Introduction to Operad Theory

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Contents

Definition (Nonsymmetric Operad)

A nonsymmetric operad X, also called operad without permutations, is a family $\{X_n\}_{n\in\mathbb{N}}$ of objects (sets in particular) whose elements are called $n-$ary operations, together with a distinguished element $I \in X_1$ and a collection of composition functions

$$\circ_{i_1,\ldots,i_k} : X_k \times X_{i_1} \times \ldots \times X_{i_k} \to X_{i_1+\ldots+i_k}$$

satisfying the associativity and identity axioms below. We denote the action of the composition maps by :

$$\mu : (f, g_1, \ldots, g_k) \mapsto \mu(f, g_1, \ldots, g_k) = f \circ_{i_1,\ldots,i_k} (g_1, \ldots, g_k),$$

(Associativity) Let $n \in \mathbb{N}$, and $f \in X_n$. For each $i \in \{1, 2, ..., n\}$, let $a_i \in \mathbb{N}$ and $g_i \in X_{a_i}$. Then for each a_i, for each $j \in \{1, 2, ..., a_i\}$, let $h_{i,j} \in X_{k_{i,j}}$ for some arbitrary $k_{i,j} \in \mathbb{N}$. Then :

$$f \circ_{(k_{1,1}+...+k_{1,a_1}),...,(k_{n,1}+...+k_{n,a_n})} (g_1 \circ_{k_{1,1},...,k_{1,a_1}} (h_{1,1}, ..., h_{1,a_1})$$

$,...,g_n \circ_{k_{n,1},...,k_{n,a_n}} (h_{n,1}, ..., h_{n,a_n})) = (f \circ_{a_1,...,a_n}$
$(g_1, ..., g_n)) \circ_{k_{1,1},...,k_{1,n},k_{2,1},...,k_{n,a_n}} (h_{1,1}, ..., h_{1,a_1}, h_{2,1}, ..., h_{n,a_n})$.
(Identity) For any $n \in \mathbb{N}$ and $f \in X_n$, we have
$f \circ_{1,...,1} (I, ..., I) = f = I \circ_n (f)$.

Definition (Symmetric Operad)

A symmetric operad (or just operad) is a nonsymmetric operad X equiped with a right action of the symmetric group S_n on each X_n satisfies the following equivariance axioms

(Equivariance 1) Let $n \in \mathbb{N}$, $f \in X_n$, and $g_1 \in X_{a_1}, \ldots, g_n \in X_{a_n}$ for some arbitrary $a_i \in \mathbb{N}$. Let $\tau \in S_n$, and $\sigma = \tau^{-1}$. Then :

$$(f.\tau) \circ_{a_1,\ldots,a_n} (g_1, \ldots, g_n) = (f \circ_{a_{\sigma(1)},\ldots,a_{\sigma(n)}})(g_{\sigma(1)}, \ldots, g_{\sigma(n)}).\tau',$$

where $\sigma \in S_n$, and $\tau' \in S_{a_1+\ldots+a_n}$

(Equivariance 2) Let n, f and the g_i and a_i be as above. Let $\sigma_1 \in S_{a_1}, \ldots, \sigma_n \in S_{a_n}$. Then :

$$f \circ_{a_1,\ldots,a_n} (g_1.\sigma_1, \ldots, g_n.\sigma_n) = (f \circ_{a_1,\ldots,a_n} (g_1, \ldots, g_n)).(\sigma_1, \ldots, \sigma_n),$$

where $(\sigma_1, \ldots, \sigma_n) \in S_{a_1,\ldots,a_n}$ is the disjoint union of the σ_i.

Example (Canonical Operad)

Let X be a set. For each $n \in \mathbb{N}$, define $\mathrm{End}_X(n) = \mathrm{Hom}(X^n, X)$, for n, f, g_i, and a_i as in the definition of an operad, we define :

$$f \circ_{a_1,\ldots,a_n} (g_1, \ldots, g_n) = f(g_1, \ldots, g_n),$$

where $f(g_1, \ldots, g_n)$ represents the function given by :

$$(x_1, \ldots, x_{i_1}, \ldots, x_{i_1+\ldots+i_n} \mapsto f(g_1(x_1, \ldots, x_{i_1}), \ldots, g_n(x_{i_{n-1}+1}, \ldots, x_{i_n})).$$

If $\tau \in S_n$, we define $(x_1, \ldots, x_n) \mapsto f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, where $\sigma = \tau^{-1}$.
End_X is a symmetric operad.

Algebra over an operad

Definition (Morphism of Nonsymmetric Operads)

Let $(X, \circ, I)$ and $(Y, \circ', J)$ be a nonsymmetric operads. Then a morphism of nonsymmetric operads $F : X \to Y$ is a family of functions $\{F_n : X_n \to Y_n\}_{n\in\mathbb{N}}$ that satisfy the following :
(i) $F_1(I) = J$. That is, F preserves the identity,
(ii) For n, f, g_i, and a_i as in the definition of an operad,

$$F_{a_1+\ldots+a_n}(f \circ_{a_1,\ldots,a_n}(g_1, \ldots, g_n)) = F_n(f)\circ'_{a_1,\ldots,a_n}(F_{a_1}(g_1), \ldots, F_{a_n}(g_n)).$$

Algebra over an operad

Definition (Morphism of Symmetric Operads)

Let X and Y be symmetric operads with group actions . and $*$ respectively. Then a morphism of operads $F : X \to Y$ is a morphism of nonsymmetric operads that additionally satisfies :
(iii) For any $n \in \mathbb{N}$, $f \in X_n$, and $\tau \in S_n$, we have

$$F_n(f.\tau) = F_n(f) * \tau.$$

Algebra over an operad

Definition (Algebra over an Operad)

Let X be an operad. An algebra over X, also called X-algebra, is a morphism of operads $F : X \to \mathrm{End}_O$ for some set O. This may either be a morphism of nonsymmetric or symmetric operads, depending on the type of X.

Definition (Morphism of Algebras over an Operad)

Let X be an operad, and let $F : X \to \mathrm{End}_O$ and $G : Y \to \mathrm{End}_P$ be two X-algebras. A morphism of algebras $M : F \to G$ is a function $M : O \to P$ such that for all $f \in X_n$ and $\omega_1, ..., \omega_n \in O$, the following equivariance property holds :

$$M([F(f)](\omega_1, ..., \omega_n)) = [G(f)](M(\omega_1), M(\omega_2), ..., M(\omega_n)).$$

Quotient Operads

Let R be a unital commutative ring and we denote $\mathcal{M} := \mathrm{Mod}_R$

Definition (Operadic Ideals)

Let X be a symmetric operad over $\mathcal{M}$. Let $\{Y_n\}_{n\in\mathbb{N}}$ be a family of object in $\mathcal{M}$ such that Y_n is a graded submodule of X_n for all $n \in \mathbb{N}$, and each Y_n is S_n-invariant with respect to the symmetric action from X. If for any $f, g_1, ..., g_n \in X$, where at least one of those elements in some Y, we have $f \circ_{a_1,...,a_n} (g_1, ..., g_n) \in Y$. Then Y is called an operadic ideal of X.

Quotient Operads

Definition (Quotient Operad)

Let Y be an operadic ideal of X. Define the quotient operad X/Y as follows : set $(X/Y)_n = X_n/Y_n$ for each $n \in \mathbb{N}$, where this represents a quotient of graded modules. We designate $I + Y_1$ as the identity, where I is the identity of X, and define the composition map by :
$(f + Y_n) \circ_{a_1,\ldots,a_n} (g_1 + Y_{a_1}, \ldots, g_n + Y_{a_n}) =$
$(f \circ_{a_1,\ldots,a_n} (g_1, \ldots, g_n)) + Y_{a_1+\ldots+a_n}$.
We define a symmetric action on X/Y by :
$(f + Y_n).\sigma = (f.\sigma) + Y_n$ for all $\sigma \in S_n$.

Quotient Operads

Theorem (Quotient Operads are Operads)

The quotient operad X/Y defined above is an operad

Theorem

Let X and Y be $\mathcal{M}$-operads, and $F : X \to Y$ a morphism of operads. Then the kernel of F, defined as $\ker_n(F) := \{f \in X_n | F(f) = 0 \in Y_n\}$, is an operadic ideal.

Differential graded modules

In the category dg-Mod$_R$, we consider a sequence $\{M_n, d_n\}_{n\geq 1}$ of dg-modules such that each M_n is equipped with an action of S_n, we rewrite the previous definition of an operad in the case of dg-Mod, as follows :
$(f, g_1, ..., g_n) \in M_n \otimes M_{i_1} \otimes ... \otimes M_{i_n} \mapsto f(g_1, ..., g_n) \in M_{i_1+...+i_n}$.
We have a map of dg-modules. Therefore, it must also satisfy the derivation relation

$$d(f(g_1, ..., g_n)) = d(f)(g_1, ..., g_n) + \sum_{k=1}^{n} \varepsilon(g_1, ..., dg_k, ..., g_n)$$

where $\varepsilon = (-1)^{|f|+|g_1|+...+|g_{k-1}|}$ for each k.

Differential graded modules

If $A \in$ dg-Mod$_R$, an algebra over the operad $\mathcal{O}$ is an object wich has the operations described by $\mathcal{O}$. Consider the map $\mu : \mathcal{O}(n) \otimes A^{\otimes n} \to A$ satisfying the usual associativity and equivariance axioms, we denote the image of $(f, x_1, ..., x_n)$ under the map by $f(x_1, ..., x_n)$ and $|f(x_1, ..., x_n)| = |x_1| + ... + |x_n|$. The structure maps are maps of dg modules and satisfy a derivation relation :

$$\delta(f(x_1, ..., x_n)) = d(f)(x_1, ..., x_n) + \sum_{k=1}^{n} \varepsilon(x_1, ..., \delta x_k, ..., x_n)$$

Theorem

Let (A, δ) be a dga over the operad $(\mathcal{O}, d)$. Then the homology $H(A, \delta)$ is an algebra over the homology operad $H(\mathcal{O}, d)$.

Homotopy Theories of Algebras over Operads

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