Differential Graded Operad and its Minimal Model

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October 05, 2019

Abstract

This note is a part of a course given as a brief and elementary introduction to operads. We give an overview of free operads and their construction on one hand, and the differential graded operad on the other hand, we present afterward a proof of theorem giving their minimal model proved by Markl in [4].

1 Free Operads

1.1 Functorial definition of operads[1],[3]:

Recall the fonctorial definition of operads as presented for example in [1] and [3], one can work firstly in the category of vector spaces, denoted `Vect`, an operad is a monad in the category `Vect`, i.e. a monoid in the category `End(Vect)`, (with monoidal structure $\circ$). More precisely, an operad $\mathcal{P}$ is an object in `End(Vect)`, i.e. a functor $\mathcal{P}:$ `Vect` $\to$ `Vect` together with two maps $\gamma : \mathcal{P} \to \mathcal{P}$ and $i : \mathcal{I} \to \mathcal{P}$, which are natural transformations, satisfying associativity and unity requirements given by the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{P}\circ(\mathcal{P}\circ\mathcal{P}) & \xrightarrow{\gamma\otimes id} & \mathcal{P}\circ\mathcal{P} \\ \downarrow{\scriptstyle id\otimes\gamma} & & \downarrow{\scriptstyle \gamma} \\ \mathcal{P}\circ\mathcal{P} & \xrightarrow{\gamma} & \mathcal{P} \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} I\circ\mathcal{P} & \xrightarrow{i\otimes id} & \mathcal{P}\circ\mathcal{P} & \xleftarrow{id\otimes i} & \mathcal{P}\circ I \\ & \searrow{\scriptstyle \simeq} & \downarrow{\scriptstyle \widetilde{\varphi}} & \swarrow{\scriptstyle \simeq} & \\ & & \mathcal{P}. & & \end{array}$$

Note that, for two endofunctors $\mathcal{P}, \mathcal{Q} \in$ `End(Vect)` the composition $\circ$ is obviously defined by

$$\mathcal{P}\circ\mathcal{Q}(V) = \mathcal{P}(\mathcal{Q}(V)) \qquad \text{and} \quad \mathcal{P}\circ\mathcal{Q}(\ell) = \mathcal{P}(\mathcal{Q}(\ell)).$$

for any vector space V and any linear map ℓ. It is also possible to define additional operations on endofunctors in `Vect`, namely the tensor product and the direct sum, by

$$(\mathcal{P}\otimes\mathcal{Q})(V) = \mathcal{P}(V)\otimes\mathcal{Q}(V) \quad \text{and} \quad (\mathcal{P}\otimes\mathcal{Q})(\ell) = \mathcal{P}(\ell)\otimes\mathcal{Q}(\ell),$$

respectively

$$(\mathcal{P}\oplus\mathcal{Q})(V) = \mathcal{P}(V)\oplus\mathcal{Q}(V) \quad \text{and} \quad (\mathcal{P}\oplus\mathcal{Q})(\ell) = \mathcal{P}(\ell)\oplus\mathcal{Q}(\ell),$$

for any vector space V and any linear map ℓ.

1.2 $S-$modules

Definition 1.1. *An $S-$module $\mathcal{P}$ is a sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of vector spaces endowed with right Σ_n-module structure.*

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In view of the classical definition, operads are defined by means of $S-$modules. To an $S-$module $\mathcal{P}$, we can associate an endofunctor $\tilde{\mathcal{P}} : \mathtt{Vect} \to \mathtt{Vect}$, called *Schur functor*, by

$$\tilde{\mathcal{P}}(V) = \bigoplus_{n\in\mathbb{N}} \mathcal{P}_n \otimes_{\Sigma_n} V^{\otimes n}$$

and

$$\tilde{\mathcal{P}}(\ell) = \bigoplus_{n\in\mathbb{N}} \mathrm{id} \otimes_{\Sigma_n} \ell^{\otimes n} : \tilde{\mathcal{P}}(V) \to \tilde{\mathcal{P}}(W),$$

for any vector space V and any linear map $\ell : V \to W$.

Particular case: The tensor algebras can be seen as Schur functor applied to $\mathcal{P}$ as $\mathcal{I}$, the identity, and we have $TV = \bigoplus_{n\in\mathbb{N}} V^{\otimes n}$

A Schur functor is thus a special kind of endofunctor in $\mathtt{Vect}$, thus defines an operad in view of the previously given functorial definition. In the following, we would like to limit ourselves to operads given by Schur functors. Showing that Schur functors are in one-to-one correspondence with $S-$modules will then allow us to use the functorial and the classical definition of operads in an equivalent manner. In particular, the identification of S-modules and Schur functors should respect the operations $\circ$, $\oplus$ and $\otimes$. Therefore, we first have to define these operations for $S-$modules. The direct sum of two $S-$modules $\mathcal{P}$ and $\mathcal{Q}$ is defined by

$$(\mathcal{P} \oplus \mathcal{Q})(n) = \mathcal{P}(n) \oplus \mathcal{Q}(n),$$

concerning vector spaces, and by $(\mu \oplus \nu).\lambda$, concerning the Σ_n-action. From this definition, it follows that

$$\widetilde{\mathcal{P} \oplus \mathcal{Q}} = \tilde{\mathcal{P}} \oplus \tilde{\mathcal{Q}}.$$

The tensor product of two $S-$modules $\mathcal{P}$ and $\mathcal{Q}$ is defined by

$$(\mathcal{P} \otimes \mathcal{Q})(n) = \bigoplus_{i+j=n} \mathrm{Ind}_{\Sigma_i\times\Sigma_j}^{\Sigma_n} \mathcal{P}_i \otimes \mathcal{Q}_j,$$

which is a right Σ_n-module. It can be shown that from this definition, it follows that

$$\widetilde{\mathcal{P} \otimes \mathcal{Q}} = \tilde{\mathcal{P}} \otimes \tilde{\mathcal{Q}}.$$

Remark:

$$(\mathcal{P} \otimes \mathcal{Q})(n) = \bigoplus_{i+j=n} \mathcal{P}_i \otimes \mathcal{Q}_j \mathbb{K}(\Sigma_n/\Sigma_i \times \Sigma_j) = \bigoplus_{i+j=n} \mathcal{P}_i \otimes \mathcal{Q}_j \mathbb{K}[(\mathrm{Sh}(i,j)],$$

as vector space, where $\mathrm{Sh}(i,j)$ denotes the space of $(i,j)-$shuffles, i.e. permutations of $i+j=n$ elements, where the first i and the last j elements are respectively in natural order, i.e. permutations $\sigma \in \Sigma_n$ with $\sigma_1 < ...\sigma_i$ and $\sigma_{i+1} < ... < \sigma_n$.

The composite of two $S-$modules $\mathcal{P}$ and $\mathcal{Q}$ is defined by

$$\begin{aligned}(\mathcal{P} \otimes \mathcal{Q})(n) &= \bigoplus_{k\in\mathbb{N}} \mathcal{P}_k \otimes \Sigma_k \mathcal{Q}^{\otimes k}(n) \\ &= \bigoplus_{k\in\mathbb{N}} \mathcal{P}_k \otimes_{\Sigma_k} (\bigoplus_{i_1+...+i_k=n} \mathrm{Ind}_{\Sigma_{i_1}\times...\times\Sigma_{i_k}}^{\Sigma_n} \mathcal{Q}(i_1) \otimes ... \otimes \mathcal{Q}(i_k))\end{aligned}$$

This is a right Σ_n-module if $\mathcal{Q}^{\otimes k}$ carries left Σ_k-module structure which is compatible with the right Σ_n-module structure.

More generally, we get

$$(\mathcal{P}\circ\mathcal{Q})(n)=\bigoplus_{\substack{k\in\mathbb{N}\\ i_1+...+i_k=n}}\mathcal{P}_k\otimes_{\Sigma_k}(\mathcal{Q}(i_1)\otimes...\otimes\mathcal{Q}(i_k))\otimes\mathbb{K}[\mathrm{Sh}(i_1,...i_k)].$$

This space is spanned by equivalence classes (for the Σ_k-action) of elements $(\mu;\nu_1,...,\nu_k;\sigma)$ where $\mu\in\mathcal{P}(k)$, $\nu_j\in\mathcal{Q}(i_j)$ and $\sigma\in\mathbb{K}[\mathrm{Sh}(i_1,...i_k)]$.

The left Σ_k-module structure on $\mathcal{Q}^{\otimes k}(n)$ is explained by the following example. Consider the case $k=2$, and let $\tau\in\Sigma_2$ be the transposition, then the action of τ on

$$\mathcal{Q}^{\otimes 2}(n)=\bigoplus_{i+j=n}\mathcal{Q}(i)\otimes\mathcal{Q}(j)\otimes\mathbb{K}[\mathrm{Sh}(i,j)]$$

is given by

$$\tau.(\nu_1,\nu_2,\sigma)=(\nu_2,\nu_1,\sigma'),$$

where

$$\sigma'=\sigma\circ\begin{pmatrix}1&\cdots&j&j+1&\cdots&i+j\\ i+1&\cdots&i+j&1&\cdots&i\end{pmatrix}$$

.

Indeed, for instance, if $i=3$, $j=2$ and $\sigma=\begin{pmatrix}1&2&3&4&5\\2&3&5&1&4\end{pmatrix}$, then σ'=$\sigma\circ\begin{pmatrix}1&2&3&4&5\\4&5&1&2&3\end{pmatrix}$. Hence, roughly, the Σ_k-action on $\mathcal{Q}^{\otimes k}(n)$ is given by the action on $\mathcal{Q}(i_1)\otimes...\otimes\mathcal{Q}(i_k)$ and by changing the shuffle appropriately.

It can be shown that from this definition, it follows that $\widetilde{\mathcal{P}\circ\mathcal{Q}}=\widetilde{\mathcal{P}}\circ\widetilde{\mathcal{Q}}$.

Remark: Operads are abstractions of algebras, however, not all results can be transferred from the algebraic to the operadic setting. The tensor product, providing the monoidal structure on `Vect` is bilinear, whereas the composition $\circ$, providing the monoidal structure on `End(Vect)` is only linear in the left factor. This is best seen in the above given formula for the composite of $S-$modules, and due to the fact that the right factor $\mathcal{Q}$ appears multiple times in this composite. This weakened form of bilinearity will be the source of several obstructions in the following.

An S-module morphism is a sequence of linear maps, commuting with the symmetric group action. $S-$modules and $S-$module morphisms form a category `S-Mod`. This category is a monoidal category with monoidal structure given by the composition $\circ$ and the unit $S-$module $I=(0,\mathbb{K},0,0,...)$.

As the map $\sim:\{S-modules\}\to\{Schur functors\}$respects all operations, we can identify S-modules and Schur functors, provided this map is injective. In order to proof injectivity, we need the following

Lemma 1.1. *$\mathcal{P}(n)$ is the $n-$multilinear part of $\widetilde{\mathcal{P}}(\mathbb{K}x_1\oplus...\oplus\mathbb{K}x_n)$.*

Proof: The $k-$th tensor power admits a basis made up by elements of the form $x_{i_1}...x_{i_k}$ (where the tensor product $\otimes$ is omitted). Multilinear means that all the x_i-s are different and $n-$multilinear thus means that we only consider basis elements of the form $x_{\sigma_1}...x_{\sigma_n},\sigma\in\Sigma_n$. The $n-$ multilinear part $\mathscr{M}^n(\mathcal{P})$ is finally given by $\mathscr{M}^n(\mathcal{P})=\mathcal{P}_n\otimes_{\Sigma_n}k^\sigma x_{\sigma_1}...x_{\sigma_n}$. Consider now an element of the form $\theta\otimes\tau.(x_1...x_n)=(\theta.\tau)\otimes(x_1...x_n)$, which can also be viewed as an element of $\mathcal{P}(k)\otimes\mathbb{K}(x_1...x_n)$, where the latter factor is a one-dimensional vector space. Finally, we can identify the considered element with $\theta.\tau\in\mathcal{P}(n)$.

Injectivity now follows immediately. Indeed, if the two Schur functors $\tilde{}\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{P}}$ are equal, they have to coincide on every vector space, in particular $\tilde{}\widetilde{\mathcal{P}}(\mathbb{K}x_1\oplus...\oplus\mathbb{K}x_n)=\widetilde{\mathcal{Q}}(\mathbb{K}x_1\oplus...\oplus\mathbb{K}x_n)$ for any n, thus their $n-$multilinear parts are equal, for any n. Finally, $\mathcal{P}(n)\simeq\mathcal{Q}(n)$, for any n, i.e. $\mathcal{P}\simeq\mathcal{Q}$.

Remark: We will now confine ourselves to operads given by Schur functors. This allows to view an operad either as an $S-$module or as a Schur functor, using the most convenient standpoint depending on the situation. There is naturally a forgetful functor `Op` $\to$ `S-Mod` from the category of Operads ans the category of $S-$module, it has obviously an adjoint functor `S-Mod` $\to$ `Op`.

It can be shown that the functorial definition of an operad is 'equivalent' to the classical definition. We will only give a rough description how the classical structure of an operad can be obtained from the functorial one in the nonsymmetric case.

Using the $S-$module viewpoint, an operad $\mathcal{P}$ provides a sequence $(\mathcal{P}(n))_{n\in\mathbb{N}}$ of vector spaces. The sequence of linear maps $\gamma_n : (\mathcal{P}\circ\mathcal{P})(n) \to \mathcal{P}(n)$, where

$$(\mathcal{P}\circ\mathcal{P})(n) = \bigoplus_{\substack{k\in\mathbb{N} \\ i_1+\ldots+i_k=n}} \mathcal{P}(k)\otimes(\mathcal{P}(i_1)\otimes\ldots\otimes\mathcal{P}(i_k)),$$

gives rise to the composition maps $\gamma_{i_1,\ldots,i_k}\mathcal{P}(k)\otimes(\mathcal{P}(i_1)\otimes\ldots\otimes\mathcal{P}(i_k)) \to \mathcal{P}(n)$, whereas the sequence of linear maps $i_n : \mathcal{I}(n) \to \mathcal{P}(n)$, where $\mathcal{I}(1) = \mathbb{K}$ and $\mathcal{I}(n) = 0$ for $n \neq 1$, giver rise to the identity $\iota : \mathbb{K} \to \mathcal{P}(1), \iota(1) =: 1_{\mathcal{P}}$.

1.3 $\mathcal{P}-$algebras

In the classical setting we considered P-algebras, which are representations of an operad $\mathcal{P}$ on a vector space, i.e. a sequence of linear maps $\mu_n : \mathcal{P}(n)\otimes_{\Sigma_n} V^{\otimes n} \to V$ that respects composition and identity. In the functorial setting, we give, using the endofunctor standpoint, the following

Definition 1.2. *A $\mathcal{P}-$algebra is a vector space V together with a linear map $\gamma_V : \mathcal{P} \to V$, such that the following diagrams commute:*

$$\begin{array}{ccc} \mathcal{P}\circ\mathcal{P}(V) = \mathcal{P}(\mathcal{P}(V)) & \xrightarrow{\gamma(V)} & \mathcal{P}(V) \quad \textit{and} \\ \downarrow{\scriptstyle \mathcal{P}(\gamma)} & & \downarrow{\scriptstyle \gamma_V} \\ \mathcal{P}(V) & \xrightarrow{\gamma} & V \end{array}$$

The classical and the functorial definition of a $\mathcal{P}-$algebra coincide (if $\mathcal{P}$ is a Schur functor). Starting from the functorial definition, we get that a $\mathcal{P}-$algebra is a vector space V together with the linear map

$$\gamma_V : \mathcal{P}(V) = \bigoplus_{n\in\mathbb{N}} \mathcal{P}(n)\otimes_{\Sigma_n} V^{\otimes n} \to V,\ n\in\mathbb{N},$$

which is made up by a sequence of linear maps
$\gamma_{V,n} : \mathcal{P}(n)\otimes_{\Sigma_n} V^{\otimes n} \to V,\ n\in\mathbb{N}$,
that respects composition and identity, which is encoded in the commutative diagrams. Indeed, the triangle diagram encodes that the abstract identity is sent to the concrete one. The square diagram encodes that 'the concrete map associated to abstract composition' (in the upper and right parts of the diagram) and 'composition of concrete maps' (in the left and lower parts) coincide. Let us roughly explain what happens in the 'composition of concrete maps'. Since composition of Schur functors coincides with the Schur functor associated to the composite of $S-$modules, we essentially have

$$\begin{array}{ccl} (\mathcal{P}\otimes\mathcal{P})(V) & = & \mathcal{P}(k)\otimes\mathcal{P}_{i_1}\otimes\ldots\otimes\mathcal{P}_{i_k}\otimes V^{\otimes(i_1+\ldots+i_k)} \\ & \simeq & \mathcal{P}(k)\otimes(\mathcal{P}(i_1)\otimes V^{\otimes i_1})\otimes\ldots\otimes(\mathcal{P}(i_k)\otimes V^{\otimes}i_k) \\ & \overset{\mathcal{P}(\gamma_V)=\mathrm{id}\otimes\gamma_V^{\otimes k}}{\longrightarrow} & \mathcal{P}(k)\otimes V\otimes\ldots\otimes V = \mathcal{P}(k)\otimes V^{\otimes k} \overset{\gamma_V}{\to} V \end{array}$$

where we omitted the direct sums in order to simplify notations.

Definition 1.3. *Let (V,γ_V) and (W,γ_W) be two $\mathcal{P}-$algebras. A $\mathcal{P}-$algebra morphism $\varphi : (V,\gamma_V) \to (W,\gamma_W)$ is a linear map $\varphi : V \to W$, such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{P}(V) & \xrightarrow{\gamma_V} & \mathcal{P}(V) \\ \downarrow{\scriptstyle \mathcal{P}(\varphi)} & & \downarrow{\scriptstyle \varphi} \\ \mathcal{P}(W) & \xrightarrow{\gamma_W} & W \end{array}$$

$\mathcal{P}-$algebras and $\mathcal{P}-$algebra morphisms form a category $\mathcal{P}-$ `Alg`.

Definition 1.4. *The free $\mathcal{P}-$algebra over a vector space V is the $\mathcal{P}-$algebra $\mathcal{F}(V)$ together with the linear map $i : V \to \mathcal{F}(V)$, such that for any $\mathcal{P}-$algebra $\mathcal{A}$ and any linear map $\varphi : V \to \mathcal{A}$, there exixts a unique $\mathcal{P}-$algebra morphism $\widetilde{\varphi} : \mathcal{F}(V) \to \mathcal{A}$ such that $\varphi = \widetilde{\varphi} \circ i$, i.e. the following diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{\ i\ } & \mathcal{F}(V) \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \widetilde{\varphi}} \\ & & \mathcal{A}. \end{array}$$

If existence of the free P-algebra is proved, we get that $\mathcal{F}$ is a functor from `Vect` to $\mathcal{P}$-`Alg` and that $i : V \to \mathcal{F}(V)$ is functorial in V, since for any linear map $\ell : V \to W$, there exists a unique $\mathcal{P}$-algebra morphism $\mathcal{F}(\ell) : \mathcal{F}(V) \to \mathcal{F}(W)$, such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\ i_V\ } & \mathcal{F}(V) \\ {\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(\ell)} \\ W & \xrightarrow[\ i_W\ ]{} & \mathcal{F}(W). \end{array}$$

Existence of the free $\mathcal{P}$-algebra is given by the following

Proposition 1.1. *The free $\mathcal{P}-$algebra over V is the vector space $\mathcal{P}(V) = \bigoplus_{n\in\mathbb{N}} \mathcal{P}(n) \otimes_{\Sigma_n} V^{\otimes n}$ given by the Schur functor $\mathcal{P}$, endowed with the $\mathcal{P}-$algebra structure $\gamma_{\mathcal{P}(V)} : \mathcal{P}(\mathcal{P}(V)) \to \mathcal{P}(V)$, given by the monoidal composition $\gamma(V) : (\mathcal{P} \circ \mathcal{P})(V) \to \mathcal{P}(V)$, together with the linear map $i_V : V \to \mathcal{P}(V)$, given by $i(V) \to \mathcal{P}(V)$.*

Remark: Operads are exactly what is needed to construct free algebras.

Example 1.1. *There are two natural example that should be mentioned, the operads $\mathscr{A}$ and $\mathscr{C}$*

1. In view of the previous proposition, the Schur functor Ass applied to a vector space V should provide the free associative nonunital algebra over V, which is the tensor algebra $T(V)$. This means that we should have

$$\mathscr{A}(V) = \bigoplus_{n\in\mathbb{N}} \mathscr{A}(n) \otimes_{\Sigma_n} V^{\otimes n} = \bigoplus_{n\in\mathbb{N}} V^{\otimes n} = T(V).$$

Recalling that the tensor product Σ_n is actually over $\mathbb{K}[\Sigma_n]$, we get that

$$\mathscr{A}(n) = \mathbb{K}[\Sigma_n],$$

for $n \geq 1$ and $\mathscr{A}(0) = 0$. Hence, we obtain the same result as previously.

2. The Schur functor $\mathscr{C}$ applied to a vector space V should provide the free commutative nonunital algebra over V, which is the symmetric algebra $S(V)$. Note that

$$S(V) = \bigoplus_{n\in\mathbb{N}^*} S^n V = \bigoplus_{n\in\mathbb{N}^*} (V^{\otimes n})_{\Sigma_n},$$

i.e. given by tensors which are invariant under the symmetric group action. This means that we should have

$$\mathscr{C}(V) = \bigoplus_{n\in\mathbb{N}} \mathscr{C}(n) \otimes_{\Sigma_n} V^{\otimes n} = \bigoplus_{n\in\mathbb{N}} S^n V = S(V)$$

In view of the previously obtained form of $\mathscr{C}$, we should obtain that $\mathbb{K} \otimes_{\Sigma_n} V^{\otimes n} = (V^{\otimes n})_{\Sigma_n}$, where $\mathbb{K}$ is the trivial representation. Indeed, elements of are of the form

$$\sum k \otimes (v_1 ... v_n) = \sum k.\sigma \otimes (v1 ... v_n) = \sum k \otimes \sigma.(v_1 ... v_n) = \sum k \otimes (v_{\sigma_1^{-1}} ... v_{\sigma_n^{-1}}),$$

which is also an element of $(V^{\otimes n})_{\Sigma_n}$, and vice versa. Hence, we have

$$\mathscr{C}(n) = \mathbb{K},$$

for $n \geq 1$ and $\mathscr{C}(0) = 0$.

1.4 Free operad

1.4.1 Construction of the free operad

As operads can be regarded as abstractions of algebras, we would like to define the free operad over an S-module in a similar way as we defined the free associative algebra over a vector space. However, due to the lack of linearity in the right factor of the composition of $S-$modules, this is not possible. Therefore, we will define the free operad using a limiting procedure. As for any free object, the free operad over an $S-$module M is defined by means of a universal property. Namely, as being the operad $\mathcal{F}(M)$ together with the $S-$module morphism $i : M \to \mathcal{F}(M)$, such that for any operad $\mathcal{P}$ and any $S-$module morphism $\varphi : M\mathcal{P}$, there exists a unique morphism of operads $\widetilde{\varphi} : \mathcal{F}(M) \to \mathcal{P}$, such that $\varphi = \widetilde{\varphi} \circ i$, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{i} & \mathcal{F}(M) \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \widetilde{\varphi}} \\ & & \mathcal{P}. \end{array}$$

Equivalently, one can define the free operad functor $S-$Mod→Oper as being the left adjoint functor to the forgetful functor Oper→ $S-$Mod.

In order to construct the free operad, we will view the S-module M as a Schur functor and define the sequence of Schur functors $(\mathscr{T}_n M)_{n\in\mathbb{N}}$ by

$$\begin{aligned} \mathscr{T}_0 M &= I \\ \mathscr{T}_1 M &= I \oplus M \\ \mathscr{T}_2 M &= I \oplus (M\circ)I \oplus M) = I \oplus (M \circ \mathscr{T}_1 M) \\ &\dots \\ \mathscr{T}_n M &= I \oplus (M \circ \mathscr{T}_{n-1} M) \end{aligned}$$

Remark: In general, we cannot develop the above expressions, since the composition is only left-additive. However, if it were biadditive, we could write $\mathscr{T}_n M = I \oplus M \oplus M^{\circ 2} \oplus ... \oplus M^{\circ n}$, which would then give the operadic analogue of the tensor algebra, which is the free associative algebra.

Moreover, we recursively define a sequence $i_n : \mathscr{T}_{n-1}M \to \mathscr{T}_n M$ of natural transformations by

$$i_1 : \mathscr{T}_0 M \to \mathscr{T}_1 M, \quad I \to I \oplus M$$

and

$$i_n : \mathscr{T}_{n-1}M = I \oplus (M \circ \mathscr{T}_{n-2}M) \to \mathscr{T}_n M = I \oplus (M \circ \mathscr{T}_{n-1}M), \quad i_n = \mathrm{id}_I \oplus (\mathrm{id}_M \circ i_{n-1}).$$

Note that in is a split monomorphism. A monomorphism is a left-cancellable morphism, i.e. a morphism f, such that $f \circ g = f \circ h \Rightarrow g = h$. In concrete categories, a monomorphism is a slightly weaker concept than an injection, which is itself a slightly weaker concept than a split monomorphism. Finally, we have a direct system $(\mathscr{T}_n M, i_n)$ and we can take the direct limit (also called inductive limit or colimit):

$$\mathscr{T} M = \varinjlim \mathscr{T}_n M = \coprod_n \mathscr{T}_n M / \sim,$$

where the equivalence relation $\sim$ is given by the identification in the disjoint union of $\mathscr{T}_{n-1}M$ and its injection in $\mathscr{T}_n M$. Thus, we can also see $\mathscr{T} M$ as being the increasing union $\bigcup_n \mathscr{T}_n M$. This direct limit $\mathscr{T} M$ will play the role of the free operad over the $S-$module M.

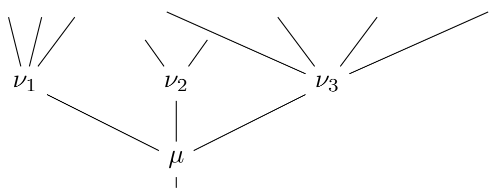
Let us detail another viewpoint, using tree diagrams, of the free operad. In order to do this we need some more information about the relationship between operads and trees.

Remark (Tree Guidelines 3): Recall that the composite $\mathcal{P} \circ \mathcal{Q}$ of two $S-$modules $\mathcal{P}$ and $\mathcal{Q}$ is defined by

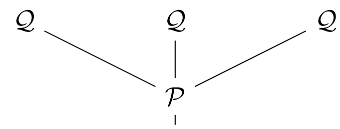
$$(\mathcal{P} \circ \mathcal{Q})(n) = \bigoplus_{\substack{k \in \mathbb{N} \\ i_1 + ... + i_k = n}} \mathcal{P}_k \otimes_{\Sigma_k} (\mathcal{Q}(i_1) \otimes ... \otimes \mathcal{Q}(i_k)) \otimes \mathbb{K}[\mathrm{Sh}(i_1, ...i_k)].$$

and that this space is spanned by (equivalence classes of) elements $(\mu; \nu_1, ..., \nu_k; \sigma)$, where $\mu \in \mathcal{P}_k$, $\nu_j \in \mathcal{Q}(i_1)$, and $\sigma \in \mathbb{K}[\mathrm{Sh}(i_1, ...i_k)]$.

In order to simplify notations, we will often omit the shuffles in the following. An element $(\mu; \nu_1, ..., \nu_k)$ will be represented by

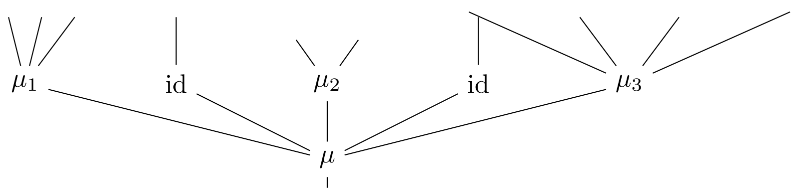


, respectively by

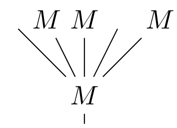


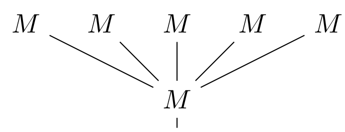
if we are not interested in the chosen operation (and its arity), but only in the corresponding space.

We will now apply this notation to the case of the free operad $\mathscr{T}M$, which is sometimes called the tree module. The unique element id of $\mathscr{T}_0 M = I$ is represented by the trivial tree. If we consider, for instance, an element $(\mu; \mu_1, \mathrm{id}, \mu_2, \mathrm{id}, \mu_3)$ of $M \circ (I \oplus M)$, it can be represented by



, or, more generally by



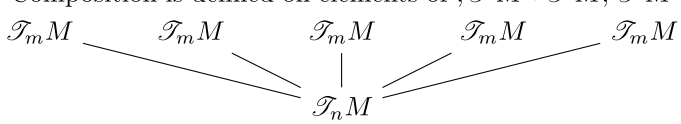


Note that, in particular, elements of $M^{\circ 2}$ are of this type, so that $M^{\circ 2} \subset \mathscr{T}_2 M$. In general, $\mathscr{T}_n M$. is the space of trees with n levels at most, whose vertices are labelled by (elements of) M. Furthermore, $M^{\circ n} \subset \mathscr{T}_n M$.

We have, by successive application of the i_k-s, morphisms $i_{n,m} : \mathscr{T}_n M \to \mathscr{T}_m M$. These give rise to a morphism $u : I \to \mathscr{T}M$ Moreover, we have morphisms $j_n : M \circ \mathscr{T}_{n-1} M$ given by inclusion of the second term in the definition of $\mathscr{T}_n M$. These give rise to a morphism $j : M \to \mathscr{T}M$.

Theorem 1.1. *There is a composition morphism γ, such that $(\mathscr{T}M, \gamma, u)$ is an operad, which together with j is the free operad over M.*

Proof: Composition is defined on elements of , $\mathscr{T}M \circ \mathscr{T}M$, $\mathscr{T}M = \bigcup_n \mathscr{T}_n M$, it is defined on elements of



the form

Therefore, we define inductively on $\mathscr{T}_n M \circ \mathscr{T}_m M$, by

$$\mathscr{T}_n M \circ \mathscr{T}_n M = (I \oplus (M \circ \mathscr{T}_{n-1} M)) \circ \mathscr{T}_m M \simeq \mathscr{T}_m M \oplus (M \circ (\mathscr{T}_{n-1} \circ \mathscr{T}_m M))$$

$$\xrightarrow{i_{m,n+m} \oplus \mathrm{id} \circ \gamma_{n-1,m}} \mathscr{T}_{n+m} M \oplus (M \circ \mathscr{T}_{n+m-1} M) \xrightarrow{\mathrm{id} + j_{n+m}} \mathscr{T}_{n+m} M.$$

Of course, one still has to check that the definition is independent of the choices of n and m, and that all other conditions (associativity, unitality and universality) are verified. □

Remark: Note that in the above definition of the composition map γ, we used left-additivity of the composition $\circ$ of $S-$modules. Moreover, we used the associativity isomorphism

$$(M \circ \mathscr{T}_{n-1}M) \circ \mathscr{T}_m M \simeq M \circ \mathscr{T}_{n-1} \circ \mathscr{T}_m M),$$

which bares some differences to its algebraic analogue. In particular, when working in a graded context, this associativity isomorphism will lead to Koszul sign, since the switching map is involved.

Example 1.2. *Consider the $S-$module $M = (0, W, 0, ...)$, where W is a vector space. The corresponding Schur functor is, applied on a vector space V, $M(V) = W \otimes V$. Note that this functor is linear, i.e. $M(V \oplus V') = M(V) \oplus M(V)$. We can thus write*

$$\begin{array}{rcl} \mathscr{T}_0 M &=& I \\ \mathscr{T}_1 M &=& I \oplus M \\ \mathscr{T}_2 M &=& I \oplus (M \circ (I \oplus M)) = I \oplus M \oplus M^{\circ 2} \\ ... && \\ \mathscr{T}_n M &=& I \oplus (M \circ \mathscr{T}_{n-1} M) = I \oplus M \oplus ... \oplus M^{\circ n} \\ ... && \end{array}$$

as Schur functors, or, equivalently

$$\begin{array}{rcl} \mathscr{T}_0 M &=& (0, \mathbb{K}, 0, ...) \\ \mathscr{T}_1 M &=& (0, \mathbb{K} \oplus W, 0, ...) \\ \mathscr{T}_2 M &=& (0, \mathbb{K} \oplus W \oplus W^{\otimes 2}, 0, ...) \\ ... && \\ \mathscr{T}_n M &=& (0, \mathbb{K} \oplus W \oplus W^{\otimes 2} \oplus ... \oplus W^{\otimes n}, 0, ...) \\ ... && \end{array}$$

as $S-$modules. Finally, we get

$$\mathscr{T} M = (0, T(W), 0, ...)$$

and we recover the tensor algebra $T(W)$, i.e. the free associative algebra over W.

It is possible to introduce a weight grading on the free operad $\mathscr{T} M$. This is done by defining the weight of an element $\mu \in M(n)$ to be equal to one, and the weight of the element $id \in I(1)$ to be zero. The weight of a general element of $T\ M$ is then given by the number of operations of M, which it is built from. In terms of trees, the weight is given by the number of vertices (decorated by M). As usually, we denote the space of elements of weight k by $\mathscr{T} M^{(k)}$. In particular, we have that $\mathscr{T} M^{(0)} = I$, $\mathscr{T} M^{(1)} = M$, and that $\mathscr{T} M^{(2)} \subset M^{\circ 2} \subset \mathscr{T}_2 M$.

1.4.2 Free operad and types of algebras

The importance of the free operad lies in the fact that any operad can be given as the quotient of a free operad by an operadic ideal. Indeed, the operad corresponding to some type of algebras can be given as the quotient $\mathscr{T} M/(R)$, where the $S-$module M is determined by the generating operations of the considered algebra, and $R \subset \mathscr{T} M$ is determined by the relations that these operations verify.

Let us be more precise. An *algebra of type P* is given by a vector space A and n-ary operations $\mu_n : A^{\otimes n} \to A$, called generating operations, satisfying certain relations $r_j = 0$. Further, we assume that the relations are multilinear, i.e. of the form $r_j = \sum_k \varphi_k = 0$, where φ_k is a composite of generating relations (and identities). The elements $r_j = \sum_k \varphi_k$ are called *relators*. The category of algebras of type `P` is denoted by `P-Alg`.

Example 1.3. *Let A be an algebra of type associative, i.e. an associative algebra, then there is only one generating operation, namely the binary multiplication $\mu : A^{\otimes 2} \to A$, satisfying the associativity relation $-\mu \circ (\mu, id) + \mu \circ (id, \mu) = 0$. The unique relator r is given by $r = \varphi_1 + \varphi_2 = -\mu \circ (\mu, id) + \mu \circ (id, \mu)$.*

Let M be the S-module, whose arity n spaces are generated by the n-ary generating operations μ_n, and where the Σ_n-module structure is given by the symmetries of these operations. Since the relators are composites of these generating relations (and identity), they span a sub- $S-$module R of the free operad $\mathscr{T}M$. Let (R) denote the operadic ideal of $\mathscr{T}M$ generated by R. The precise definition of operadic ideals is given as follows:

Definition 1.5. *An operadic ideal I of an operad $\mathcal{P}$ is a sub$-S-$module of* P*, such that for any family of operations $\{\mu; \nu_1, ..., \nu_k\}$ of $\mathcal{P}$, we have that if one of these operations is in I, then the composite $\gamma(\mu; \nu_1, ..., \nu_k)$ is also in I.*

This way, we have naturally constructed the operad $\mathscr{T}M/(R)$, which corresponds to algebras of type `P`.

For algebras of type `P`, there exists the notion of free algebras of type `P` over a vector space V. Let $\mathcal{P}$ denote the functor $\mathcal{P} : V \mapsto \mathcal{P}(V)$, which gives the free algebra of type `P` over V. As we have seen in the previous chapter, this functor $\mathcal{P}$ is a Schur functor, and more precisely an operad. By construction also gives the free algebra of type `P` over the vector space V. Since both constructions are functorial in V, the operads $\mathcal{P}$ and $\mathscr{T}M/(R)$ coincide. We get the following

Proposition 1.2. *A type* `P` *of algebras (whose relations are multilinear) determines an operad $(\mathscr{T}M/(R))(V)$. Moreover, the category $\mathcal{P}-$*`Alg` *of algebras over this operad is equivalent to the category* `P-Alg` *of algebras of the given type* `P`*.*

2 Differential Graded Operads

Definition 2.1. *Differential graded operad $\mathcal{A}$ is a sequence of chain complexes*

$$\begin{array}{r} ... \overset{d}{\leftarrow} A(1)_0 \overset{d}{\leftarrow} A(1)_1 \overset{d}{\leftarrow} A(1)_2 \overset{d}{\leftarrow} ... \overset{d}{\leftarrow} A(1)_n \overset{d}{\leftarrow} ... \\ ... \overset{d}{\leftarrow} A(2)_0 \overset{d}{\leftarrow} A(2)_1 \overset{d}{\leftarrow} A(2)_2 \overset{d}{\leftarrow} ... \overset{d}{\leftarrow} A(2)_n \overset{d}{\leftarrow} ... \\ .... \\ ... \overset{d}{\leftarrow} A(k)_0 \overset{d}{\leftarrow} A(k)_1 \overset{d}{\leftarrow} A(k)_2 \overset{d}{\leftarrow} ... \overset{d}{\leftarrow} A(k)_n \overset{d}{\leftarrow} ... \\ .... \end{array}$$

equipped with the following structure:
(i) Each $A(n)_*$ is a Σ_n-module, that is the symmetric group Σ_n acts on the chain complex $A(n)_*$.
(ii) There is defined a (partial) composition product

$$a(k)_m \circ_i b(l)_n \in A(k + l - 1)_{m+n}, \; i = 1, 2, ..., k$$

so that the following conditions are satisfied:

Associativity. *For $f \in A(p)_*$, $g \in A(q)$, $h \in A(r)_*$*

$$f \circ_i (g \circ_j h) = (f \circ_i g) \circ_{i+j-1} h, \quad 1 \leq i \leq p, \; 1 \leq j \leq q;$$
$$(f \circ_j h) \circ_i g = (f \circ_i g) \circ_{q+j-1} h, \quad 1 \leq i < j \leq p.$$

Compatibility with differential. *The composition operation*

$$\circ_i : A(k)_* \otimes A(l)_* \to A(k + l)_*$$

is a chain map

$$d(f \circ_i g) = d(f) \circ_i g + (-1)^n f \circ_i d(g).$$

Compatibility with symmetric group action. Unit. *There exists $e \in A(1)_0$ such that*

$$a \circ_i e = e \circ_1 a = a.$$

By a *degree p derivation on an operad* $\mathcal{P}$ we mean a sequence $\theta = \{\theta(n) : \mathcal{P}(n) \to \mathcal{P}(n)\}$ of equivariant degree p maps such that

$$\theta(m+n-1)(f \circ_i g) = \theta(m)(f) \circ_i g + (-1)^{p.|f|}.f \circ_i \theta(n)(g),$$

for $f \in \mathcal{P}(m),\ g \in \mathcal{P}(n),\ m, n \geq 1$ and $1 \leq i \leq n$. We denote by $\mathrm{Der}_p(\mathcal{P})$ the vector space of all degree g derivations of $\mathcal{P}$.

Lemma 2.1. *Let $\phi \in Der_p(\mathcal{P})$ and $\psi \in Der_q(\mathcal{P})$. The the formula*

$$[\phi, \psi](n) := \phi(n)\psi(n) - (-1)^{pq}\psi(n)\phi(n),\ n \geq 2$$

defines on $Der_*(\mathcal{P})$ the structure of a graded Lie algebra.

Proof. An easy exercice. □

By a differential operad we mean a couple $(\mathcal{S}, d)$ where $\mathcal{S}$ is an operad and d is a degree -1 derivation of $\mathcal{S}$, $d \in \mathrm{Der}_{-1}(\mathcal{S})$, with $d^2 = 0$. It is clear that for such a differential operad the collection $\mathcal{H} = \mathcal{H}(\mathcal{S}, d) := H(S(n), d(n)); n \geq 2$, where $H(S(n), d(n))$ denotes the homology of the differential space $(\mathcal{S}(n), d(n))$, has a natural structure of an operad, called the homology (operad) of $(\mathcal{S}, d)$. Observe that what we call a differential operad here is exactly an operad in the monoidal category of differential spaces. Of course, any nondifferential operad $\mathcal{S}$ can be considered as a differential operad with the trivial differential. If we wish to stress that we consider S in this way, we write $(\mathcal{S}, 0)$ instead of $\mathcal{S}$.

Generic example. Any dg module (V, d) has its (nonunital) endomorphism operad (End(V),D) defined by

$$End(V)(n) = \begin{cases} \mathrm{Hom}_*(V^{\otimes n}, V), & \text{for } n \geq 2, \quad \text{and} \\ 0, & \text{for } n = 1, \end{cases}$$

where $\mathrm{Hom}_p(V^{\otimes n}, V)$ denotes the vector space of homogeneous degree p linear maps $f : V^{\otimes n} \to V$. The composition maps are defined in the obvious way and the differential D is, for $f \in \mathrm{Hom}_p(V^{\otimes n}, V)$ given by

$$D(f) := d \circ f - (-1)^p.f \circ d^{\otimes n},$$

where $d^{\otimes n}$ is the usual differential induced by d on $V^{\otimes n}$.

For $f \in \mathrm{Hom}_p(V^{\otimes n}, V)$ and $g \in \mathrm{Hom}_q(V^{\otimes n}, V)$, the product $f \circ_i g \in \mathrm{Hom}_{p+q}(V^{\otimes n+m-1}, V),\ i = 1, 2, ...$ is defined by

$$f \circ_i g(v_1 \otimes ... \otimes v_{m+n-1}) = f(v_1 \otimes ... \otimes v_{i-1} \otimes g(v_i \otimes ... \otimes v_{i+n-1}) \otimes ... \otimes v_{m+n-1}).$$

The circle product is the substitution

$$f \circ (g_1, ..., g_m)(v_1 \otimes ... \otimes v_{k_1+...+k_m}) = \sum f(g_1(v_1 \otimes ... \otimes v_{k_1}) \otimes ... \otimes g_{k_m}(v_{k_1+...+k_{m-1}} \otimes ... \otimes v_{k_1+...+k_m})).$$

The defining conditions of an operad mimic the properties of the endomorphism opearad.

The endomorphism operad inspires the following terminology: An element $v \in \mathcal{P}(n)_k$ has the degree k and the arity n.

An algebra over a differential operad $(\mathcal{S}, d_{\mathcal{S}})$ (or an $(\mathcal{S}, d_{\mathcal{S}})-$algebra) is then a differential operad homomorphism $(\mathcal{S}, d_{\mathcal{S}}) \to (End(V), D)$.

Return back to free operad, we can construct a graded as follows; consider a sequence $\mathcal{M} := \{M(n);\ n \geq 2\}$ of graded Σ_n-modules, from the homogeneity of the axioms of an (nonunital) operad, each $\mathcal{F}(M)(n)$ is naturally graded, $\mathcal{F}(M)(n) = \bigoplus_{l \geq 1} \mathcal{F}^l(M)(n)$ and that grading has the following properties.

(i) Each $\mathcal{F}^l(M)(n)$ is a Σ_n-invariant submodule of $\mathcal{F}(M)(n)$,

(ii) $\mathcal{F}^1(M)(n) = M(n)$ and $\mathcal{F}^{\geq n}(M)(n)$,

(iii) If $f \in \mathcal{F}^l(M)(n)$ and $g \in \mathcal{F}^k(M)(n)$ then $f \circ_i g \in \mathcal{F}^{k+l}(M)(m+n-1)$.

Lemma 2.2. *The correspondence $\mathcal{H} : (\mathcal{S}, d_{\mathcal{S}}) \to \mathcal{H}(\mathcal{S}, d_{\mathcal{S}})$ induces a functor from the category of differential operads to the category of nondifferential operads.*

If (V, dV) is a differential vector space and $(End(V), D)$ the endomorphism operad, then there exists a noncanonical map $\Phi : (End(H(V, dV), 0) \to (End(V), D)$ of differential operads inducing the canonical isomorphism

$$(End(H(V, dV), 0) \cong \mathcal{H}(End(V), D)$$

given by the Künneth formula.

Proof. The first part of the lemma is an easy exercise. To prove the second part, choose a decomposition $V = H \oplus B \oplus C$ with $H \oplus B = Ker(d)$ and $B = Im(d)$. Let $\iota : H \to V$ be the corresponding inclusion and $\pi : V \to H$ be the corresponding projection. For an element $f \in End(H)(n)$, $f : H^{\otimes n} \to H$, let $\Phi(f) \in End(V)$ be defined as $\Phi(f) := \iota \circ f(\pi^{\otimes n})$. Let us verify that $\Phi : (End(H(V, d), 0) \to (End(V), D)$ thus defined is a homomorphism of differential operads. For $f \in End(H)(m), g \in End(H)(n) \text{and} 1 \leq i \leq n$, we have

$$\begin{aligned}
\Phi(f \circ_i g) &= \iota \circ g(\mathbb{I}^{\otimes(i-1)} \otimes f \otimes \mathbb{I}^{\otimes(n-i)})(\pi^{\otimes(m+n-1)}) = \iota \circ g(\pi^{\otimes(i-1)} \otimes f(\pi^{\otimes m}) \otimes \pi^{\otimes(n-i)}) \\
&= \iota \circ g(\pi^{\otimes(i-1)} \otimes \pi\iota f(\pi^{\otimes m}) \otimes \pi^{\otimes(n-i)}) = \Phi(f) \circ_i \Phi(g),
\end{aligned}$$

because $\pi\iota = \mathbb{I}$. We also easily have $D(\Phi(f)) = d\iota f \pi^{\otimes m} - (-1)^p \iota f(\pi d)^{\otimes n} = 0$, because $d\iota = \pi d = 0$, thus Φ is indeed a map of differential operads. The rest of the statement follows from the Künneth formula. □

Let $A : (\mathcal{S}, d_{\mathcal{S}}) \to (End(V), D)$ be an $(\mathcal{S}, d_{\mathcal{S}})-$algebra structure on (V, d). By the above lemma, A induces an $\mathcal{H}(\mathcal{S}, d_{\mathcal{S}})-$algebra structure on $H(V, d)$,

$$\mathcal{H}(A) : \mathcal{H}(\mathcal{S}, d_{\mathcal{S}}) \to End(H(V, d)$$

which we call the homology algebra structure induced by A.

3 Model for Operads (Markl Theorem)

The following theorem show the existence of a minimal model for differential operads.

Theorem 3.1. *Let $(\mathcal{S}, d_{\mathcal{S}})$ be a differential graded operad (with $\mathcal{S}(1) = 0$ as usual). Then there exist a $S-$modules $\mathcal{M} = \{M(n); n \geq 2\}$, a differential d on $\mathcal{F}(\mathcal{M})$ and a homomorphism $\nu : (\mathcal{F}(\mathcal{M}), d) \to (\mathcal{S}, d_{\mathcal{S}})$ such that*

(i) the differential d is minimal, $d(\mathcal{M}) \subset \mathcal{F}^{\geq 2}(\mathcal{M})$, and

(ii) the map ν induces an isomorphism in homology.

The object $\nu : (\mathcal{F}(\mathcal{M}, d) \to (\mathcal{S}, d_{\mathcal{S}})$ is called the minimal model of the differential operad $(\mathcal{S}, d_{\mathcal{S}})$.

It is unique in the sense that if $\nu' : (\mathcal{F}(\mathcal{M}', d) \to (\mathcal{S}, d_{\mathcal{S}})$ is another minimal model of $(\mathcal{S}, d_{\mathcal{S}})$, then the differential operads $(\mathcal{F}(\mathcal{M}, d)$ and $(\mathcal{F}(\mathcal{M}', d)$ are isomorphic.

Proof.

We denote by s and s^{-1} respectively the suspension and the desuspension of a graded module or graded vector space and $((sV)_p = V_{p-1}, (s^{-1}V)_p = V_{p+1})$.

Let $X(2) := \mathcal{H}(\mathcal{S}, d_{\mathcal{S}})(2)$ and let $\sigma(2) : X(2) \to Z(\mathcal{S}, d_{\mathcal{S}})(2) \subset \mathcal{S}(2)$ be an equivariant splitting of the projection $cl(2) : Z(\mathcal{S}, d_{\mathcal{S}})(2) \to \mathcal{H}(\mathcal{S}, d_{\mathcal{S}})(2)$ (cl will denote the projection of a chain onto its homology class). Define a differential d_2 on $\mathcal{F}(X(2))$ and a map $\nu_2 : (\mathcal{F}(X(2), d_2) \to (\mathcal{S}, d_{\mathcal{S}})$ by

$$d_2 = 0 \quad \text{and } \nu_{|X(2)} := \sigma(2).$$

We finish the construction by induction. Suppose we have already constructed a collection $X(< n) = \{X(k); 2 \leq k < n\}$, a differential d_{n-1} on $\mathcal{F}(X(< n))$ and a map $\nu_{n-1} : (\mathcal{F}(X(< n)), d_{n-1}) \to (\mathcal{S}, d_{\mathcal{S}})$ such that

$(i)_{n-1}$ the differential d_{n-1} is minimal, $d_{n-1}(X(< n)) \subset \mathcal{F}^{\geq 2}(X(< n))$ and

$(ii)_{n-1}$ the map $\mathcal{H}(\nu_{n-1})(k) : \mathcal{H}(\mathcal{F}(X(< n)), d_{n-1})(k) \to \mathcal{H}(\mathcal{S}, d_{\mathcal{S}})(k)$ is an isomorphism for any $k \leq n-1$.

Let $A(n) := \mathcal{H}(\mathcal{S}, d_{\mathcal{S}})(n)/(Im(\mathcal{H}(\nu_{n-1}))(n), \overline{B}(n) := Ker(H(\nu_{n-1}))(n)$ and $B(n) := s\overline{B}(n)$.

Let $\sigma(n) : A(n) \to Z(\mathcal{S}, d_{\mathcal{S}})(n)$ be an equivariant section of the composition $Z(\mathcal{S}, d_{\mathcal{S}})(n) \stackrel{cl}{\to} A(n)$ and let $r'(n) : \mathcal{H}(\mathcal{F}(X(<n)), d_{n-1})(n) \to Z(\mathcal{F}(X(<n)), d_{n-1})(n)$ be an equivariant section of the projection $cl_n : Z(\mathcal{F}(X(<n)), d_{n-1})(n) \to \mathcal{H}(\mathcal{F}(X(<n)), d_{n-1})(n)$ and let $r(n) : \overline{B}(n) \to Z(\mathcal{F}(X(<n)), d_{n-1})(n)$ be the composition of the inclusion $\overline{B}(n) \hookrightarrow \mathcal{H}(\mathcal{F}(X(<n)), d_{n-1})(n)$ and $r'(n)$. Define

$$X(n) := A(n) \oplus B(n), X(\leq n) := X(<n) \oplus X(n),$$
$$d_{n|X(<n)} := d_{n-1|X(<n)}, d_{n|A(n)} := 0,\ d_{n|B(n)} := r(n) \circ s^{-1},$$
$$\nu_{n|X(<n)} := \nu_{n-1|X(<n)}, \nu_{n|A(n)} := \sigma(n), \text{and} \nu_{n|B(n)} := 0.$$

Then d_n is obviously minimal since $Z(\mathcal{F}(X(<n)), d_{n-1})(n) \subset \mathcal{F}(X(<n))(n) \subset \mathcal{F}^{\geq 2}(X(<n))(n)$.

Also $\mathcal{H}(\nu_n)(n)$ is obviously an epimorphism, by construction. Let us prove that $\mathcal{H}(\nu_n)(n)$ is a monomorphism.

Let $z \in Z(\mathcal{F}(X(\leq n)), d_n)(n)$ and write it in the form $z = a + b + \omega$, with $a \in A(n)$, $b \in B(n)$ and $\omega \in \mathcal{F}^{\geq 2}(X(<n))$. By definition $d_n(a) = 0$, therefore $-d_n(b) = -r(n)(s^{-1}b) = d_n(\omega)$. Since ω is decomposable we have, in fact, $d_n(\omega) = d_{n-1}(\omega)$, therefore $d_n(\omega)$ represents a trivial homological class in $H(F(X(<n)), dn-1)(n)$ and $b = 0$. If z represents an element of $Ker(\mathcal{H}(\nu_n))(n)$, then $a = 0$, so $z \in Z(\mathcal{F}(X(<n)), d_{n-1})(n)$ and obviously $cl(z) \in Ker(\mathcal{H}(\nu_{n-1}))(n)$. But then $z = d_n(scl_n(z))$ by the construction of d_n which implies that z represents a trivial homology class in $\mathcal{H}(\mathcal{F}(X(\leq n)), d_n)$. We see that our data satisfy the conditions $(i)_n$–$(ii)_n$ and we finish the construction by induction.

We leave the proof of the uniqueness to the reader. □

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