la même période. Nous envisageons poursuivre les travaux entrepris depuis ces quatre dernières années et qui se situent dans la suite des papiers suivants:

[**Ref1**] Exponential growth and an asymptotic formula for the ranks of homotopy groups of a finite 1-connected complex (en collaboration avec Y. Félix et S. Halperin) Annals of Mathematics **171** (2009) .

[**Ref2**] The structure of the homotopy Lie algebra (en collaboration avec Y. Félix et S. Halperin) Commentarii Mathematici Helvetici (à paraître 2010).

[**Ref3**] The ranks of homotopy groups of a space of finite complex. (en collaboration avec Y. Félix et S. Halperin) Journal of the Amer. Math Soc. (soumis).

Le thème scientifique de ces recherches en cours est le suivant:

Recall that any finitely generated abelian group, G, has the form $G \cong \mathbb{Z}^k \oplus T$ where T is a finite group; k is called the *rank* of G, $\mathrm{rk}\, G$. Evidently $\mathrm{rk}\, G = \dim G \otimes_{\mathbb{Z}} \mathbb{Q}$ and so the definition may be extended to all abelian groups :

Definition : The *rank* of an arbitrary abelian group, G, is defined by $\mathrm{rk}\, G = \dim G \otimes_{\mathbb{Z}} \mathbb{Q}$.

In particular, since for any pointed topological space X the groups $\pi_i(X)$, $i \geq 2$, are abelian, the sequences $(\mathrm{rk}\, \pi_i(X))_{i\geq 2}$ are well defined.

It is a classical result that if $(k_i)_{i\geq 2}$ is an arbitrary sequence with each k_i a non-negative integer or ∞ then there is a simply connected CW complex Y with $\mathrm{rk}\, \pi_i(Y) = k_i$, $i \geq 2$. Thus in this paper we shall be concerned with the following

Question : What are the restrictions on the sequences $(\mathrm{rk}\, \pi_i(X))_{i\geq 2}$ imposed by the condition that X be a finite dimensional connected CW complex ?

First note that the class of all pointed topological spaces, X, may be divided into the three distinct groups characterized by the following conditions :

(i) $\sum_{i\geq 2} \operatorname{rk} \pi_i(X) < \infty$.

(ii) For $i \geq 2$ each $\operatorname{rk} \pi_i(X) < \infty$, but $\sum_{i\geq 2} \operatorname{rk} \pi_i(X) = \infty$.

(iii) For some $i \geq 2$, $\operatorname{rk} \pi_i(X) = \infty$.

Definition. A pointed topological space, X, is called *rationally elliptic* (resp. *rationally hyperbolic*, *π-rank infinite*) if X belongs to group (i) (resp. group (ii), group (iii)) above.

Now for any connected CW complex, X, a classical spectral sequence argument applied to Postnikov decompositions for the universal cover, $\widetilde{X}$, establishes the following equivalences :

$$\text{(1)} \qquad \operatorname{rk} \pi_i(X) < \infty \ \text{ for } 2 \leq i \leq k \quad \Longleftrightarrow \quad \dim H_{\leq k}(\widetilde{X}; \mathbb{Q}) < \infty \,.$$

It follows that X is rationally elliptic (resp. rationally hyperbolic) if and only if $\widetilde{X}$ is rationally elliptic (resp. rationally hyperbolic) in the sense of [Ref1].

Now consider the question above. In the elliptic case it is completely resolved by Friedlander and Halperin in 1979, where the authors establish a simple algorithm that decides whether any finite sequence $k_1, \ldots, k_r$ of non-negative integers appears as the sequence $(\operatorname{rk} \pi_i(X))_{i\geq 2}$ for a rationally elliptic finite dimensional CW complex. For the rationally hyperbolic and π-rank infinite cases, however, such a characterization seems out of reach, especially given the fact that when n is odd the space $S^n \vee S^n$ and $S^n \vee S^1$ satisfy $\operatorname{rk} \pi_i(X) = 0$ unless $i \equiv 1(\operatorname{mod}(n-1))$. Thus instead we consider the sequence

$$\mu_k(X) = \max_{k+2\leq i\leq k+n} \operatorname{rk} \pi_i(X) \,.$$

Our principal result deals with the hyperbolic case, and we need first to recall the

Definition. The *homotopy log index*, α_X, of a pointed topological space X is given by $\alpha_X = \limsup_k \frac{\log \operatorname{rk} \pi_k(X)}{k}$.

This invariant, which provides one measure of the growth of the sequence $\operatorname{rk} \pi_k(X)$ was introduced in a very different context by Gelfand and Kirillov.

Now if X is a rationally hyperbolic connected n-dimensional CW complex we have (1) that $\dim H(\widetilde{X}; \mathbb{Q}) < \infty$ and so we may set $h = \max_i \dim H_i(\widetilde{X}; \mathbb{Q})$. To state our main theorem we introduce the notation :

$$\beta(n, h) = 40\,(2n \log n + \log(h+1) + 1) \log nh$$

and

$$\gamma(n, h) = (n+1) \log(h+1) + 2n \log 2n \,.$$

Then our first main theorem reads :

Theorem A. [Ref3] *Suppose X is an n-dimensional connected rationally hyperbolic CW complex. Then $0 < \alpha_X < \infty$, and for some K, and for every $k \geq K$,*

$$e^{\left(\alpha_X - \frac{\beta(n,h)}{\log k}\right)k} \leq \max_{k+2\leq i\leq k+n} rk\, \pi_i(X) \leq e^{\left(\alpha_X + \frac{\gamma(n,h)}{k}\right)k} \,.$$

This leaves the π-rank infinite case, and here we have a complete answer :

Theorem B. [Ref3] *Suppose X is an n-dimensional connected CW complex. If X is π-rank infinite then for all $k \geq 0$,*

$$\max_{k+2\leq i\leq k+n} rk\,\pi_i(X) = \infty\,.$$

Remark. The principal result of [R1] is equivalent to the assertion that (for X as in Theorem A) if k is sufficiently large then $\max_{k+2\leq i\leq k+n} \mathrm{rk}\,\pi_i(X) = e^{(\alpha_X+\varepsilon_k)k}$ with $\varepsilon_k \to 0$ as $k \to \infty$. Now in Theorem A we give precise estimates for ε_k depending only on n, h and k. Not surprisingly, while the result of [R1] generalizes to spaces of finite Lusternik-Schnirelmann category, Theorem A does not, as we shall see in Theorem D, below.

When combined with previously established results Theorems A and B have the following immediate corollaries :

Corollary 1. *Let X be an n-dimensional connected CW complex. Then,*

(i) X is rationally elliptic $\Longleftrightarrow rk\,\pi_i(X) = 0\,,\quad i \geq 2n$.

(ii) X is rationally hyperbolic $\Longleftrightarrow 1 \leq \max_{k+2\leq i\leq k+n} rk\,\pi_i(X) < \infty$ for all $k \geq 0$.

(iii) X is π-rank infinite $\Longleftrightarrow \max_{k+2\leq i\leq k+n} rk\,\pi_i(X) = \infty$ for all $k \geq 0$.

Corollary 2. *Let X be an n-dimensional connected CW complex. Then*

(i) X is rationally elliptic $\Longleftrightarrow \alpha_X = -\infty$

(ii) X is rationally hyperbolic $\Longleftrightarrow 0 < \alpha_X < \infty$

(iii) X is π-rank infinite $\Longleftrightarrow \alpha_X = \infty$.

Corollary 3. *Let X be an n-dimensional connected CW complex. Then X is rationally elliptic (resp. rationally hyperbolic, π-rank infinite) if and only if $\max_{2n\leq i\leq 3n-2} rk\,\pi_i(X) = 0$ (resp. $\in (0,\infty)$, resp. $= \infty$).*

The asymptotic formula of Theorem A provides a good estimate of the homotopy log index α_X in terms of $\max_{k+2\leq i\leq k+n} \mathrm{rk}\,\pi_i(X)$, provided $k \geq K$ for sufficiently large K. Unfortunately we are not able to give any estimate for K and, indeed, nothing we know gives any suggestion that this might be possible.

By contrast it is possible to directly estimate α_X from the integers $\mathrm{rk}\,\pi_i(X)$, $i \leq r \dim X$, or equivalently from the integers $\mathrm{rk}\,H_i(\Omega X)$, $i \leq r\dim X$, with an error bound depending explicitly in r. This, our third main result, reads.

Theorem C. [Ref3] *Let X be a rationally hyperbolic n-dimensional CW complex and set $h = \max_i$ dim $H^i(\widetilde{X};\mathbb{Q})$. Then for $\log r > 2^n n^{2n+5} \log nh$,*

$$\max_{r<i\leq nr} \frac{rk\,\pi_i(X)}{i} - \frac{n\log 2n}{r} \leq \alpha_X \leq \max_{r<i\leq 2r} \frac{rk\,\pi_i(X)}{i} + \frac{\beta(n,h)}{10\log r}\,.$$

The main part of Theorem A asserts that for the 'universal sequence' $\delta_k = 1/k$, given any n-dimensional rationally hyperbolic CW complex X there is a constant $c = c(n,h)$ such that for k sufficiently large $\max_{k+2\leq i\leq k+n} \frac{\log \mathrm{rk}\, \pi_i(X)}{k} \geq \alpha_X - c\delta_k$. This is the assertion that does not generalize to rationally hyperbolic spaces of finite Lusternik Schnirelmann category. Our final main theorem reads:

Theorem D. [Ref3] *Let $\delta_k \to 0$ be any sequence of non-negative numbers and let $\alpha \in (0,\infty)$ be any number. Then there is a simply connected rationally hyperbolic wedge of spheres X such that $\alpha_X = \alpha$, and for any $c > 0$ and any integer $d > 0$ there are infinitely many k for which*

$$\max_{k\leq i\leq k+d} \frac{\log rk\, \pi_i(X)}{k} < \alpha_X - c\delta_k\,.$$

The proof of Theorems A,B,C and D proceeds by a careful analysis of the homotopy Lie algebra $L_X = \pi_*(\Omega X)\otimes \mathbb{Q}$ with Lie bracket given by the Samelson product.

We work over a ground field $\mathbb{k}$ of characteristic $\neq 2$. A graded Lie algebra, L, is a graded vector space equipped with a Lie bracket $[\ ,\ ] : L\otimes L \to L$, satisfying

$$[x,y] + (-1)^{\deg x\cdot \deg y}[y,x] = 0 \quad \text{and} \quad [x,[y,z]] = [[x,y],z] + (-1)^{\deg x\cdot\deg y}[y,[x,z]]\,,$$

and $[x,[x,x]] = 0$, $x\in L_{\mathrm{odd}}$ if char $\mathbb{k} = 3$ (This condition is automatic if char $\mathbb{k}$ is not 3.)

As in the classical case, L has a universal enveloping algebra UL, and we define

$$\mathrm{depth}\, L = \text{least } m \text{ (or } \infty) \text{ such that } \mathrm{Ext}^m_{UL}(\mathbb{k}, UL) \neq 0\,.$$

Similarly, if M is an L-module, then

$$\mathrm{grade}_L M = \text{ least } q \text{ (or } \infty) \text{ such that } \mathrm{Ext}^q_{UL}(M, UL) \neq 0\,.$$

The graded Lie algebra, L, is *connected* if $L = \{L_i\}_{i\geq 0}$ and of *finite type* if each $\dim L_i < \infty$; graded Lie algebras satisfying both condition are called cft graded Lie algebras.

Suppose now X is a simply connected CW complex of finite type. Then the rational homotopy Lie algebra, $L_X = \pi_*(\Omega X)\otimes\mathbb{Q}$ (with Lie bracket given by the Samelson product) is a cft graded Lie algebra. The motivation for the study of cft graded Lie algebras of finite depth is the following result;

Theorem. *If X is a simply connected CW complex of finite type, then*

$$\mathit{depth}\, L_X \leq \mathit{cat}_0 X$$

where $\mathit{cat}_0 X$ denotes the rational Lusternik-Schnirelmann category of X. In particular, if X is a finite CW complex, then $\mathit{depth}\, L_X$ is finite.

An important question connected with the Lie algebra L_X is the behavior of the integers $\dim\,(L_X)_i$, since

$$\dim(L_X)_i = \mathrm{rank}\, \pi_{i+1}(X)\,,$$

as explained above. We also focus on the structure of cft graded Lie algebras of finite depth, with particular attention to the interplay between depth and growth of the integers $\dim L_i$, and to the structure of the ideals in L. Our aim is a classification theory for the ideals in a cft graded Lie algebra of finite depth, and in particular for the homotopy Lie algebras L_X of a space of finite Lusternik-Schnirelmann category. A crucial notion is that of full subspace.

Definition : A subspace W of a graded vector space $V = \{V_i\}_{i\geq 0}$ is *full* in V if for some fixed λ, q and N (all positive)

$$\dim V_k \leq \lambda \sum_{i=k}^{k+q} \dim W_i \qquad , \ k \geq N\,.$$

An easy argument then shows that an equivalence relation on the subspaces of V is defined by

$$U \sim W \qquad \Leftrightarrow \qquad U \text{ and } W \text{ are full in } U + W\,.$$

Two subspaces V, W in a graded Lie algebra L are called *L-equivalent* ($V \sim_L W$) if for all ideals $K \subset L$, $V \cap K \sim W \cap K$. As we show in section 5, the set $\mathcal{L}$ of L-equivalence classes $[I]$ of ideals $I \subset L$ is a distributive lattice under the operations $[I] \leq [J]$ if $I \cap J \sim_L I$, $[I] \vee [J] = [I + J]$ and $[I] \wedge [J] = [I \cap J]$. In such a lattice each maximal chain of strict inequalities $0 < [I(1)] < \cdots < [I(r)] = [I]$ has the same length r; the number r is the height of $[I]$, $ht[I]$.

Now our main result in [Ref2] reads

Theorem. *Let L be a cft graded Lie algebra of finite depth m and suppose $ht[L] = r$. Then $r \leq m$. Moreover, the number ν_L of L-equivalence classes of ideals in L satisfies $\nu_L \leq 2^r$ and equality holds if and only if $L \sim_L I(1) \oplus \cdots \oplus I(r)$ where the $I(i)$ are ideals of height 1.*

Un CV et une liste de publications de S. Halperin sont joints.

Jean-Claude THOMAS