Journal of Mathematical Sciences: Advances and Applications
Volume 59, 2019, Pages 9-17
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jmsaa_7100122090

ON THE HILALI CONJECTURE FOR ODD GRADED HOMOTOPY GROUPS

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Abstract

The Hilali Conjecture (also known as conjecture H) [5] predicts that for any rationally elliptic and simply connected topological space X we always have $\dim(\pi_*(X) \otimes \mathbb{Q}) \leqslant \dim H^*(X; \mathbb{Q})$.

The goal of the present paper is to prove the Hilali conjecture for the topological spaces with odd homotopy groups satisfying some conditions, over the last years many works have tried to solve this conjecture in particular cases, e.g., [1], [4], [7], [8].

2010 Mathematics Subject Classification: 55P62.
Keywords and phrases: Hilali conjecture, rational homotopy, Sullivan minimal model, homotopy, cohomology, commutative differential graded algebra, elliptic space.
Received August 21, 2019; Revised October 17, 2019

1. Introduction

Let X be a simply connected topological space, X is called rationally elliptic if $\dim(\pi_*(X) \otimes \mathbb{Q}) < \infty$ and $\dim(H^*(X; \mathbb{Q})) < \infty$.

Conjecture H (Topological version). Let X be a rationally elliptic and simply connected topological space, then $\dim(\pi_*(X) \otimes \mathbb{Q}) \leqslant \dim(H^*(X; \mathbb{Q}))$.

By the theory of minimal models of Sullivan [2], the rational homotopy type of X is encoded in a differential algebra (A, d) called *the minimal model* of X. This is a free graded algebra $A = \Lambda V$, generated by a graded vector space $V = \bigoplus_{k \geqslant 2} V^k$, and with decomposable differential, i.e., $d : V^k \to (\Lambda^{\geqslant 2} V)^{k+1}$, it satisfies:

$$\begin{cases} V^k = (\pi_k(X) \otimes \mathbb{Q}), \\ H^k(\Lambda V, d) = H^k(X, \mathbb{Q}). \end{cases}$$

A well detailed chapter about Sullivan minimal models is contained in ([2], pp. 138-160). X is rationally elliptic if and only if $(\Lambda V, d)$ is rationally elliptic.

Because of the correspondence between simply connected topological spaces and their minimal models the Hilali conjecture has an algebraic version:

Conjecture H (Algebraic version). If $(\Lambda V, d)$ is a Sullivan minimal model of a rationally elliptic and simply connected topological space X, then $\dim H^*(\Lambda V, d) \geqslant \dim V$.

The Hilali conjecture was formulated in 1990 [5] by Hilali who proved the case of rationally elliptic spaces of pure type, many other works ([1], [4], [5], [6], [7], [8]) proved the conjecture for several kind of topological spaces: H-spaces, nilmanifolds, symplectic and cosymplectic manifolds, coformal spaces with only odd-degree generators, 2-stage spaces, formal and coformal spaces and for hyperelliptic spaces.

In the present paper, we start in Section 2 by recalling the Sullivan minimal models which are used in this work, in Section 3 we establish the main result: $\dim H^*(\Lambda V, d) \geqslant n$, where $V = \mathbb{Q}(a_1, \cdots, a_n)$ and $|a_i|$ are odd numbers satisfying $(d(a_i))^2 = dQ_i$ with $Q_i \in \Lambda(a_1, \cdots, a_{i-1})$, our proof is based on a recurrence on dim V. And finally in Section 4 we give an example to illustrating our theorem.

2. Sullivan Minimal Models

We recall some definitions and results about minimal models [2]. Let (A, d) be a differential algebra, that is, A is a (positively) graded commutative algebra over the rational numbers, with a differential d which is a derivation, i.e., $d(a \cdot b) = (da) \cdot b + (-1)^{\deg(a)} a \cdot (db)$, where $\deg(a)$ is the degree of a. We say that A is connected if $A^0 = \mathbb{Q}$, and simply-connected if moreover $A^1 = 0$.

A simply-connected differential algebra (A, d) is said to be *minimal* if:

(1) A is free as an algebra, that is, A is the free algebra ΛV over a graded vector space $V = \oplus_{k \geq 2} V^k$, and

(2) For $x \in V^k$, $dx \in (\Lambda V)^{k+1}$ has no linear term, i.e., it lives in $\Lambda V^{>0} \cdot \Lambda V^{>0} \subset \Lambda V$.

Let (A, d) be a simply-connected differential algebra. A minimal model for (A, d) is a minimal algebra $(\Lambda V, d)$ together with a quasi-isomorphism $\rho : (\Lambda V, d) \to (A, d)$ (that is, a map of differential algebras such that $\rho_* : H^*(\Lambda V, d) \to H^*(A, d)$ is an isomorphism). A minimal model for (A, d) exists and it is unique up to isomorphism.

Now consider a simply-connected CW-complex X. There is an algebra of piecewise polynomial rational differential forms $(\Omega^*_{PL}(X), d)$ defined in [3]. A minimal model of X is a minimal model $(\Lambda V_X, d)$ for $(\Omega^*_{PL}(X), d)$. We have that

$$\begin{cases} V^k = (\pi_k(X) \otimes \mathbb{Q})^*, \\ H^k(\Lambda V, d) = H^k(X, \mathbb{Q}). \end{cases}$$

3. Main Theorem

Let X be a rationally elliptic and simply connected topological space with a Sullivan minimal model $(\Lambda V, d)$, such that $V = \mathbb{Q}(a_1, \cdots, a_n)$ with $|a_i|$ are odd integers for all i, $1 \leq i \leq n$.

Let $\{\alpha_1, \cdots, \alpha_r\}$ be a homogeneous basis of $H^*(\Lambda(a_1, \cdots, a_{n-1}), d)$, for all i, $1 \leq i \leq n$, we put $A_i = \Lambda(a_1, \cdots, a_i)$, $da_i = P_i \in A_i$ and for all k, $1 \leq k \leq r$, $\alpha_k = [\omega_k]$.

Then we have the following result:

Theorem 3.1. *Let X be a rationally elliptic and simply connected topological space with a Sullivan minimal model $(\Lambda V, d)$, where $V = \mathbb{Q}(a_1, \cdots, a_n)$ with $|a_i|$ are odd integers. If we have for all i, $1 \leq i \leq n$, $A_i = \Lambda(a_1, \cdots, a_i)$, $da_i = P_i \in A_i$, such that $[P_i^2] = 0$ in $H^*(A_{i-1})$, then* $\dim H^*(A_n) \geqslant n$.

Our goal is to prove that $\dim H^*(\Lambda(a_1, \cdots, a_n), d) \geqslant n$ such that $V = \mathbb{Q}(a_1, \cdots, a_n)$ with $|a_i|$ odd integers and $(da_i)^2 = dQ_i$ for all i, where $Q_i \in \Lambda(a_1, \cdots, a_{i-1})$.

We suppose by recurrence that $\dim H^*(\Lambda(a_1, \cdots, a_{n-1}) \geq n-1$, and let us prove that $\dim H^*(\Lambda(a_1, \cdots, a_n)) \geq n$, the case $n = 1$ is obvious.

Let $\{\alpha_1, \cdots, \alpha_r\}$ be a homogeneous basis of $H^*(\Lambda(a_1, \cdots, a_{n-1}))$; we put $da_n = P_n \in \Lambda(a_1, \cdots, a_{n-1})$, $\alpha_k = [\omega_k]$, $1 \leq k \leq r$ and $A_i = \Lambda(a_1, \cdots, a_i)$, $1 \leq i \leq n$.

We have two cases:

Case 1. $[P_n] = 0$ in $H^*(A_{n-1})$:

Let us write $da_n = P_n$. If $[P_n] = 0$ and $P_n = dQ$, then we can do the following exchange of variables $a_n' = a_n - Q$. In this case $((\Lambda(a_1, \cdots, a_n), d) \simeq (\Lambda(a_1, \cdots, a_{n-1}), d) \otimes (\Lambda a_n', 0)$ and $H^*(\Lambda(a_1, \cdots, a_n)) = H^*(\Lambda(a_1, \cdots, a_{n-1}) \otimes \Lambda a_n'$. Therefore $\dim H^*(\Lambda(a_1, \cdots, a_n)) \geq 2(n-1)$.

Case 2. $[P_n] \neq 0$ in $H^*(\Lambda(a_1, \cdots, a_n))$:

Let us write $\Im : A_{n-1} \to A_n$ for the canonic injection and $\Im_* : H^*(A_{n-1}) \to H^*(A_n)$ the induced morphism in cohomology.

Proposition 3.2. *We have* $\ker \Im_* = H^*(A_{n-1}).[P_n]$, *which is the ideal generated by* $[P_n]$.

Proof. (a) One has $\Im_*([\omega P_n]) = [\omega P_n] = [d(\omega a_n)] = 0$, hence $H^*(A_{n-1}) \cdot [P_n] \subset \ker \Im_*$.

(b) Let $\alpha \in \ker \Im_*$, then $\alpha = 0$ in $H^*(A_n)$ so there exists ω in A_{n-1} and a polynomial R in A_n such that $\omega = dR$ and $[\omega] = \alpha$.

$$\exists (P, Q) \in A_{n-1}^2 \text{ such that } R = Pa_n + Q \text{ so } \begin{cases} \alpha = [\omega], \\ d\omega = 0, \\ \omega = d(Pa_n + Q). \end{cases}$$

Then $\omega = (dP)a_n + (-1)^{|P|} PP_n + dQ$.

We also have $\omega \in A_{n-1}$ then $\exists (P', Q') \in A_{n-2}^2$ such that $\omega = P' a_{n-1} + Q'$. Then

$$\omega = dR = dPa_n + (-1)^{|P|} PP_n + dQ = P' a_{n-1} + Q' \text{ so } \begin{cases} dP = 0, \\ \omega = ((-1)^{|P|} PP_n + dQ). \end{cases}$$

Therefore $\alpha = [\omega' P_n]$, where $[\omega'] = [(-1)^{|P|} P]$ so $\alpha \in H^*(A_{n-1}).[P_n]$, then $\ker \Im_* = H^*(A_{n-1}).[P_n]$. □

Proposition 3.3. *One has* $\dim H^*(A_n) \geq \dim H^*(A_{n-1})$.

Proof. Let us denote by $B_1 = \{[\omega_i P_n]$ such that $1 \leq i \leq p$ and $|\omega_1| \leq \cdots, \leq |\omega_p|\}$ a basis of $\ker \Im_*$, and $B_2 = \{\beta_j = [\rho_j] / 1 \leq j \leq q\}$ a basis of one complementary F of $\ker \Im_*$ in $H^*(A_{n-1})$. Let us prove that $B = \{[\omega_i P_n a_n - \omega_i Q_n], \beta_j / 1 \leq i \leq p, 1 \leq j \leq q\}$ is an additively free family in $H^*(A_n)$. Indeed, let $(\lambda_1, \cdots, \lambda_p) \in \mathbb{Q}^p$ and $(\mu_1, \cdots, \mu_q) \in \mathbb{Q}^q$ be rational coefficients such that $\sum_{i=1}^p \lambda_i [\omega_i P_n a_n - \omega_i Q_n] + \sum_{j=1}^q \mu_j [\rho_j] = 0$. Then $\exists (P, Q) \in A_{n-1}^2$ such that $\sum_{i=1}^p \lambda_i \omega_i P_n a_n - \sum_{i=1}^p \lambda_i \omega_i Q_n + \sum_{j=1}^q \mu_j \rho_j = d(Pa_n + Q) = (dP) a_n + (-1)^{|P|} PP_n + dQ$, therefore

$$\begin{cases} \sum_{i=1}^p \lambda_i \omega_i P_n = dP, \\ \sum_{j=1}^q \mu_j \rho_j - \sum_{i=1}^p \lambda_i \omega_i Q_n = (-1)^{|P|} PP_n + dQ. \end{cases}$$

Hence

$$\begin{cases} \sum_{i=1}^p \lambda_i [\omega_i P_n] = 0 \Rightarrow \lambda_i = 0, \\ \sum_{j=1}^q \mu_j \beta_j \in H^*(A_{n-1}) \cdot [P_n] \cap F. \end{cases}$$

Then for all $1 \leq i \leq p, 1 \leq j \leq q, (\lambda_i, \mu_j) = (0, 0)$, so we can conclude that, $\dim H^*(A_n) \geq \operatorname{card}(B) = \dim H^*(A_{n-1})$. □

Let us denote by $Z(A_{n-1}) := \ker(d : A_{n-1} \to A_{n-1})$ and $B(A_{n-1}) := \mathrm{Im}$ $(d : A_{n-1} \to A_{n-1})$, and let us consider the morphism

$$\phi : H^*(A_{n-1}) \to H^*(A_{n-1}),$$

$$[\omega] \mapsto [\omega P_n] = [P_n \omega].$$

Lemma 3.4. *One has* $\mathrm{Im}\, \phi \subset \ker \phi$.

Proof. Indeed if $\theta \in Z(A_{n-1})$, then $\phi([\theta P_n]) = [\theta P_n^2] = [d(\theta Q_n)] = 0$.

Lemma 3.5. *If* $\ker \phi = \mathrm{Im}\, \phi$, *then* $\dim H^*(A_n) \geq \dim H^*(A_{n-1}) + 1$.

Proof. (a) If $\forall 1 \leq k \leq n-1$, $da_k = 0$, then $\dim H^*(A_n) \geq \dim H^*$ $(A_{n-1}) \geq 2^{n-1} \geq n$.

(b) If there exists $l \in \{1, \cdots, n-1\}$ such that $da_l \neq 0$, then $da_l \in B$ $(A_{n-1}) \subset \ker \phi$, hence there exists $\lambda \in \mathbb{Q}^*$ such that $da_l = \lambda P_n$, since $|a_1| \leq, \cdots, \leq |a_n|$, therefore $|da_l| \leq |P_n|$.

Let $\beta = [a_l - \lambda a_n] \in H^*(A_n)$ and let's prove that the family $B \cup \{\beta\}$ is free in $H^*(A_n)$.

Indeed, if we suppose by contradiction that β can be written as a linear composition of the form

$\beta = [\sum_{i=1}^{p} \lambda_i \omega_i P_n a_n - \sum_{i=1}^{p} \lambda_i \omega_i Q_n + \sum_{j=1}^{q} \mu_j \rho_j]$, where λ_i and μ_j are rational numbers, then $a_l - \lambda a_n = \sum_{i=1}^{p} \lambda_i \omega_i P_n a_n - \sum_{i=1}^{p} \lambda_i \omega_i Q_n + \sum_{j=1}^{q} \mu_j \rho_j + dQ$, where $Q \in A_n$.

But this is impossible since the second member does not contain the term a_l, hence, $\dim H^*(A_n) \geq \dim H^*(A_{n-1}) + 1$. □

Lemma 3.6. *If* $\ker\phi \neq \operatorname{Im}\phi$, *then* $\dim H^*(A_n) \geq \dim H^*(A_{n-1}) + 1$.

Proof. We know by the Lemma 2.5 that B is an additively free family in $H^*(A_n)$, let $[\omega] \in \ker\phi \setminus \operatorname{Im}\phi$.

Then we have $\phi([\omega]) = 0$.

If there exists $\rho \in A_{n-1}$ such that $\omega P_n = d\rho$, let $\alpha = [a_n\omega - \rho] \in H^*(A_n)$ and let us prove that $B \cup \{\alpha\}$ is an additively free family in $H^*(A_n)$.

Indeed if $\alpha = [\sum_{i=1}^{p}\lambda_i\omega_i P_n a_n - \sum_{i=1}^{p}\lambda_i\omega_i Q_n + \sum_{j=1}^{q}\mu_j\rho_j]$ with λ_i, μ_j are rational numbers, then there exist $(P, Q) \in A_{n-1}^2$ such that $a_n\omega - \rho = a_n(\sum_{i=1}^{p}\lambda_i\omega_i P_n) - \sum_{i=1}^{p}\lambda_i\omega_i Q_n + \sum_{j=1}^{q}\mu_j\rho_j + d(a_nP + Q) = a_n(\sum_{i=1}^{p}\lambda_i \omega_i P_n - dP) + \sum_{j=1}^{q}\mu_j\rho_j + P_nP + dQ$, we get the following equalities:

$$\begin{cases} \omega = \sum_{i=1}\lambda_i\omega_i P_n - dP(\star), \\ \rho = -(\sum_{j=1}^{q}\mu_j\rho_j + P_nP + dQ) + \sum_{i=1}^{p}\lambda_i\omega_i Q_n. \end{cases}$$

But $(\star)$ implies that $[\omega] \in \operatorname{Im}\phi, ([\omega] = \phi(\sum_{i=1}^{p}\lambda_i[\omega_i]))$, which is a contradiction since $[\omega] \in \ker\phi \setminus \operatorname{Im}\phi$. □

4. An Example

Let V be a graded vector space with a basis $\{a_1, \cdots, a_6\}$ such that $a_1, a_2 \in V^3$, $a_3 \in V^5$, $a_4, a_5 \in V^7$, and $a_6 \in V^9$

Now we define a linear map d of degree 1 by:

$da_1 = da_2 = 0$, $da_3 = a_1a_2$, $da_4 = a_1a_3$, $da_5 = a_2a_3$, and $da_6 = a_1a_5 + a_2a_4$.

We have $(da_6)^2 = 2a_1a_2a_4a_5 = d(2a_3a_4a_5)$, so $[(da_6)^2] = 0$ in $H^*(\Lambda(a_1, \cdots, a_6), d)$.

The family $\{[1], [a_1], [a_2], [a_1a_4], [a_2a_5], [a_1a_2a_3a_4a_5a_6]\}$ is free in $H^*(\Lambda(a_1, \cdots, a_6), d)$. Therefore, $\dim H^*(\Lambda(a_1, \cdots, a_6), d) \geq 6$.

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