An introduction to finite type invariants of knots and 3-manifolds

ICPAM-ICTP Research School, Symplectic Geometry and Geometric Topology

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Abstract

The finite type invariant concept for knots was introduced in the 90's in order to classify knot invariants, with the work of Vassiliev, Goussarov and Bar-Natan, shortly after the birth of numerous quantum knot invariants. This very useful concept was extended to 3–manifold invariants by Ohtsuki.

These lectures are an introduction to finite type invariants of links and 3-manifolds. The linking number is the simplest finite type invariant for 2–component links. It is defined in many equivalent ways in the first section. For an important example, we present it as the algebraic intersection of a torus and a 4-chain called a *propagator* in a configuration space.

In the second section, we introduce the simplest finite type 3–manifold invariant that is the Casson invariant of integral homology spheres. It is defined as the algebraic intersection of three propagators in a two-point configuration space.

In the third section, we explain the general notion of finite type invariants and introduce relevant spaces of Feynman Jacobi diagrams.

In Sections 4 and 5, we sketch a construction based on configuration space integrals of universal finite type invariants for links in rational homology spheres and we state open problems.

In Section 6, we present the needed properties of parallelizations of 3–manifolds and associated Pontrjagin classes, in details.

Keywords: Knots, 3-manifolds, finite type invariants, homology 3–spheres, linking number, Theta invariant, Casson-Walker invariant, Feynman Jacobi diagrams, perturbative expansion of Chern-Simons theory, configuration space integrals, parallelizations of 3–manifolds, first Pontrjagin class

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Foreword

These notes are the notes of five lectures presented in the ICPAM-ICTP research school of Meknès in May 2012. They contain some technical proofs that were not presented in the oral lectures. They also contain background material that was only quickly and informally recalled in the lectures. I thank the organizers of this great research school. I also thank Catherine Gille and Kévin Corbineau for useful comments on these notes.

1 Various aspects of the linking number

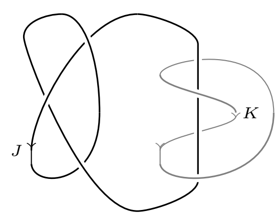
1.1 The Gauss linking number of two disjoint knots in $\mathbb{R}^3$

Let S^1 denote the unit circle of $\mathbb{C}$.

$$S^1 = \{z; z \in \mathbb{C}, |z| = 1\}.$$

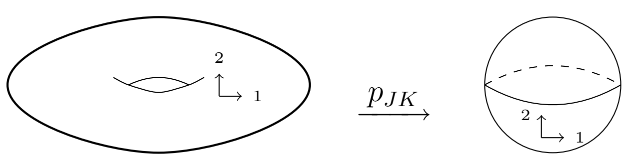
Consider two C^∞ embeddings

$$J\colon S^1 \hookrightarrow \mathbb{R}^3 \quad \text{and} \quad K\colon S^1 \hookrightarrow \mathbb{R}^3 \setminus J(S^1)$$



and the associated *Gauss map*

$$\begin{array}{llll} p_{JK}\colon & S^1 \times S^1 & \hookrightarrow & S^2 \\ & (w,z) & \mapsto & \frac{1}{\| K(z)-J(w) \|}(K(z)-J(w)) \end{array}$$



Denote the standard area form of S^2 by $4\pi\omega_{S^2}$ so that ω_{S^2} is the homogeneous volume form of S^2 such that $\int_{S^2} \omega_{S^2} = 1$. In 1833, Gauss defined the *linking number* of the disjoint *knots* $J(S^1)$ and $K(S^1)$, simply denoted by J and K, as an integral [Gau77]. With modern notation, his definition reads

$$lk_G(J,K) = \int_{S^1 \times S^1} p^*_{JK}(\omega_{S^2}).$$

It can be rephrased as $lk_G(J,K)$ *is the degree of the Gauss map* p_{JK}.

1.2 Some background material on manifolds without boundary, orientations, and degree

A *topological n–dimensional manifold M without boundary* is a Hausdorff topological space that is a union of open subsets U_i indexed in a set I $(i \in I)$, where every U_i is identified with an open subset V_i of $\mathbb{R}^n$ by a homeomorphism $\phi_i : U_i \to V_i$, called a *chart.* Manifolds are considered up to homeomorphism so that homeomorphic manifolds are considered identical.

For $r = 0, \ldots, \infty$, the topological manifold M *has a C^r–structure* or *is a C^r–manifold*, if, for each pair $\{i, j\} \subset I$, the map $\phi_j \circ \phi_i^{-1}$ defined on $\phi_i(U_i \cap U_j)$ is a C^r–diffeomorphism to its image. The notion of C^s–maps, $s \leq r$, from such a manifold to another one can be naturally deduced from the known case where the manifolds are open subsets of some $\mathbb{R}^n$, thanks to the local identifications provided by the charts. C^r–manifolds are considered up to C^r–diffeomorphisms.

An *orientation* of a real vector space V of positive dimension is a basis of V up to a change of basis with positive determinant. When $V = \{0\}$, an orientation of V is an element of $\{-1, 1\}$. For $n > 0$, an orientation of $\mathbb{R}^n$ identifies $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}; \mathbb{R})$ with $\mathbb{R}$. (In these notes, we freely use basic algebraic topology, see [Gre67] for example.) A homeomorphism h from an open subset U of $\mathbb{R}^n$ to another such V is *orientation-preserving* at a point x, if $h_*\colon H_n(U, U \setminus \{x\}) \to H_n(V, V \setminus \{h(x)\})$ is orientation-preserving. If h is a diffeomorphism, h is orientation-preserving at x if and only if the determinant of the Jacobian $D_x h$ is positive. If $\mathbb{R}^n$ is oriented and if the transition maps $\phi_j \circ \phi_i^{-1}$ are orientation-preserving (at every point) for $\{i, j\} \subset I$, the manifold M is *oriented.*

According to Theorem 7.1, for $n = 0$, 1, 2 or 3, any topological n-manifold may be equipped with a unique smooth structure (up to diffeomorphism). See Subsection 7.1. Unless otherwise mentioned, our manifolds are *smooth* (i. e. C^∞), oriented and compact, and considered up oriented diffeomorphisms. Products are oriented by the order of the factors. More generally, unless otherwise mentioned, the order of appearance of coordinates or parameters orients manifolds.

A point y is *a regular value* of a smooth map $p\colon M \to N$ between two smooth manifolds M and N, if for any $x \in p^{-1}(y)$ the tangent map $D_x p$ at x is surjective. According to the Morse-Sard theorem [Hir94, p. 69], the set of regular values of such a map is dense. If M is compact, it is furthermore open.

When M is oriented, compact and when the dimension of M coincides with the dimension of N, the *differential degree* of p at a regular value y of N is the (finite) sum running over the $x \in p^{-1}(y)$ of the signs of the determinants of $D_x p$. In our case where M has no boundary, this differential degree is locally constant on the set of regular values, and it is the *degree* of p, if N is connected. See [Mil97, Chapter 5].

Finally, recall a homological definition of the degree. Let $[M]$ denote the class of an oriented *closed* (i.e. compact, connected, without boundary) n–manifold in $H_n(M; \mathbb{Z})$. $H_n(M; \mathbb{Z}) = \mathbb{Z}[M]$. If M and N are two closed oriented n–manifolds and if $f\colon M \to N$ is a (continuous) map, then $H_n(f)([M]) = \deg(f)[N]$. In particular, for the Gauss map p_{JK} of Subsection 1.1,

$$H_2(p_{JK})([S^1 \times S^1]) = lk(J, K)[S^2].$$

1.3 The Gauss linking number as a degree

Since the differential degree of the Gauss map p_{JK} is locally constant, $lk_G(J,K) = \int_{S^1\times S^1} p^*_{JK}(\omega)$ for any 2-form ω on S^2 such that $\int_{S^2}\omega = 1$.

Let us compute $lk_G(J,K)$ as the differential degree of p_{JK} at the vector Y that points towards us. The set $p^{-1}_{JK}(Y)$ is made of the pairs of points (w,z) where the projections of $J(w)$ and $K(z)$ coincide, and $J(w)$ is under $K(z)$. They correspond to the *crossings* $^{J}\times^{K}$ and $^{K}\times^{J}$ of the diagram.

In a diagram, a crossing is *positive* if we turn counterclockwise from the arrow at the end of the upper strand to the arrow of the end of the lower strand like $\times$. Otherwise, it is *negative* like $\times$.

For the positive crossing $^{J}\times^{K}$, moving $J(w)$ along J following the orientation of J, moves $p_{JK}(w,z)$ towards the South-East direction, while moving $K(z)$ along K following the orientation of K, moves $p_{JK}(w,z)$ towards the North-East direction, so that the local orientation induced by the image of p_{JK} around $Y\in S^2$ is $\circlearrowleft^{Dpdz}_{Dpdw}$ that is $\circlearrowleft^{2}_{1}$. Therefore, the contribution of a positive crossing to the degree is 1. It is easy to see that the contribution of a negative crossing is (-1).

We have proved the following formula

$$\deg_Y(p_{JK}) = \sharp\, ^{J}\times^{K} - \sharp\, ^{K}\times^{J}$$

where $\sharp$ stands for the cardinality –here $\sharp\, ^{J}\times^{K}$ is the number of occurences of $^{J}\times^{K}$ in the diagram– so that

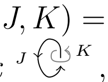
$$lk_G(J,K) = \sharp\, ^{J}\times^{K} - \sharp\, ^{K}\times^{J}.$$

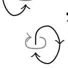
Similarly, $\deg_{-Y}(p_{JK}) = \sharp\, ^{K}\times^{J} - \sharp\, ^{J}\times^{K}$ so that

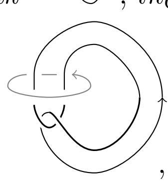
$$lk_G(J,K) = \sharp\, ^{K}\times^{J} - \sharp\, ^{J}\times^{K} = \frac{1}{2}\left(\sharp\, ^{J}\times^{K} + \sharp\, ^{K}\times^{J} - \sharp\, ^{K}\times^{J} - \sharp\, ^{J}\times^{K}\right)$$

and $lk_G(J,K) = lk_G(K,J)$.

In our first example, $lk_G(J,K) = 2$. Let us draw some further examples.

For the *positive Hopf link* , $lk_G(J,K) = 1$.

For the *negative Hopf link* , $lk_G(J,K) = -1$.

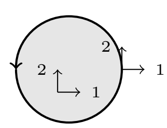
For the *Whitehead link* , $lk_G(J,K) = 0$.

1.4 Some background material on manifolds with boundary and algebraic intersections

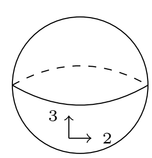
A *topological n–dimensional manifold M with possible boundary* is a Hausdorff topological space that is a union of open subsets U_i indexed in a set I, $(i \in I)$, where every U_i is identified with an open subset V_i of $]-\infty,0]^k \times \mathbb{R}^{n-k}$ by a chart $\phi_i : U_i \to V_i$. The *boundary* of $]-\infty,0]^k \times \mathbb{R}^{n-k}$ is made of the points $(x_1,\dots,x_n)$ of $]-\infty,0]^k \times \mathbb{R}^{n-k}$ such that there exists $i \leq k$ such that $x_i = 0$. The *boundary* of M is made of the points that are mapped to the boundary of $]-\infty,0]^k \times \mathbb{R}^{n-k}$.

For $r = 1,\dots,\infty$, the topological manifold M *is a C^r–manifold with ridges (or with corners) (resp. with boundary)*, if, for each pair $\{i,j\} \subset I$, the map $\phi_j \circ \phi_i^{-1}$ defined on $\phi_i(U_i \cap U_j)$ is a C^r–diffeomorphism to its image (resp. and if furthermore $k \leq 1$, for any i). Then the *ridges* of M are made of the points that are mapped to points $(x_1,\dots,x_n)$ of $]-\infty,0]^k \times \mathbb{R}^{n-k}$ so that there are at least two $i \leq k$ such that $x_i = 0$.

The tangent bundle of an oriented submanifold A in a manifold M at a point x is denoted by T_xA. The *normal bundle* T_xM/T_xA of A in M at x is denoted by N_xA. It is oriented so that (a lift of an oriented basis of) N_xA followed by (an oriented basis of) T_xA induce the orientation of T_xM. The boundary ∂M of an oriented manifold M is oriented by the *outward normal first* convention. If $x \in \partial M$ is not in a ridge, the outward normal of M at x followed by an oriented basis of $T_x\partial M$ induce the orientation of M. For example, the standard orientation of the disk in the plane induces the standard orientation of the circle, counterclockwise, as the following picture shows.



As another example the sphere S^2 is oriented as the boundary of the ball B^3 that has the standard orientation induced by (Thumb, index finger (2), middle finger (3)) of the right hand.



Two submanifolds A and B in a manifold M are transverse if at each intersection point x, $T_xM = T_xA + T_xB$. The transverse intersection of two submanifolds A and B in a manifold M is oriented so that the normal bundle of $A \cap B$ is $(N(A) \oplus N(B))$, fiberwise. If the two manifolds are of complementary dimensions, then the sign of an intersection point is $+1$ if the orientation of its normal bundle coincides with the orientation of the ambient space, that is if $T_xM = N_xA \oplus N_xB$ (as oriented vector spaces), this is equivalent to $T_xM = T_xA \oplus T_xB$ (as oriented vector spaces again, exercise). Otherwise, the sign is -1. If A and B are compact and if A and B are of complementary dimensions in M, their *algebraic intersection* is the sum of the signs of the intersection points, it is denoted by $\langle A,B\rangle_M$.

When M is an oriented manifold, $(-M)$ denotes the same manifold, equipped with the opposite orientation. In a manifold M, a k-dimensional *chain (resp. rational chain)* is a finite combination with coefficients in $\mathbb{Z}$ (resp. in $\mathbb{Q}$) of smooth k-dimensional oriented submanifolds C of M with boundary and ridges, up to the identification of $(-1)C$ with $(-C)$.

Again, unless otherwise mentioned, manifold are oriented. The boundary ∂ of chains is a linear map that maps a smooth submanifold to its oriented boundary. The canonical orientation of a point is the sign $+1$ so that $\partial[0,1] = \{1\} - \{0\}$.

Lemma 1.1 *Let A and B be two transverse submanifolds of a d–dimensional manifold M, of respective dimensions α and β, with disjoint boundaries. Then*

$$\partial(A \cap B) = (-1)^{d-\beta} \partial A \cap B + A \cap \partial B.$$

Proof: Note that $\partial(A \cap B) \subset \partial A \cup \partial B$. At a point $a \in \partial A$, $T_a M$ is oriented by $(N_a A, o, T_a \partial A)$, where o is the outward normal of A. If $a \in \partial A \cap B$, then o is also an outward normal for $A \cap B$, and $\partial(A \cap B)$ is cooriented by $(N_a A, N_a B, o)$ while $\partial A \cap B$ is cooriented by $(N_a A, o, N_a B)$. At a point $b \in A \cap \partial B$, $\partial(A \cap B)$ is cooriented by $(N_a A, N_a B, o)$ like $A \cap \partial B$. $\diamond$

1.5 A general definition of the linking number

Lemma 1.2 *Let J and K be two rationally null-homologous disjoint cycles of respective dimensions j and k in a d-manifold M, where $d = j + k + 1$. There exists a rational $(j+1)$–chain Σ_J bounded by J transverse to K, and a rational $(k+1)$–chain Σ_K bounded by K transverse to J and for any two such rational chains Σ_J and Σ_K, $\langle J, \Sigma_K \rangle_M = (-1)^{j+1} \langle \Sigma_J, K \rangle_M$. In particular, $\langle J, \Sigma_K \rangle_M$ is a topological invariant of (J, K) that is denoted by $lk(J, K)$ and called the* linking number *of J and K.*

$$lk(J, K) = (-1)^{(j+1)(k+1)} lk(K, J).$$

Proof: Since K is rationally null-homologous, K bounds a rational $(k+1)$–chain Σ_K. Without loss, Σ_K is assumed to be transverse to Σ_J so that $\Sigma_J \cap \Sigma_K$ is a rational 1–chain (that is a rational combination of intervals). According to Lemma 1.1,

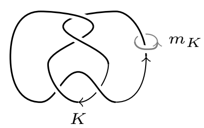
$$\partial(\Sigma_J \cap \Sigma_K) = (-1)^{d+k+1} J \cap \Sigma_K + \Sigma_J \cap K.$$

Furthermore, the sum of the coefficients of the points in the left-hand side must be zero, since this sum vanishes for the boundary of an interval. This shows that $\langle J, \Sigma_K \rangle_M = (-1)^{d+k} \langle \Sigma_J, K \rangle_M$, and therefore that this rational number is independent of the chosen Σ_J and Σ_K. Since $(-1)^{d+k} \langle \Sigma_J, K \rangle_M = (-1)^{j+1} (-1)^{k(j+1)} \langle K, \Sigma_J \rangle_M$, $lk(J, K) = (-1)^{(j+1)(k+1)} lk(K, J)$. $\diamond$

In particular, the *linking number* of two rationally null-homologous disjoint links J and K in a 3-manifold M is the algebraic intersection of a rational chain bounded by one of the knots and the other one.

For $\mathbb{K} = \mathbb{Z}$ or $\mathbb{Q}$, a $\mathbb{K}$-*sphere or (integral or rational) homology sphere* (resp. a $\mathbb{K}$-*ball*) is a smooth, compact, oriented 3-manifold, without ridges, with the same $\mathbb{K}$-homology as the sphere S^3 (resp. as a point). In such a manifold, any knot is rationally null-homologous so that the linking number of two disjoint knots always makes sense.

A *meridian* of a knot K is the (oriented) boundary of a disk that intersects K once with a positive sign. Since a chain Σ_J bounded by a knot J disjoint from K provides a rational cobordism between J and a combination of meridians of K, $[J] = lk(J,K)[m_K]$ in $H_1(M \setminus K; \mathbb{Q})$ where m_K is a meridian of K.



Lemma 1.3 *When K is a knot in a $\mathbb{Q}$-sphere or a $\mathbb{Q}$-ball M, $H_1(M \setminus K; \mathbb{Q}) = \mathbb{Q}[m_K]$, so that the equation $[J] = lk(J,K)[m_K]$ in $H_1(M \setminus K; \mathbb{Q})$ provides an alternative definition for the linking number.*

PROOF: Exercise. $\diamond$

The reader is also invited to check that $lk_G = lk$ as an exercise though it will be proved in the next subsection, see Proposition 1.6.

1.6 Generalizing the Gauss definition of the linking number and identifying the definitions

Lemma 1.4 *The map*

$$\begin{array}{llll} p_{S^2}\colon & ((\mathbb{R}^3)^2 \setminus diag) & \rightarrow & S^2 \\ & (x,y) & \mapsto & \frac{1}{\|y-x\|}(y-x) \end{array}$$

is a homotopy equivalence. In particular

$$H_i(p_{S^2})\colon H_i((\mathbb{R}^3)^2 \setminus diag; \mathbb{Z}) \rightarrow H_i(S^2; \mathbb{Z})$$

is an isomorphism for all i, $((\mathbb{R}^3)^2 \setminus diag)$ is a homology S^2, and $[S] = H_2^{-1}(p_{S^2})[S^2]$ is a canonical generator of

$$H_2((\mathbb{R}^3)^2 \setminus diag; \mathbb{Z}) = \mathbb{Z}[S].$$

PROOF: $((\mathbb{R}^3)^2 \setminus \text{diag})$ is homeomorphic to $\mathbb{R}^3 \times]0,\infty[\times S^2$ via the map

$$(x,y) \mapsto (x, \parallel y-x \parallel, p_{S^2}(x,y)).$$

$\diamond$

Like in Subsection 1.1, consider a two-component link $J \sqcup K : S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$. This embedding induces an embedding

$$\begin{array}{rccl} J \times K\colon & S^1 \times S^1 & \hookrightarrow & ((\mathbb{R}^3)^2 \setminus \text{diag}) \\ & (z_1, z_2) & \mapsto & (J(z_1), K(z_2)) \end{array}$$

the map p_{JK} of Subsection 1.1 reads $p_{S^2} \circ (J \times K)$, and since $H_2(p_{JK})[S^1 \times S^1] = \deg(p_{JK})[S^2] = lk_G(J, K)[S^2]$ in $H_2(S^2; \mathbb{Z}) = \mathbb{Z}[S^2]$,

$$[J \times K] = H_2(J \times K)[S^1 \times S^1] = lk_G(J, K)[S]$$

in $H_2((\mathbb{R}^3)^2 \setminus \text{diag}; \mathbb{Z}) = \mathbb{Z}[S]$. We shall see that this definition of lk_G generalizes to links in rational homology spheres and then prove that our generalized definition coincides with the general definition of linking numbers in this case.

For a 3-manifold M, the normal bundle of the diagonal of M^2 in M^2 is identified with the tangent bundle of M, fiberwise, by the map

$$(u, v) \in \frac{(T_x M)^2}{\text{diag}((T_x M)^2)} \mapsto (v - u) \in T_x M.$$

A *parallelization* τ of an oriented 3-manifold M is a bundle isomorphism $\tau\colon M \times \mathbb{R}^3 \longrightarrow TM$ that restricts to $x \times \mathbb{R}^3$ as an orientation-preserving linear isomorphism from $x \times \mathbb{R}^3$ to $T_x M$, for any $x \in M$. It has long been known that any oriented 3-manifold is parallelizable (i.e. admits a parallelization, see Subsection 6.2). Therefore, a tubular neighborhood of the diagonal in M^2 is diffeomorphic to $M \times \mathbb{R}^3$.

Lemma 1.5 *Let M be a rational homology sphere, let ∞ be a point of M. Let $\check{M} = (M \setminus \{\infty\})$. Then $\check{M}^2 \setminus diag$ has the same rational homology as S^2. Let B be a ball in $\check{M}$ and let x be a point inside B, then the class $[S]$ of $x \times \partial B$ is a canonical generator of $H_2(\check{M}^2 \setminus diag; \mathbb{Q}) = \mathbb{Q}[S]$.*

PROOF: In this proof, the homology coefficients are in $\mathbb{Q}$. Since $\check{M}$ has the homology of a point, the Künneth Formula implies that $\check{M}^2$ has the homology of a point. Now, by excision,

$$H_*(\check{M}^2, \check{M}^2 \setminus \text{diag}) \cong H_*(\check{M} \times \mathbb{R}^3, \check{M} \times (\mathbb{R}^3 \setminus 0))$$

$$\cong H_*(\mathbb{R}^3, S^2) \cong \left\{ \begin{array}{ll} \mathbb{Q} & \text{if } * = 3, \\ 0 & \text{otherwise.} \end{array} \right.$$

Using the long exact sequence of the pair $(\check{M}^2, \check{M}^2 \setminus \text{diag})$, we get that $H_*(\check{M}^2 \setminus \text{diag}; \mathbb{Q}) = H_*(S^2)$. $\diamond$

Define the *Gauss linking number* of two disjoint links J and K in $\check{M}$ so that

$$[(J \times K)(S^1 \times S^1)] = lk_G(J, K)[S]$$

in $H_2(\check{M}^2 \setminus \text{diag}; \mathbb{Q})$. Note that the two definitions of lk_G coincide when $\check{M} = \mathbb{R}^3$.

Proposition 1.6

$$lk_G = lk$$

Proof: First note that both definitions make sense when J and K are disjoint links: $[J \times K] = lk_G(J,K)[S]$ and $lk(J,K)$ is the algebraic intersection of K and a rational chain Σ_J bounded by J.

If K is a knot, a chain Σ_J provides a rational cobordism C between J and a combination of meridians of K, and a rational cobordism $C \times K$ in $\check{M}^2 \setminus \text{diag}$ that allow us to see that $lk_G(.,K)$ and $lk(.,K)$ linearly depend on $[J] \in H_1(\check{M} \setminus K)$. Thus we are left with the proof that $lk_G(m_K,K) = lk(m_K,K) = 1$. Since $lk_G(m_K,.)$ linearly depends on $[K] \in H_1(\check{M} \setminus m_K)$, we are left with the proof $lk_G(m_K,K) = 1$ when K is a meridian of m_K. Now, there is no loss in assuming that our link is a Hopf link in $\mathbb{R}^3$ so that the equality follows from the equality for links in $\mathbb{R}^3$. $\diamond$

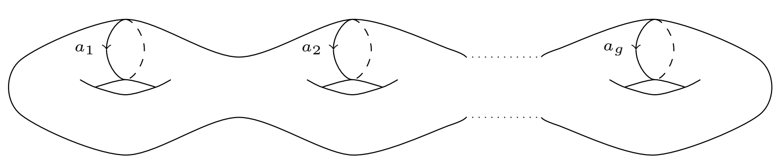
For a 2–component link (J,K) in $\mathbb{R}^3$, the definition of $lk(J,K)$ can be rewritten as

$$lk(J,K) = \int_{J \times K} p_{S^2}^*(\omega) = \langle J \times K, p_{S^2}^{-1}(Y) \rangle_{(\mathbb{R}^3)^2 \setminus \text{diag}}$$

for any regular value Y of p_{JK}, and for any 2-form ω of S^2 such that $\int_{S^2} \omega = 1$. Thus, $lk(J,K)$ is the evaluation of a 2-form $p_{S^2}^*(\omega)$ of $(\mathbb{R}^3)^2 \setminus \text{diag}$ at the 2-cycle $[J \times K]$ or it is the intersection of the 2-cycle $[J \times K]$ with a 4-manifold. We shall adapt these definitions to rational homology spheres.

1.7 Lagrangian-preserving surgeries

Definition 1.7 An *integral (resp. rational)* homology handlebody of genus g is a compact oriented 3-manifold A that has the same integral (resp. rational) homology as the usual solid handlebody H_g below.



Exercise 1.8 Show that if A is a rational homology handlebody of genus g, then ∂A is a genus g surface.

The *Lagrangian* $\mathcal{L}_A$ of a compact 3-manifold A is the kernel of the map induced by the inclusion from $H_1(\partial A;\mathbb{Q})$ to $H_1(\partial A;\mathbb{Q})$.

In the figure, the Lagrangian of H_g is freely generated by the classes of the curves a_i.

Definition 1.9 An *integral (resp. rational) Lagrangian-Preserving (or LP) surgery* (A'/A) is the replacement of an integral (resp. rational) homology handlebody A embedded in the interior of a 3-manifold M by another such A' whose boundary is identified with ∂A by an orientation-preserving diffeomorphism that sends $\mathcal{L}_A$ to $\mathcal{L}_{A'}$. The manifold $M(A'/A)$ obtained by such an LP-surgery reads

$$M(A'/A) = (M \setminus \mathrm{Int}(A)) \cup_{\partial A} A'.$$

(This only defines the topological structure of $M(A'/A)$, but we equip $M(A'/A)$ with its unique smooth structure.)

Lemma 1.10 *If (A'/A) is an integral (resp. rational) LP-surgery, then the homology of $M(A'/A)$ with $\mathbb{Z}$-coefficients (resp. with $\mathbb{Q}$-coefficients) is canonically isomorphic to $H_*(M;\mathbb{Z})$ (resp. to $H_*(M;\mathbb{Q})$). If M is a $\mathbb{Q}$-sphere, if (A'/A) is a rational LP-surgery, and if (J,K) is a two-component link of $M \setminus A$, then the linking number of J and K in M and the linking number of J and K in $M(A'/A)$ coincide.*

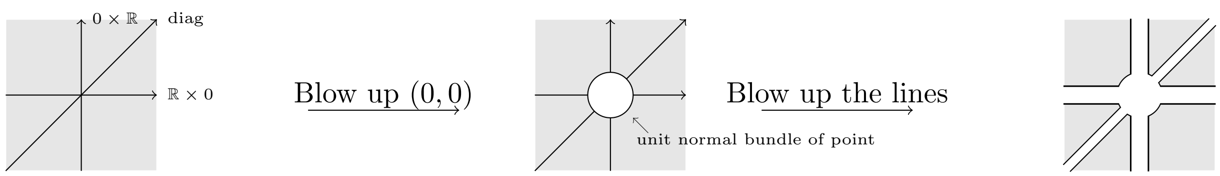
PROOF: Exercise. $\diamond$

2 Propagators and the Θ-invariant

2.1 Blowing up in real differential topology

Here, *blowing up* a submanifold A in a smooth manifold B means replacing it by its unit normal bundle $UN(A)$. The fiber $UN_a(A) = (N_a(A) \setminus \{0\})/\mathbb{R}^{+*}$ is oriented as the boundary of a unit ball of $N_a(A)$. Locally, $((\mathbb{R}^c = \{0\} \cup]0,\infty[\times S^{c-1}) \times A)$ is replaced by $([0,\infty[\times S^{c-1} \times A)$ so that the blown-up manifold $B(A)$ is homeomorphic to the complement in B of an open tubular neighborhood (thought of as infinitely small) of A. In particular, $B(A)$ is homotopy equivalent to $B \setminus A$. Furthermore, the blow up is canonical, so that the created boundary is the unit normal bundle $UN(A)$ and there is a canonical smooth projection from $B(A)$ to B such that the preimage of $a \in A$ is $UN_a(A)$. If A and B are compact, then $B(A)$ is compact, it is a smooth compactification of $B \setminus A$.

In the following figure, we see the result of blowing up $(0,0)$, the closures of $\{0\}\times\mathbb{R}$, $\mathbb{R}\times\{0\}$ and the diagonal, successively, in $\mathbb{R}^2$.



2.2 The configuration space $C_2(M)$

See S^3 as $\mathbb{R}^3 \cup \infty$ or as two copies of $\mathbb{R}^3$ identified along $\mathbb{R}^3 \setminus \{0\}$ by the (exceptionally orientation-reversing) diffeomorphism $x \mapsto x/ \parallel x \parallel^2$. Then $S^3(\infty) = \mathbb{R}^3 \cup S^2_\infty$ where the unit normal bundle S^2_∞ of ∞ in S^3 is canonically diffeomorphic to S^2 via $p_\infty \colon S^2_\infty \to S^2$, $x \in S^2_\infty$ is the limit of a sequence of points of $\mathbb{R}^3$ approaching ∞ along a line directed by $p_\infty(x) \in S^2$. Note that p_∞ reverses the orientation, too.

Fix a rational homology sphere M, a point ∞ of M, and $\check{M} = M \setminus \{\infty\}$. Identify a neighborhood of ∞ in M with the complement $\check{B}_{1,\infty}$ of the closed ball $B(1)$ of radius 1 in $\mathbb{R}^3$. Let $\check{B}_{2,\infty}$ be the complement of the closed ball $B(2)$ of radius 2 in $\mathbb{R}^3$ that is a smaller neighborhood of ∞ in M via the understood identification. Then $B_M = M \setminus \check{B}_{2,\infty}$ is a compact rational homology ball diffeomorphic to $M(\infty)$.

Define the *configuration space* $C_2(M)$ as the compact 6–manifold with boundary and ridges obtained from M^2 by blowing up (∞,∞), the closures of $\{\infty\}\times\check{M}$, $\check{M}\times\{\infty\}$ and the diagonal of $\check{M}^2$, successively. Then $\partial C_2(M)$ contains the unit normal bundle $(\frac{T\check{M}^2}{\mathrm{diag}} \setminus \{0\})/\mathbb{R}^{+*}$ of the diagonal of $\check{M}^2$. This bundle is canonically isomorphic to the unit tangent bundle $U\check{M}$ of $\check{M}$ (again via the map $([(x,y)] \mapsto [y-x])$).

Lemma 2.1 *Let $\check{C}_2(M) = \check{M}^2 \setminus \mathrm{diag}$. The open manifold $C_2(M) \setminus \partial C_2(M)$ is $\check{C}_2(M)$ and the inclusion $\check{C}_2(M) \hookrightarrow C_2(M)$ is a homotopy equivalence. In particular, $C_2(M)$ is a compactification of $\check{C}_2(M)$ homotopy equivalent to $\check{C}_2(M)$. The manifold $C_2(M)$ is a smooth compact*

6-dimensional manifold with boundary and ridges. There is a canonical smooth projection $p_{M^2}: C_2(M) \rightarrow M^2$.

$$\partial C_2(M) = p_{M^2}^{-1}(\infty,\infty) \cup (S^2_\infty \times \check{M}) \cup (-\check{M} \times S^2_\infty) \cup U\check{M}.$$

PROOF: Let $B_{1,\infty}$ be the complement of the open ball of radius one of $\mathbb{R}^3$ in S^3. Blowing up (∞,∞) in $B^2_{1,\infty}$ transforms a neighborhood of (∞,∞) into the product $[0,1[\times S^5$. Explicitly, there is a map

$$\begin{array}{llll} \psi: & [0,1[\times S^5 & \rightarrow & B^2_{1,\infty}(\infty,\infty) \subset M^2(\infty,\infty) \\ & (\lambda \in]0,1[, (x \neq 0, y \neq 0) \in S^5 \subset (\mathbb{R}^3)^2) & \mapsto & (\frac{1}{\lambda \|x\|^2}x, \frac{1}{\lambda \|y\|^2}y) \end{array}$$

that is a diffeomorphism onto its image, that is a neighborhood of the preimage of (∞,∞) under the blow up map $M^2(\infty,\infty) \stackrel{p_1}{\longrightarrow} M^2$. This neighborhood intersects $\infty \times \check{M}$, $\check{M} \times \infty$, and $\mathrm{diag}(\check{M}^2)$ as $\psi(]0,1[\times 0 \times S^2)$, $\psi(]0,1[\times S^2 \times 0)$ and $\psi(]0,1[\times(S^5 \cap \mathrm{diag}((\mathbb{R}^3)^2)))$, respectively. In particular, the closures of $\infty \times \check{M}$, $\check{M} \times \infty$, and $\mathrm{diag}(\check{M}^2)$ in $M^2(\infty,\infty)$ intersect the boundary $\psi(0 \times S^5)$ of $M^2(\infty,\infty)$ as three disjoint spheres in S^5, and they read $\infty \times M(\infty)$, $M(\infty) \times \infty$ and $\mathrm{diag}(M(\infty))$. Thus, the next steps will be three blow-ups along these three disjoint smooth manifolds.

These blow-ups will preserve the product structure $\psi([0,1[\times .)$. Therefore, $C_2(M)$ is a smooth compact 6-dimensional manifold with boundary, with three *ridges* $S^2 \times S^2$ in $p_{M^2}^{-1}(\infty,\infty)$. A neighborhood of these ridges in $C_2(M)$ is diffeomorphic to $[0,1[^2 \times S^2 \times S^2$. Recall Lemma 1.5 in order to conclude the proof as an exercise. ◇

Lemma 2.2 *The map* p_{S^2} *of Lemma 1.4 smoothly extends to* $C_2(S^3)$*, and its extension* p_{S^2} *satisfies:*

$$p_{S^2} = \begin{cases} -p_\infty \circ p_1 & \text{on } S^2_\infty \times \mathbb{R}^3 \\ p_\infty \circ p_2 & \text{on } \mathbb{R}^3 \times S^2_\infty \\ p_2 & \text{on } U\mathbb{R}^3 = \mathbb{R}^3 \times S^2 \end{cases}$$

where p_1 *and* p_2 *denote the projections on the first and second factor with respect to the above expressions.*

PROOF: Near the diagonal of $\mathbb{R}^3$, we have a chart of $C_2(S^3)$

$$\psi_d : \mathbb{R}^3 \times [0,\infty[\times S^2 \longrightarrow C_2(S^3)$$

that maps $(x \in \mathbb{R}^3, \lambda \in]0,\infty[, y \in S^2)$ to $(x, x+\lambda y) \in (\mathbb{R}^3)^2$. Here, p_{S^2} extends as the projection onto the S^2 factor.
Consider the orientation-reversing embedding ϕ_∞

$$\begin{array}{llll} \phi_\infty : & \mathbb{R}^3 & \longrightarrow & S^3 \\ & \mu(x \in S^2) & \mapsto & \begin{cases} \infty & \text{if } \mu = 0 \\ \frac{1}{\mu}x & \text{otherwise.} \end{cases} \end{array}$$

Note that this chart induces the already mentioned identification of the unit normal bundle S^2_∞ of $\{\infty\}$ in S^3 with S^2. When $\mu \neq 0$,

$$p_{S^2}(\phi_\infty(\mu x), y \in \mathbb{R}^3) = \frac{\mu y - x}{\parallel \mu y - x \parallel}.$$

Then p_{S^2} can be smoothly extended on $S^2_\infty \times \mathbb{R}^3$ (where $\mu = 0$) by

$$p_{S^2}(x \in S^2_\infty, y \in \mathbb{R}^3) = -x.$$

Similarly, set $p_{S^2}(x \in \mathbb{R}^3, y \in S^2_\infty) = y$. Now, with the map ψ of the proof of Lemma 2.1, when x and y are not equal to zero and when they are distinct,

$$p_{S^2} \circ \psi((\lambda, (x, y))) = \frac{\frac{y}{\|y\|^2} - \frac{x}{\|x\|^2}}{\parallel \frac{y}{\|y\|^2} - \frac{x}{\|x\|^2} \parallel} = \frac{\parallel x \parallel^2 y - \parallel y \parallel^2 x}{\parallel \parallel x \parallel^2 y - \parallel y \parallel^2 x \parallel}$$

when $\lambda \neq 0$. This map naturally extends to $M^2(\infty, \infty)$ outside the boundaries of $\infty \times M(\infty)$, $M(\infty) \times \infty$ and $\mathrm{diag}(M(\infty))$ by keeping the same formula when $\lambda = 0$.

Let us check that p_{S^2} smoothly extends over the boundary of the diagonal of $M(\infty)$. There is a chart of $C_2(M)$ near the preimage of this boundary in $C_2(M)$

$$\psi_2 : [0, \infty[\times [0, \infty[\times S^2 \times S^2 \longrightarrow C_2(S^3)$$

that maps $(\lambda \in]0, \infty[, \mu \in]0, \infty[, x \in S^2, y \in S^2)$ to $(\phi_\infty(\lambda x), \phi_\infty(\lambda(x + \mu y)))$ where p_{S^2} reads

$$(\lambda, \mu, x, y) \mapsto \frac{y - 2\langle x, y\rangle x - \mu x}{\parallel y - 2\langle x, y\rangle x - \mu x \parallel},$$

and therefore smoothly extends when $\mu = 0$. We similarly check that p_{S^2} smoothly extends over the boundaries of $(\infty \times M(\infty))$ and $(M(\infty) \times \infty)$. $\diamond$

Let τ_s denote the standard parallelization of $\mathbb{R}^3$. Say that a parallelization

$$\tau \colon \check{M} \times \mathbb{R}^3 \to T\check{M}$$

of $\check{M}$ that coincides with τ_s on $\check{B}_{1,\infty}$ is *asymptotically standard.* According to Subsection 6.2, such a parallelization exists. Such a parallelization identifies $U\check{M}$ with $\check{M} \times S^2$.

Proposition 2.3 *For any asymptotically standard parallelization τ of $\check{M}$, there exists a smooth map $p_\tau \colon \partial C_2(M) \to S^2$ such that*

$$p_\tau = \begin{cases} p_{S^2} & \text{on } p_{M^2}^{-1}(\infty, \infty) \\ -p_\infty \circ p_1 & \text{on } S^2_\infty \times \check{M} \\ p_\infty \circ p_2 & \text{on } \check{M} \times S^2_\infty \\ p_2 & \text{on } U\check{M} \stackrel{\tau}{=} \check{M} \times S^2 \end{cases}$$

where p_1 and p_2 denote the projections on the first and second factor with respect to the above expressions.

PROOF: This is a consequence of Lemma 2.2. ◇

Since $C_2(M)$ is homotopy equivalent to $(\check{M}^2\setminus\text{diag})$, according to Lemma 1.5, $H_2(C_2(M);\mathbb{Q}) = \mathbb{Q}[S]$ where the canonical generator $[S]$ is the homology class of a fiber of $U\check{M} \subset \partial C_2(M)$. For a 2-component link (J,K) of $\check{M}$, the homology class $[J \times K]$ of $J \times K$ in $H_2(C_2(M);\mathbb{Q})$ reads $lk(J,K)[S]$, according to Subsection 1.6 and to Proposition 1.6.

Define an *asymptotic rational homology* $\mathbb{R}^3$ as a pair $(\check{M},\tau)$ where $\check{M}$ is 3-manifold that reads as the union over $]1,2] \times S^2$ of a rational homology ball B_M and the complement $\check{B}_{1,\infty}$ of the unit ball of $\mathbb{R}^3$, and τ is an asymptotically standard parallelization of $\check{M}$. Since such a pair $(\check{M},\tau)$ canonically defines the rational homology sphere $M = \check{M} \cup \{\infty\}$, "Let $(\check{M},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$" is a shortcut for "Let M be a rational homology sphere equipped with an asymptotically standard parallelization τ of $\check{M}$".

2.3 On propagators

Definition 2.4 Let $(\check{M},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. A *propagating chain* of $(C_2(M),\tau)$ is a 4–chain F of $C_2(M)$ such that $\partial F = p_\tau^{-1}(a)$ for some $a \in S^2$. A *propagating form* of $(C_2(M),\tau)$ is a closed 2-form ω_p on $C_2(M)$ whose restriction to $\partial C_2(M)$ reads $p_\tau^*(\omega)$ for some 2-form ω of S^2 such that $\int_{S^2}\omega = 1$. Propagating chains and propagating forms will simply be called *propagators* where their nature is clear from the context.

Example 2.5 Recall the map $p_{S^2}\colon C_2(S^3) \to S^2$ of Lemma 2.2. For any $a \in S^2$ $p_{S^2}^{-1}(a)$ is a propagating chain of $(C_2(S^3),\tau_s)$, and for any 2-form ω of S^2 such that $\int_{S^2}\omega = 1$, $p_{S^2}^*(\omega)$ is a propagating form of $(C_2(S^3),\tau_s)$.

Propagating chains exist because the 3-cycle $p_\tau^{-1}(a)$ of $\partial C_2(M)$ bounds in $C_2(M)$ since $H_3(C_2(M);\mathbb{Q}) = 0$. Dually, propagating forms exist because the restriction induces a surjective map $H^2(C_2(M);\mathbb{R}) \to H^2(\partial C_2(M);\mathbb{R})$ since $H^3(C_2(M),\partial C_2(M);\mathbb{R}) = 0$.

Lemma 2.6 *Let $(\check{M},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let C be a two-cycle of $C_2(M)$. For any propagating chain F of $(C_2(M),\tau)$ transverse to C and for any propagating form ω_p of $(C_2(M),\tau)$, $[C] = \int_C \omega_p[S] = \langle C,F\rangle_{C_2(M)}[S]$ in $H_2(C_2(M);\mathbb{Q}) = \mathbb{Q}[S]$. In particular, for any two-component link (J,K) of $\check{M}$.*

$$lk(J,K) = \int_{J\times K}\omega_p = \langle J \times K, F\rangle_{C_2(M)}.$$

PROOF: Fix a propagating chain F, the algebraic intersection $\langle C,F\rangle_{C_2(M)}$ only depends on the homology class $[C]$ of C in $C_2(M)$. Similarly, since ω_p is closed, $\int_C \omega_p$ only depends on $[C]$. (Indeed, if C and C' cobound a chain D, $C \cap F$ and $C' \cap F$ cobound $\pm(D \cap F)$, and $\int_{\partial D = C' - C}\omega_p = \int_D d\omega_p$ according to the Stokes theorem.) Furthermore, the dependance on $[C]$ is linear. Therefore it suffices to check the lemma for a chain that represents the canonical generator $[S]$ of $H_2(C_2(M);\mathbb{Q})$. Any fiber of $U\check{M}$ is such a chain. ◇

2.4 The Θ-invariant of (M, τ)

Note that the intersection of transverse (oriented) submanifolds is an associative operation, so that $A \cap B \cap C$ is well defined. Furthermore, for a connected manifold N, the class of a 0-cycle in $H_0(M; \mathbb{Q}) = \mathbb{Q}[m] = \mathbb{Q}$ is a well-defined number, so that the *algebraic intersection* of several transverse submanifolds whose codimension sum is the dimension of the ambient manifold is well defined as the homology class of their (oriented) intersection. This extends to rational chains, multilinearly. Thus, for three such transverse submanifolds A, B, C in a manifold D, their algebraic intersection $\langle A, B, C\rangle_D$ is the sum over the intersection points a of the associated signs, where the sign of a is positive if and only if the orientation of D is induced by the orientation of $N_aA \oplus N_aB \oplus N_aC$.

Theorem 2.7 *Let $(\check{M}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let F_a, F_b and F_c be three pairwise transverse propagators of $(C_2(M), \tau)$ with respective boundaries $p_\tau^{-1}(a)$, $p_\tau^{-1}(b)$ and $p_\tau^{-1}(c)$ for three distinct points a, b and c of S^2, then*

$$\Theta(M, \tau) = \langle F_a, F_b, F_c\rangle_{C_2(M)}$$

does not depend on the chosen propagators F_a, F_b and F_c. It is a topological invariant of (M, τ). For any three propagating chains ω_a, ω_b and ω_c of $(C_2(M), \tau)$,

$$\Theta(M, \tau) = \int_{C_2(M)} \omega_a \wedge \omega_b \wedge \omega_c.$$

PROOF: Since $H_4(C_2(M)) = 0$, if the propagator F_a is changed to a propagator F'_a with the same boundary, $(F'_a - F_a)$ bounds a 5-dimensional chain W transverse to $F_b \cap F_c$. The 1-dimensional chain $W \cap F_b \cap F_c$ does not meet $\partial C_2(M)$ since $F_b \cap F_c$ does not meet $\partial C_2(M)$. Therefore, up to a well-determined sign, the boundary of $W \cap F_b \cap F_c$ is $F'_a \cap F_b \cap F_c - F_a \cap F_b \cap F_c$. This shows that $\langle F_a, F_b, F_c\rangle_{C_2(M)}$ is independent of F_a when a is fixed. Similarly, it is independent of F_b and F_c when b and c are fixed. Thus, $\langle F_a, F_b, F_c\rangle_{C_2(M)}$ is a rational function on the connected set of triples (a, b, c) of distinct point of S^2. It is easy to see that this function is continuous. Thus, it is constant.

Let us similarly prove that $\int_{C_2(M)} \omega_a \wedge \omega_b \wedge \omega_c$ is independent of the propagating forms ω_a, ω_b and ω_c. Assume that the form ω_a that restricts to $\partial C_2(M)$ as $p_\tau^*(\omega_A)$ is changed to ω'_a that restricts to $\partial C_2(M)$ as $p_\tau^*(\omega'_A)$.

Lemma 2.8 *There exists a one-form η_A on S^2 such that $\omega'_A = \omega_A + d\eta_A$. For any such η_A, there exists a one-form η on $C_2(M)$ such that $\omega'_a - \omega_a = d\eta$, and the restriction of η to $\partial C_2(M)$ is $p_\tau^*(\eta_A)$.*

PROOF OF THE LEMMA: Since ω_a and ω'_a are cohomologous, there exists a one-form η on $C_2(M)$ such that $\omega'_a = \omega_a + d\eta$. Similarly, since $\int_{S^2} \omega'_A = \int_{S^2} \omega_A$, there exists a one-form η_A on S^2 such that $\omega'_A = \omega_A + d\eta_A$. On $\partial C_2(M)$, $d(\eta - p_\tau^*(\eta_A)) = 0$. Thanks to the exact sequence

$$0 = H^1(C_2(M)) \longrightarrow H^1(\partial C_2(M)) \longrightarrow H^2(C_2(M), \partial C_2(M)) \cong H_4(C_2(M)) = 0,$$

$H^1(\partial C_2(M))=0$. Therefore, there exists a function f from $\partial C_2(M)$ to $\mathbb{R}$ such that

$$df=\eta-p(\tau)^*(\eta_A)$$

on $\partial C_2(M)$. Extend f to a C^∞ map on $C_2(M)$ and change η into $(\eta-df)$. $\diamond$

Then

$$\begin{aligned}\int_{C_2(M)}\omega'_a\wedge\omega_b\wedge\omega_c-\int_{C_2(M)}\omega_a\wedge\omega_b\wedge\omega_c &= \int_{C_2(M)}d(\eta\wedge\omega_b\wedge\omega_c)=\int_{\partial C_2(M)}\eta\wedge\omega_b\wedge\omega_c\\ &= \int_{\partial C_2(M)}p(\tau)^*(\eta_A\wedge\omega_B\wedge\omega_C)=0\end{aligned}$$

since any 5-form on S^2 vanishes. Thus, $\int_{C_2(M)}\omega_a\wedge\omega_b\wedge\omega_c$ is independent of the propagating forms ω_a, ω_b and ω_c. Now, we can choose the propagating forms ω_a, ω_b and ω_c supported in very small neighborhoods of F_a, F_b and F_c, respectively, so that the intersection of the three supports is a very small neighborhood of $F_a\cap F_b\cap F_c$, where it can easily be seen that $\int_{C_2(M)}\omega_a\wedge\omega_b\wedge\omega_c=\langle F_a,F_b,F_c\rangle_{C_2(M)}$. $\diamond$

In particular, $\Theta(M,\tau)$ reads $\int_{C_2(M)}\omega^3$ for any propagating chain ω of $(C_2(M),\tau)$. Since such a propagating chain represents the linking number, $\Theta(M,\tau)$ can be thought of as the *cube of the linking number with respect to* τ.

When τ varies continuously, $\Theta(M,\tau)$ varies continuously in $\mathbb{Q}$ so that $\Theta(M,\tau)$ is an invariant of the homotopy class of τ.

2.5 Parallelisations of 3-manifolds and Pontrjagin classes

In this subsection, M denotes a smooth, compact oriented 3-manifold with possible boundary ∂M. It has long been known that such a 3-manifold is parallelizable. It is proved in Subsection 6.2.

Let $GL^+(\mathbb{R}^3)$ denote the group of orientation-preserving linear isomorphisms of $\mathbb{R}^3$. Let $[(M,\partial M),(GL^+(\mathbb{R}^3),1)]_m$ denote the set of maps

$$g:(M,\partial M)\longrightarrow(GL^+(\mathbb{R}^3),1)$$

from M to $GL^+(\mathbb{R}^3)$ that send ∂M to the unit 1 of $GL^+(\mathbb{R}^3)$. Let $[(M,\partial M),(GL^+(\mathbb{R}^3),1)]$ denote the group of homotopy classes of such maps, with the group structure induced by the multiplication of maps, using the multiplication in $GL^+(\mathbb{R}^3)$. For a map g in $[(M,\partial M),(GL^+(\mathbb{R}^3),1)]_m$, set

$$\begin{array}{rccc}\psi_{\mathbb{R}}(g): & M\times\mathbb{R}^3 & \longrightarrow & M\times\mathbb{R}^3\\ & (x,y) & \mapsto & (x,g(x)(y)).\end{array}$$

Let $\tau_M\colon M\times\mathbb{R}^3\to TM$ be a parallelization of M. Then any parallelization τ of M that coincides with τ_M on ∂M reads

$$\tau=\tau_M\circ\psi_{\mathbb{R}}(g)$$

for some $g\in[(M,\partial M),(GL^+(\mathbb{R}^3),1)]_m$.

Thus, fixing τ_M identifies the set of homotopy classes of parallelizations of M fixed on ∂M with the group $[(M, \partial M), (GL^+(\mathbb{R}^3), 1)]$. Since $GL^+(\mathbb{R}^3)$ deformation retracts onto the group $SO(3)$ of orientation-preserving linear isometries of $\mathbb{R}^3$, $[(M, \partial M), (GL^+(\mathbb{R}^3), 1)]$ is isomorphic to $[(M, \partial M), (SO(3), 1)]$.

See S^3 as $B^3/\partial B^3$ where B^3 is the standard ball of radius 2π of $\mathbb{R}^3$ seen as $([0, 2\pi] \times S^2)/(0 \sim \{0\} \times S^2)$. Let $\rho: B^3 \to SO(3)$ map $(\theta \in [0, 2\pi], v \in S^2)$ to the rotation $\rho(\theta, v)$ with axis directed by v and with angle θ. This map induces the double covering $\tilde{\rho}: S^3 \to SO(3)$, that orients $SO(3)$ and that allows one to deduce the first three homotopy groups of $SO(3)$ from the ones of S^3. They are $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$, $\pi_2(SO(3)) = 0$ and $\pi_3(SO(3)) = \mathbb{Z}[\tilde{\rho}]$. For $v \in S^2$, $\pi_1(SO(3))$ is generated by the class of the loop that maps $\exp(i\theta) \in S^1$ to the rotation $\rho(\theta, v)$.

Note that a map g from $(M, \partial M)$ to $(SO(3), 1)$ has a degree $\deg(g)$, that may be defined as the differential degree at a regular value (different from 1) of g. It can also be defined homologically, by $H_3(g)[M, \partial M] = \deg(g)[SO(3), 1]$.

The following theorem is proved in Section 6.

Theorem 2.9 *For any smooth compact connected oriented 3-manifold M, the group*

$$[(M, \partial M), (SO(3), 1)]$$

is abelian, and the degree

$$deg: [(M, \partial M), (SO(3), 1)] \longrightarrow \mathbb{Z}$$

is a group homomorphism, that induces an isomorphism

$$deg: [(M, \partial M), (SO(3), 1)] \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}.$$

When $\partial M = \emptyset$, (resp. when $\partial M = S^2$), there exists a canonical map p_1 from the set of homotopy classes of parallelizations of M (resp. that coincide with τ_s near S^2) such that for any map g in $[(M, \partial M), (SO(3), 1)]_m$, for any trivialisation τ of TM

$$p_1(\tau \circ \psi_{\mathbb{R}}(g)) - p_1(\tau) = 2deg(g).$$

The definition of the map p_1 is given in Subsection 6.5, it involves relative Pontrjagin classes. When $\partial M = \emptyset$, the map p_1 coincides with the map h that is studied by Kirby and Melvin in [KM99] under the name of *Hirzebruch defect.*

Since $[(M, \partial M), (SO(3), 1)]$ is abelian, the set of parallelizations of M that are fixed on ∂M is an affine space with translation group $[(M, \partial M), (SO(3), 1)]$.

Recall that $\rho: B^3 \to SO(3)$ maps $(\theta \in [0, 2\pi], v \in S^2)$ to the rotation with axis directed by v and with angle θ. Let M be an oriented connected 3-manifold with possible boundary. For a ball B^3 embedded in M, let $\rho_M(B^3) \in [(M, \partial M), (SO(3), 1)]_m$ be a (continuous) map that coincides with ρ on B^3 and that maps the complement of B^3 to the unit of $SO(3)$. The homotopy class of $\rho_M(B^3)$ is well-defined.

Lemma 2.10 $deg(\rho_M(B^3)) = 2$

PROOF: Exercise. $\diamond$

2.6 Defining a $\mathbb{Q}$-sphere invariant from Θ

Recall that an asymptotic rational homology $\mathbb{R}^3$ is a pair $(\check{M},\tau)$ where $\check{M}$ is 3-manifold that reads as the union over $]1,2]\times S^2$ of a rational homology ball B_M and the complement $\check{B}_{1,\infty}$ of the unit ball of $\mathbb{R}^3$, and that is equipped with an asymptotically standard parallelization τ.

In this subsection, we prove the following proposition.

Proposition 2.11 *Let $(\check{M},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. For any map g in $[(B_M, B_M\cap\check{B}_{1,\infty}),(SO(3),1)]_m$ trivially extended to $\check{M}$,*

$$\Theta(M,\tau\circ\psi_{\mathbb{R}}(g))-\Theta(M,\tau)=\frac{1}{2}deg(g).$$

Theorem 2.9 allows us to derive the following corollary from Proposition 2.11.

Corollary 2.12 $\Theta(M)=\Theta(M,\tau)-\frac{1}{4}p_1(\tau)$ *is an invariant of $\mathbb{Q}$-spheres.*

$\diamond$

Lemma 2.13 $\Theta(M,\tau\circ\psi_{\mathbb{R}}(g))-\Theta(M,\tau)$ *is independent of τ. Set $\Theta'(g)=\Theta(M,\tau\circ\psi_{\mathbb{R}}(g))-\Theta(M,\tau)$. Then Θ' is a homomorphism from $[(B_M, B_M\cap\check{B}_{1,\infty}),(SO(3),1)]$ to $\mathbb{Q}$.*

Proof: For $d=a$, b or c, the propagator F_a of $(C_2(M),\tau)$ can be assumed to be a product $[-1,0]\times p^{-1}_{\tau|UB_M}(d)$ on a collar $[-1,0]\times UB_M$ of UB_M in $C_2(M)$. Since $H_3([-1,0]\times UB_M;\mathbb{Q})=0$, $\left(\partial([-1,0]\times p^{-1}_{\tau|UB_M}(d))\setminus(0\times p^{-1}_{\tau|UB_M}(d))\right)\cup(0\times p^{-1}_{\tau\circ\psi_{\mathbb{R}}(g)|UB_M}(d))$ bounds a chain G_d.

The chains G_a, G_b and G_c can be assumed to be transverse. Construct the propagator $F_d(g)$ of $(C_2(M),\tau\circ\psi_{\mathbb{R}}(g))$ from F_d by replacing $[-1,0]\times p^{-1}_{\tau|UB_M}(d)$ by G_d on $[-1,0]\times UB_M$. Then

$$\Theta(M,\tau\circ\psi_{\mathbb{R}}(g))-\Theta(M,\tau)=\langle G_a,G_b,G_c\rangle_{[-1,0]\times UB_M}.$$

Using τ to identify UB_M with $B_M\times S^2$ allows us to see that $\Theta(M,\tau\circ\psi_{\mathbb{R}}(g))-\Theta(M,\tau)$ is independent of τ. Then it is easy to observe that Θ' is a homomorphism from $[(B_M,\partial B_M),(SO(3),1)]$ to $\mathbb{Q}$. $\diamond$

According to Theorem 2.9 and to Lemma 2.10, it suffices to prove that $\Theta'(\rho_M(B^3))=1$ in order to prove Proposition 2.11. It is easy to see that $\Theta'(\rho_M(B^3))=\Theta'(\rho)$. Thus, we are left with the proof of the following lemma.

Lemma 2.14 $\Theta'(\rho)=1$.

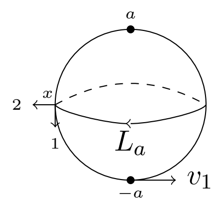
Again, see B^3 as $([0,2\pi]\times S^2)/(0\sim\{0\}\times S^2)$. We first prove the following lemma:

Lemma 2.15 *Let a be the North Pole. The point $(-a)$ is regular for the map*

$$\begin{array}{rccc}\rho_a\colon & B^3 & \rightarrow & S^2 \\ & m & \mapsto & \rho(m)(a)\end{array}$$

and its preimage (cooriented by S^2 via ρ_a) is the knot $L_a = \{\pi\} \times E$, where E is the equator that bounds the Southern Hemisphere.

PROOF: It is easy to see that $\rho_a^{-1}(-a) = \{\pi\} \times E$.



Let $x \in \{\pi\} \times E$. When m moves along the great circle that contains a and x from x towards $(-a)$ in $\{\pi\} \times S^2$, $\rho(m)(a)$ moves from $(-a)$ in the same direction, that will be the direction of the tangent vector v_1 of S^2 at $(-a)$, counterclockwise in our picture, where x is on the left. Then in our picture, S^2 is oriented at $(-a)$ by v_1 and by the tangent vector v_2 at $(-a)$ towards us. In order to move $\rho(\theta, v)(a)$ in the v_2 direction, one increases θ so that L_a is cooriented and oriented like in the figure. ◇

PROOF OF LEMMA 2.14: We use the notation of the proof of Lemma 2.13 and we construct an explicit G_a in $[-1, 0] \times UB^3 \overset{\tau_s}{=} [-1, 0] \times B^3 \times S^2$.

When $\rho(m)(a) \neq -a$, there is a unique geodesic arc $[a, \rho(m)(a)]$ with length ($\ell \in [0, \pi[$) from a to $\rho(m)(a) = \rho_a(m)$. For $t \in [0, 1]$, let $X_t(m) \in [a, \rho_a(m)]$ be such that the length of $[X_0(m) = a, X_t(m)]$ is $t\ell$. This defines X_t on $(M \setminus L_a)$, $X_1(m) = \rho_a(m)$. Let us show how the definition of X_t smoothly extends on the manifold $B^3(L_a)$ obtained from B^3 by blowing up L_a.

The map ρ_a maps the normal bundle of L_a to a disk of S^2 around $(-a)$, by an orientation-preserving diffeomorphism on every fiber (near the origin). In particular, ρ_a induces a map from the unit normal bundle of L_a to the unit normal bundle of $(-a)$ in S^2 that preserves the orientation of the fibers. Then for an element y of the unit normal bundle of L_a in M, define $X_t(y)$ as before on the half great circle $[a, -a]_{\rho_a(-y)}$ from a to $(-a)$ that is tangent to $\rho_a(-y)$ at $(-a)$ (so that $\rho_a(-y)$ is an outward normal of $[a, -a]_{\rho_a(-y)}$ at $(-a)$). This extends the definition of X_t , continuously.

The whole sphere is covered with degree (-1) by the image of $([0, 1] \times UN_x(L_a))$, where the fiber $UN_x(L_a)$ of the unit normal bundle of L_a is oriented as the boundary of a disk in the fiber of the normal bundle. Let $G_h(a)$ be the closure of $\left(\cup_{t \in [0,1], m \in (B^3 \setminus L_a)} (m, X_t(m))\right)$ in UB^3.

$$G_h(a) = \cup_{t \in [0,1], m \in B^3(L_a)} \left(p_{B^3}(m), X_t(m)\right).$$

Then

$$\partial G_h = -(B^3 \times a) + \cup_{m \in B^3}(m, \rho_a(m)) + \cup_{t \in [0,1]} X_t(-\partial S^3(L_a))$$

where $(-\partial S^3(L_a))$ is oriented like $\partial N(L_a)$ so that the last summand reads $(-L_a \times S^2)$ because the sphere is covered with degree (-1) by the image of $([0,1] \times UN_x(L_a))$.

Let D_a be a disk bounded by L_a in B^3. Set $G(a) = G_h(a) + D_a \times S^2$ so that $\partial G(a) = -(B^3 \times a) + \cup_{m \in B^3}(m, \rho_a(m))$. Now let ι be the endomorphism of UB^3 over B^3 that maps a unit vector to the opposite one. Set

$$\begin{array}{llll} & G_a & = [-1,-2/3] \times B^3 \times a & +\{-2/3\} \times G(a) & +[-2/3,0] \times \cup_{m \in B^3}(m, \rho_a(m)) \\ \text{and} & G_{-a} & = [-1,-1/3] \times B^3 \times (-a) & +\{-1/3\} \times \iota(G(a)) & +[-1/3,0] \times \cup_{m \in B^3}(m, \rho(m)(-a)). \end{array}$$

Then

$$G_a \cap G_{-a} = [-2/3,-1/3] \times L_a \times (-a) + \{-2/3\} \times D_a \times (-a) - \{-1/3\} \times \cup_{m \in D_a}(m, \rho_a(m)).$$

Finally, $\Theta'(\rho)$ is the algebraic intersection of $G_a \cap G_{-a}$ with $F_c(\rho)$ in $C_2(M)$. This intersection coincides with the algebraic intersection of $G_a \cap G_{-a}$ with any propagator of $(C_2(M), \tau)$ according to Lemma 2.6. Therefore

$$\Theta'(\rho) = \langle F_a, G_a \cap G_{-a} \rangle_{[-1,0] \times S^2 \times B^3} = -\deg_a(\rho_a \colon D_a \to S^2).$$

The orientation of L_a allows us to choose $(-D_a)$ as the Northern Hemisphere, the image of this hemisphere under ρ_a covers the sphere with degree 1 so that $\Theta'(\rho) = 1$. $\diamond$

3 Introduction to finite type invariants

3.1 Definition of finite type invariants

Let $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$.

A $\mathbb{K}$–valued *invariant* of oriented 3-manifolds is a function from the set of 3-manifolds, considered up to orientation-preserving diffeomorphisms to $\mathbb{K}$. Let $\coprod_{i=1}^n S_i^1$ denote a disjoint union of n circles, where each S_i^1 is a copy of S^1. Here, an *n–component link* in a 3-manifold M is an equivalence class of embeddings $L\colon \coprod_{i=1}^n S_i^1 \hookrightarrow M$ under the equivalence relation that identifies two embeddings L and L' if and only if there is an orientation-preserving diffeomorphism h of M such that $h(L) = L'$. A *knot* is a one-component link. A *link invariant* (resp. a *knot invariant*) is a function of links (resp. knots). For example, Θ is an invariant of $\mathbb{Q}$-spheres and the linking number is a rational invariant of two-component links in rational homology spheres

In order to study a function, it is common to study its derivative, and the derivatives of its derivative. The derivative of a function is defined from its variations. For a function f from $\mathbb{Z}^d = \oplus_{i=1}^d \mathbb{Z}e_i$ to $\mathbb{K}$, one can define its first order derivatives $\frac{\partial f}{\partial e_i}\colon \mathbb{Z}^d \to \mathbb{K}$ by

$$\frac{\partial f}{\partial e_i}(z) = f(z+e_i) - f(z)$$

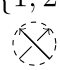
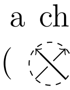
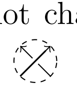
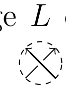
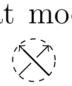
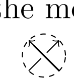
and check that all the first order derivatives of f vanish if and only if f is constant. Inductively define an n-order derivative as a first order derivative of an $(n-1)$-order derivative for a positive integer n. Then it can be checked that all the $(n+1)$-order derivatives of a function vanish if and only if f is a polynomial of degree not greater than n. In order to study topological invariants, we can similarly study their variations under *simple operations.*

Below, X denotes one of the following sets

- $\mathbb{Z}^d$,
- the set $\mathcal{K}$ of knots in $\mathbb{R}^3$, the set $\mathcal{K}_n$ of n-component links in $\mathbb{R}^3$,
- the set $\mathcal{M}$ of $\mathbb{Z}$-spheres, the set $\mathcal{M}_Q$ of $\mathbb{Q}$-spheres.

and $\mathcal{O}(X)$ denotes a set of *simple operations* acting on some elements of X.

For $X = \mathbb{Z}^d$, $\mathcal{O}(X)$ will be made of the operations $(z \to z \pm e_i)$

For knots or links in $\mathbb{R}^3$, the *simple operations* will be *crossing changes.* A *crossing change ball* of a link L is a ball B of the ambient space, where $L \cap B$ is a disjoint union of two arcs α_1 and α_2 properly embedded in B, and there exist two disjoint topological disks D_1 and D_2 embedded in B, such that, for $i \in \{1,2\}$, $\alpha_i \subset \partial D_i$ and $(\partial D_i \setminus \alpha_i) \subset \partial B$. After an isotopy, the projection of (B, α_1, α_2) reads  or  , a *crossing change* is a change that does not change L outside B and that modifies it inside B by a local move ( $\to$ ) or ( $\to$ ). For the move ( $\to$ ), the crossing change is *positive*, it is *negative* for the move ( $\to$ ).

For integral (resp. rational) homology spheres, the simple operations will be integral (resp. rational) *LP-surgeries of genus* 3.

Say that crossing changes are *disjoint* if they sit inside disjoint 3-balls. Say that LP-surgeries (A'/A) and (B'/B) in a manifold M are *disjoint* if A and B are disjoint in M. Two operations on $\mathbb{Z}^d$ are always *disjoint* (even if they look identical). In particular, disjoint operations commute, (their result does not depend on which one is performed first). Let $\underline{n} = \{1, 2, \dots, n\}$. Consider the vector space $\mathcal{F}_0(X)$ freely generated by X over $\mathbb{K}$. For an element x of X and n pairwise disjoint operations $o_1, \dots, o_n$ acting on x, define

$$[x; o_1, \dots, o_n] = \sum_{I \subseteq \underline{n}} (-1)^{\sharp I} x((o_i)_{i \in I}) \in \mathcal{F}_0(X)$$

where $x((o_i)_{i\in I})$ denotes the element of X obtained by performing the operations o_i for $i \in I$ on x. Then define $\mathcal{F}_n(X)$ as the $\mathbb{K}$-subspace of $\mathcal{F}_0(X)$ generated by the $[x; o_1, \dots, o_n]$, for all $x \in X$ equipped with n pairwise disjoint simple operations. Since

$$[x; o_1, \dots, o_n, o_{n+1}] = [x; o_1, \dots, o_n] - [x(o_{n+1}); o_1, \dots, o_n],$$

$\mathcal{F}_{n+1}(X) \subseteq \mathcal{F}_n(X)$, for all $n \in \mathbb{N}$.

Definition 3.1 A $\mathbb{K}$–valued function f on X, uniquely extends as a $\mathbb{K}$–linear map of

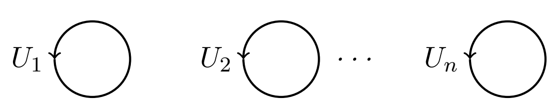
$$\mathcal{F}_0(X)^* = \mathrm{Hom}(\mathcal{F}_0(X); \mathbb{K})$$

that is still denoted by f. For an integer $n \in \mathbb{N}$, the invariant (or function) f is of *degree* $\leq n$ if and only if $f(\mathcal{F}_{n+1}(X)) = 0$. The *degree* of such an invariant is the smallest integer $n \in \mathbb{N}$ such that $f(\mathcal{F}_{n+1}(X)) = 0$. An invariant is of *finite type* if it is of degree n for some $n \in \mathbb{N}$. This definition depends on the chosen set of operations $\mathcal{O}(X)$. We fixed our choices for our sets X, but other choices could lead to different notions. See [GGP01].

Let $\mathcal{I}_n(X) = (\mathcal{F}_0(X)/\mathcal{F}_{n+1}(X))^*$ be the space of invariants of degree at most n. Of course, for all $n \in \mathbb{N}$, $\mathcal{I}_n(X) \subseteq \mathcal{I}_{n+1}(X)$.

Example 3.2 $\mathcal{I}_n(\mathbb{Z}^d)$ is the space of polynomials of degree n on $\mathbb{Z}^d$. (Exercise).

Lemma 3.3 *Any n-component link in $\mathbb{R}^3$ can be transformed to the trivial n-component link below by a finite number of disjoint crossing changes.*

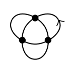
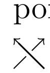
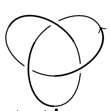


Proof: Let L be an n-component link in $\mathbb{R}^3$. Since $\mathbb{R}^3$ is simply connected, there is a homotopy that carries L to the trivial link. Such a homotopy $h\colon [0,1] \times \coprod_{i=1}^n S^1_i \to \mathbb{R}^3$ can be chosen, so that $h(t,.)$ is an embedding except for finitely many times t_i, $0 < t_1 < \ldots < t_i < t_{i+1} < t_k < 1$ where $h(t_i,.)$ is an immersion with one double point and no other multiple points, and the link $h(t,.)$ changes exactly by a crossing change when t crosses a t_i. For an elementary proof of this fact, see [Les05, Subsection 7.1]. $\diamond$

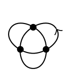
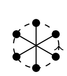
In particular, a degree 0 invariant of n-component links of $\mathbb{R}^3$ must be constant, since it is not allowed to vary under a crossing change.

Exercise 3.4 1. Check that $\mathcal{I}_1(\mathcal{K}) = \mathbb{K}c_0$, where c_0 is the constant map that maps any knot to 1.
2. Check that the linking number is a degree 1 invariant of 2–component links of $\mathbb{R}^3$.
3. Check that $\mathcal{I}_1(\mathcal{K}_2) = \mathbb{K}c_0 \oplus \mathbb{K}lk$, where c_0 is the constant map that maps any two-component link to 1.

3.2 Introduction to chord diagrams

Let f be a knot invariant of degree at most n. We want to evaluate $f([K; o_1, \ldots, o_n])$ where the o_i are disjoint negative crossing changes  $\to$  to be performed on a knot K. Such a $[K; o_1, \ldots, o_n]$ is usually represented as a *singular knot with n double points* that is an immersion of a circle with n transverse double points , where each double point  can be desingularized in two ways the positive one  and the negative one , and K is obtained from the singular knot by desingularizing all the crossings in the positive way, that is  in our example. Note that the sign of the desingularization is defined from the orientations of the ambient space and the knot.

Define the *chord diagram* $\Gamma([K; o_1, \ldots, o_n])$ associated to $[K; o_1, \ldots, o_n]$ as follows. Draw the preimage of the associated singular knot with n double points as an oriented dashed circle equipped with the $2n$ preimages of the double points and join the pairs of preimages of a double point by a plain segment called a *chord*.

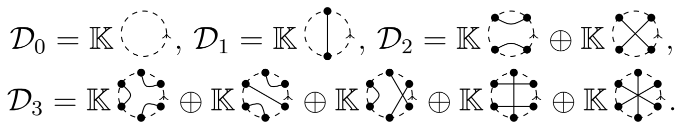
$\Gamma($  $) =$ 

Formally, a *chord diagram* with n chords is a cyclic order of the $2n$ ends of the n chords, up to a permutation of the chords and up to exchanging the two ends of a chord.

Lemma 3.5 *When f is a knot invariant of degree at most n, $f([K; o_1, \ldots, o_n])$ only depends on $\Gamma([K; o_1, \ldots, o_n])$.*

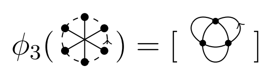
Proof: Since f is of degree n, $f([K; o_1, \dots, o_n])$ is invariant under a crossing change outside the balls of the o_i, that is outside the double points of the associated singular knot. Therefore, $f([K; o_1, \dots, o_n])$ only depends on the cyclic order of the $2n$ arcs involved in the o_i on K. $\diamond$

Let $\mathcal{D}_n$ be the $\mathbb{K}$-vector space freely generated by the n chord diagrams on S^1.



Lemma 3.6 *The map ϕ_n from $\mathcal{D}_n$ to $\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}$ that maps Γ to some $[K; o_1, \dots, o_n]$ whose diagram is Γ is well-defined and surjective.*

Proof: Use the arguments of the proof of Lemma 3.5. $\diamond$



The kernel of the composition of ϕ_n^* and the restriction below

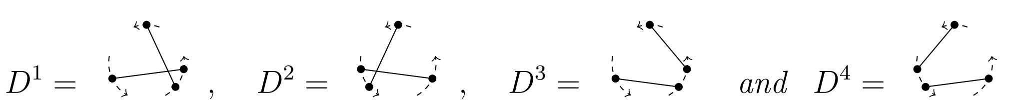
$$\mathcal{I}_n(\mathcal{K}) = \left(\frac{\mathcal{F}_0(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}\right)^* \to \left(\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}\right)^* \xrightarrow{\phi_n^*} \mathcal{D}_n^*$$

is $\mathcal{I}_{n-1}(\mathcal{K})$. Thus, $\frac{\mathcal{I}_n(\mathcal{K})}{\mathcal{I}_{n-1}(\mathcal{K})}$ injects into $\mathcal{D}_n^*$ and $\mathcal{I}_n(\mathcal{K})$ is finite dimensional for all n. Furthermore,

$$\frac{\mathcal{I}_n(\mathcal{K})}{\mathcal{I}_{n-1}(\mathcal{K})} = \mathrm{Hom}(\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}; \mathbb{K}).$$

An *isolated chord* in a chord diagram is a chord between two points of S^1 that are consecutive on the circle.

Lemma 3.7 *Let D be a diagram on S^1 that contains an isolated chord. Then $\phi_n(D) = 0$. Let D^1, D^2, D^3, D^4 be four n-chord diagrams that are identical outside three portions of circles where they look like:*



then

$$\phi_n(-D^1 + D^2 + D^3 - D^4) = 0.$$

Proof: For the first assertion, observe that $\phi_n($  $) = [$  $] - [$  $]$. For the second one, see [Les05, Lemma 2.21]. $\diamond$

Let $\mathcal{A}_n$ denote the quotient of $\mathcal{D}_n$ by the *four–term relation,* that is the quotient of $\mathcal{D}_n$ by the vector space generated by the $(-D^1 + D^2 + D^3 - D^4)$ for all the 4-tuples (D^1, D^2, D^3, D^4) as above. Call $(1T)$ the relation that identifies a diagram with an isolated chord with 0 so that $\mathcal{A}_n/(1T)$ is the quotient of $\mathcal{A}_n$ by the vector space generated by diagrams with an isolated chord.

According to Lemma 3.7 above, the map ϕ_n induces a map

$$\overline{\phi}_n \colon \mathcal{A}_n/(1T) \longrightarrow \frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}$$

The fundamental theorem of *Vassiliev invariants* (that are finite type knot invariants) can now be stated.

Theorem 3.8 *There exists a family of linear maps* $\left(Z_n^K \colon \mathcal{F}_0(\mathcal{K}) \to \mathcal{A}_n\right)_{n\in\mathbb{N}}$ *such that*

- $Z_n^K(\mathcal{F}_{n+1}(\mathcal{K})) = 0,$
- Z_n^K *induces the inverse of* $\overline{\phi}_n$ *from* $\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}$ *to* $\mathcal{A}_n/(1T)$.

In particular $\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})} \cong \mathcal{A}_n/(1T)$ *and* $\frac{\mathcal{I}_n(\mathcal{K})}{\mathcal{I}_{n-1}(\mathcal{K})} \cong (\mathcal{A}_n/(1T))^*$.

This theorem has been proved by Kontsevich and Bar-Natan using the *Kontsevich integral* $Z^K = (Z_n^K)_{n\in\mathbb{N}}$ [BN95], for $\mathbb{K} = \mathbb{R}$. It is also true when $\mathbb{K} = \mathbb{Q}$.

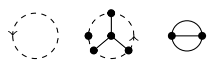
3.3 More spaces of diagrams

Definition 3.9 A *uni-trivalent graph* Γ is a 6-tuple $(H(\Gamma), E(\Gamma), U(\Gamma), T(\Gamma), p_E, p_V)$ where $H(\Gamma)$, $E(\Gamma)$, $U(\Gamma)$ and $T(\Gamma)$ are finite sets, that are called the set of half-edges of Γ, the set of edges of Γ, the set of univalent vertices of Γ and the set of trivalent vertices of Γ, respectively, $p_E \colon H(\Gamma) \to E(\Gamma)$ is a two-to-one map (every element of $E(\Gamma)$ has two preimages under p_E) and $p_V \colon H(\Gamma) \to U(\Gamma) \coprod T(\Gamma)$ is a map such that every element of $U(\Gamma)$ has one preimage under p_V and every element of $T(\Gamma)$ has three preimages under p_V, up to isomorphism. In other words, Γ is a set $H(\Gamma)$ equipped with two partitions, a partition into pairs (induced by p_E), and a partition into singletons and triples (induced by p_V), up to the bijections that preserve the partitions. These bijections are the *automorphisms* of Γ.

Definition 3.10 Let C be an oriented one-manifold. A *Jacobi diagram* Γ *with support* C*,* also called Jacobi diagram on C, is a finite uni-trivalent graph Γ equipped with an isotopy class of injections i_Γ of the set $U(\Gamma)$ of univalent vertices of Γ into the interior of C. A *vertex-orientation* of a Jacobi diagram Γ is an *orientation* of every trivalent vertex of Γ, that is a cyclic order on the set of the three half-edges which meet at this vertex. A Jacobi diagram is *oriented* if it is equipped with a vertex-orientation.

Such an oriented Jacobi diagram Γ is represented by a planar immersion of $\Gamma \cup C$ where the univalent vertices of U are located at their images under i_Γ, the one-manifold C is represented by dashed lines, whereas the diagram Γ is plain. The vertices are represented by big points. The local orientation of a vertex is represented by the counterclockwise order of the three half-edges that meet at it.

Here is an example of a picture of a Jacobi diagram Γ on the disjoint union $M = S^1 \coprod S^1$ of two circles:

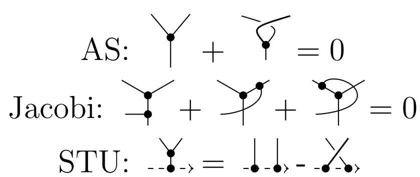


The *degree* of such a diagram is half the number of all the vertices of Γ.

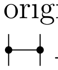
Of course, a chord diagram of $\mathcal{D}_n$ is a degree n Jacobi diagram on S^1 without trivalent vertices.

Let $\mathcal{D}^t_n(C)$ denote the $\mathbb{K}$-vector space generated by the degree n oriented Jacobi diagrams on C.

Let $\mathcal{A}^t_n(C)$ denote the quotient of $\mathcal{D}^t_n(C)$ by the following relations AS, Jacobi and STU:



As before, each of these relations relate oriented Jacobi diagrams which are identical outside the pictures where they are like in the pictures.

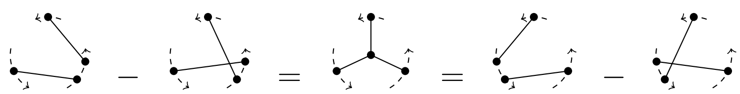
Remark 3.11 Lie algebras provide nontrivial linear maps, called *weight systems* from $\mathcal{A}^t_n(C)$ to $\mathbb{K}$, see [BN95] and [Les05, Section 6]. In the weight system constructions, the Jacobi relation for the Lie bracket ensures that the maps defined for oriented Jacobi diagrams factor through the Jacobi relation. In [Vog11], Pierre Vogel proved that the maps associated to Lie (super)algebras are sufficient to detect nontrivial elements of $\mathcal{A}^t_n(C)$ until degree 15, and he exhibited a non trivial element of $\mathcal{A}^t_{16}(\emptyset)$ that cannot be detected by such maps. The Jacobi relation was originally called IHX by Bar-Natan in [BN95] because, up to AS, it can be written as  =  − .

Set $\mathcal{A}_n(\emptyset) = \mathcal{A}_n(\emptyset;\mathbb{K}) = \mathcal{A}^t_n(\emptyset)$.

When $C \neq \emptyset$, let $\mathcal{A}_n(C) = \mathcal{A}_n(C;\mathbb{K})$ denote the quotient of $\mathcal{A}^t_n(C) = \mathcal{A}^t_n(C;\mathbb{K})$ by the vector space generated by the diagrams that have at least one connected component without univalent vertices. Then $\mathcal{A}_n(C)$ is generated by the oriented Jacobi diagrams whose (plain) connected components contain at least one univalent vertex.

Proposition 3.12 *The natural map from $\mathcal{D}_n$ to $\mathcal{A}_n(S^1)$ induces an isomorphism from $\mathcal{A}_n$ to $\mathcal{A}_n(S^1)$.*

PROOF: The natural map from $\mathcal{D}_n$ to $\mathcal{A}_n(S^1)$ factors though $4T$ since, according to STU,



in $\mathcal{A}^t_n(S^1)$. Since STU allows us to inductively write any oriented Jacobi diagram whose connected components contain at least a univalent vertex as a combination of chord diagrams, the induced map from $\mathcal{A}_n$ to $\mathcal{A}_n(S^1)$ is surjective. In order to prove injectivity, one constructs an inverse map. See [Les05, Subsection 3.4]. $\diamond$

The Le fundamental theorem on *finite type invariants of* $\mathbb{Z}$*-spheres* is the following one.

Theorem 3.13 *There exists a family $\left(Z_n^{LMO}\colon \mathcal{F}_0(\mathcal{M}) \to \mathcal{A}_n(\emptyset)\right)_{n\in\mathbb{N}}$ of linear maps such that*

- $Z_n^{LMO}(\mathcal{F}_{2n+1}(\mathcal{M})) = 0$,
- Z_n^{LMO} *induces an isomorphism from* $\frac{\mathcal{F}_{2n}(\mathcal{M})}{\mathcal{F}_{2n+1}(\mathcal{M})}$ *to* $\mathcal{A}_n(\emptyset)$,
- $\frac{\mathcal{F}_{2n-1}(\mathcal{M})}{\mathcal{F}_{2n}(\mathcal{M})} = \{0\}$.

In particular $\frac{\mathcal{F}_{2n}(\mathcal{M})}{\mathcal{F}_{2n+1}(\mathcal{M})} \cong \mathcal{A}_n(\emptyset)$ *and* $\frac{\mathcal{I}_{2n}(\mathcal{M})}{\mathcal{I}_{2n-1}(\mathcal{M})} \cong \mathcal{A}^*_n(\emptyset)$.

This theorem has been proved by Le [Le97] using the Le-Murakami-Ohtsuki invariant $Z^{LMO} = (Z_n^{LMO})_{n\in\mathbb{N}}$ of [LMO98].

In [Mou12], Delphine Moussard obtained a similar fundamental theorem for *finite type invariants of* $\mathbb{Q}$*-spheres* using the configuration space integral Z_{KKT} described in [KT99] and [Les04a].

Like in the knot case, the hardest parts of these theorems is the construction of an invariant $Z = (Z_n)_{n\in\mathbb{N}}$ that has the required properties.

3.4 Multiplying diagrams

Set $\mathcal{A}^t(C) = \prod_{n\in\mathbb{N}} \mathcal{A}^t_n(C)$ and $\mathcal{A}(C) = \prod_{n\in\mathbb{N}} \mathcal{A}_n(C)$.

Assume that a one-manifold C is decomposed as a union of two one-manifolds $C = C_1 \cup C_2$ whose interiors in C do not intersect. Define the *product associated to this decomposition*:

$$\mathcal{A}^t(C_1) \times \mathcal{A}^t(C_2) \longrightarrow \mathcal{A}^t(C)$$

as the continuous bilinear map which maps $([\Gamma_1],[\Gamma_2])$ to $[\Gamma_1 \coprod \Gamma_2]$, if Γ_1 is a diagram with support C_1 and if Γ_2 is a diagram with support C_2, where $\Gamma_1 \coprod \Gamma_2$ denotes their disjoint union.

In particular, the disjoint union of diagrams turns $\mathcal{A}(\emptyset)$ into a commutative algebra graded by the degree, and it turns $\mathcal{A}^t(C)$ into a $\mathcal{A}(\emptyset)$-module, for any 1-dimensional manifold C.

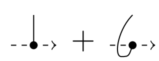
An orientation-preserving diffeomorphism from a manifold C to another one C' induces an isomorphism from $\mathcal{A}_n(C)$ to $\mathcal{A}_n(C')$, for all n.

Let $I = [0,1]$ be the compact oriented interval. If $I = C$, and if we identify I with $C_1 = [0,1/2]$ and with $C_2 = [1/2,1]$ with respect to the orientation, then the above process turns $\mathcal{A}(I)$ into an algebra where the elements with non-zero degree zero part admit an inverse.

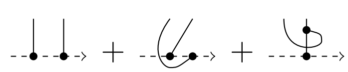
Proposition 3.14 *The algebra $\mathcal{A}([0,1])$ is commutative. The projection from $[0,1]$ to $S^1 = [0,1]/(0 \sim 1)$ induces an isomorphism from $\mathcal{A}_n([0,1])$ to $\mathcal{A}_n(S^1)$ for all n, so that $\mathcal{A}(S^1)$ inherits a commutative algebra structure from this isomorphism. The choice of a connected component C_j of C equips $\mathcal{A}(C)$ with an $\mathcal{A}([0,1])$-*module structure *$\sharp_j$, induced by the inclusion from $[0,1]$ to a little part of C_j outside the vertices, and the insertion of diagrams with support $[0,1]$ there.*

In order to prove this proposition, we present a useful trick in diagram spaces.

First adopt a convention. So far, in a diagram picture, or in a chord diagram picture, the plain edge of a univalent vertex, has always been attached on the left-hand side of the oriented one-manifold. Now, if k plain edges are attached on the other side on a diagram picture, then we agree that the corresponding represented element of $\mathcal{A}^t_n(M)$ is $(-1)^k$ times the underlying diagram. With this convention, we have the new antisymmetry relation in $\mathcal{A}^t_n(M)$:

 $= 0$

and we can draw the STU relation like the Jacobi relation:

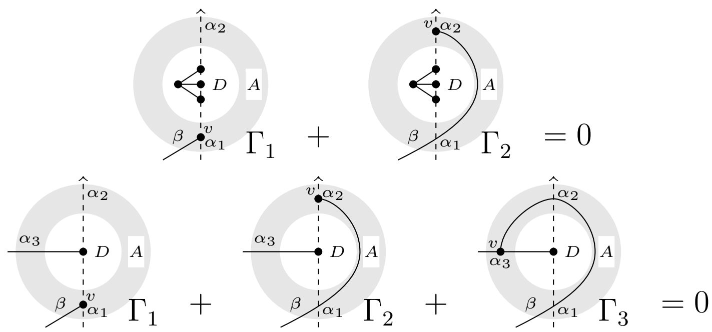
 $= 0.$

Lemma 3.15 *Let Γ_1 be a Jacobi diagram with support C. Assume that $\Gamma_1 \cup C$ is immersed in the plane so that $\Gamma_1 \cup C$ meets an open annulus A embedded in the plane exactly along $n+1$ embedded arcs α_1, α_2, ..., α_n and β, and one vertex v so that:*

1. *The α_i may be dashed or plain, they run from a boundary component of A to the other one,*
2. *β is a plain arc which runs from the boundary of A to $v \in \alpha_1$,*
3. *The bounded component D of the complement of A does not contain a boundary point of C.*

Let Γ_i be the diagram obtained from Γ_1 by attaching the endpoint v of β to α_i instead of α_1 on the same side, where the side of an arc is its side when going from the outside boundary component of A to the inside one ∂D. Then $\sum_{i=1}^n \Gamma_i = 0$ in $\mathcal{A}^t(C)$.

Examples 3.16

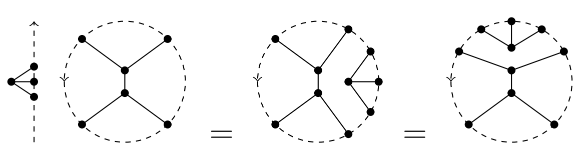


Proof: The second example shows that the STU relation is equivalent to this relation when the bounded component D of $\mathbb{R}^2 \setminus A$ intersects Γ_1 in the neighborhood of a univalent vertex on C. Similarly, the Jacobi relation is easily seen as given by this relation when D intersects Γ_1 in the neighborhood of a trivalent vertex. Also note that AS corresponds to the case when D intersects Γ_1 along a dashed or plain arc. Now for the Bar-Natan [BN95, Lemma 3.1] proof. See also [Vog11, Lemma 3.3]. Assume without loss that v is always attached on the right-hand-side of the α's. Add to the sum the trivial (by Jacobi and STU) contribution of the sum of the diagrams obtained from Γ_1 by attaching v to each of the three (dashed or plain) half-edges of each vertex w of $\Gamma_1 \cup C$ in D on the left-hand side when the half-edges are oriented towards w. Now, group the terms of the obtained sum by edges of $\Gamma_1 \cup C$ where v is attached, and observe that the sum is zero edge by edge by AS. $\diamond$

Proof of Proposition 3.14: With each choice of a connected component C_j of C, we associate an $\mathcal{A}(I)$-*module structure* $\sharp_j$ *on* $\mathcal{A}(C)$, that is given by the continuous bilinear map:

$$\mathcal{A}(I) \times \mathcal{A}(C) \longrightarrow \mathcal{A}(C)$$

such that: If Γ' is a diagram with support C and if Γ is a diagram with support I, then $([\Gamma], [\Gamma'])$ is mapped to the class of the diagram obtained by inserting Γ along C_j outside the vertices of Γ, according to the given orientation. For example,



As shown in the first example that illustrates Lemma 3.15, the independence of the choice of the insertion locus is a consequence of Lemma 3.15 where Γ_1 is the disjoint union $\Gamma \coprod \Gamma'$ and intersects D along $\Gamma \cup I$. This also proves that $\mathcal{A}(I)$ is a commutative algebra. Since the morphism from $\mathcal{A}(I)$ to $\mathcal{A}(S^1)$ induced by the identification of the two endpoints of I amounts to quotient out $\mathcal{A}(I)$ by the relation that identifies two diagrams that are obtained from one another by moving the nearest univalent vertex to an endpoint of I near the other endpoint, a similar application of Lemma 3.15 also proves that this morphism is an isomorphism from $\mathcal{A}(I)$ to $\mathcal{A}(S^1)$. (In this application, β comes from the inside boundary of the annulus.) $\diamond$

4 Configuration space construction of universal finite type invariants

4.1 Configuration spaces of links in 3-manifolds

Let $(\check{M},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$.

Let C be a disjoint union of k circles S^1_i, $i \in \underline{k}$ and let

$$L : C \longrightarrow \check{M}$$

denote a C^∞ embedding from C to $\check{M}$. Let Γ be a Jacobi diagram with support C. Let $U = U(\Gamma)$ denote the set of univalent vertices of Γ, and let $T = T(\Gamma)$ denote the set of trivalent vertices of Γ. A *configuration* of Γ is an embedding

$$c : U \cup T \hookrightarrow \check{M}$$

whose restriction $c_{|U}$ to U may be written as $L \circ j$ for some injection

$$j : U \hookrightarrow C$$

in the given isotopy class $[i_\Gamma]$ of embeddings of U into the interior of C. Denote the set of these configurations by $\check{C}(L;\Gamma)$,

$$\check{C}(L;\Gamma) = \left\{ c : U \cup T \hookrightarrow \check{M}\ ; \exists j \in [i_\Gamma], c_{|U} = L \circ j \right\}.$$

In $\check{C}(L;\Gamma)$, the univalent vertices move along $L(C)$ while the trivalent vertices move in the ambient space, and $\check{C}(L;\Gamma)$ is naturally an open submanifold of $C^U \times \check{M}^T$.

An *orientation* of a set of cardinality at least 2 is a total order of its elements up to an even permutation.

Cut each edge of Γ into two half-edges. When an edge is oriented, define its *first* half-edge and its *second* one, so that following the orientation of the edge, the first half-edge is met first. Let $H(\Gamma)$ denote the set of half-edges of Γ.

Lemma 4.1 *When Γ is equipped with a vertex-orientation, orientations of the manifold $\check{C}(L;\Gamma)$ are in canonical one-to-one correspondence with orientations of the set $H(\Gamma)$.*

PROOF: Since $\check{C}(L;\Gamma)$ is naturally an open submanifold of $C^U \times \check{M}^T$, it inherits $\mathbb{R}^{\sharp U + 3\sharp T}$-valued charts from $\mathbb{R}$-valued orientation-preserving charts of C and $\mathbb{R}^3$-valued orientation-preserving charts of $\check{M}$. In order to define the orientation of $\mathbb{R}^{\sharp U + 3\sharp T}$, one must identify its factors and order them (up to even permutation). Each of the factors may be labeled by an element of $H(\Gamma)$: the $\mathbb{R}$-valued local coordinate of an element of C corresponding to the image under j of an element of U sits in the factor labeled by the half-edge of U; the 3 cyclically ordered (by the orientation of $\check{M}$) $\mathbb{R}$-valued local coordinates of the image under a configuration c of an element of T live in the factors labeled by the three half-edges that are cyclically ordered by the vertex-orientation of Γ, so that the cyclic orders match. $\diamond$

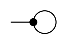
The dimension of $\check{C}(L;\Gamma)$ is

$$\sharp U(\Gamma) + 3\sharp T(\Gamma) = 2\sharp E(\Gamma)$$

where $E = E(\Gamma)$ denotes the set of edges of Γ. Since $n = n(\Gamma) = \frac{1}{2}(\sharp U(\Gamma) + \sharp T(\Gamma))$,

$$\sharp E(\Gamma) = 3n - \sharp U(\Gamma).$$

4.2 Configuration space integrals

A *numbered* degree n Jacobi diagram is a degree n Jacobi diagram Γ whose edges are oriented, equipped with an injection $j_E \colon E(\Gamma) \hookrightarrow \underline{3n}$. Such an injection numbers the edges. Note that this injection is a bijection when $U(\Gamma)$ is empty. Let $\mathcal{D}^e_n(C)$ denote the set of numbered degree n Jacobi diagrams with support C without *looped edges* like .

Let Γ be a numbered degree n Jacobi diagram. The orientations of the edges of Γ induce the following orientation of the set $H(\Gamma)$ of half-edges of Γ: Order $E(\Gamma)$ arbitrarily, and order the half-edges as (First half-edge of the first edge, second half-edge of the first edge, ..., second half-edge of the last edge). The induced orientation is called the *edge-orientation* of $H(\Gamma)$. Note that it does not depend on the order of $E(\Gamma)$. Thus, as soon as Γ is equipped with a vertex-orientation $o(\Gamma)$, the edge-orientation of Γ orients $\check{C}(L;\Gamma)$.

An edge e oriented from a vertex v_1 to a vertex v_2 of Γ induces the following canonical map

$$\begin{array}{llll} p_e\colon & \check{C}(L;\Gamma) & \to & C_2(M) \\ & c & \mapsto & (c(v_1), c(v_2)). \end{array}$$

For any $i \in \underline{3n}$, let $\omega(i)$ be a propagating form of $(C_2(M), \tau)$. Define

$$I(\Gamma, o(\Gamma), (\omega(i))_{i\in\underline{3n}}) = \int_{(\check{C}(L;\Gamma),o(\Gamma))} \bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$$

where $(\check{C}(L;\Gamma), o(\Gamma))$ denotes the manifold $\check{C}(L;\Gamma)$ equipped with the orientation induced by the vertex-orientation $o(\Gamma)$ and by the edge-orientation of Γ.

The convergence of this integral is a consequence of the following proposition that will be proved in Subsection 5.1.

Proposition 4.2 *There exists a smooth compactification $C(L;\Gamma)$ of $\check{C}(L;\Gamma)$ where the maps p_e smoothly extend.*

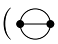
According to this proposition, $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$ smoothly extends to $C(L;\Gamma)$, and

$$\int_{(\check{C}(L;\Gamma),o(\Gamma))} \bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e))) = \int_{(C(L;\Gamma),o(\Gamma))} \bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e))).$$

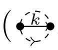
Examples 4.3 For any three propagating forms $\omega(1)$, $\omega(2)$ and $\omega(3)$ of $(C_2(M),\tau)$,

$$I(\ S_i^1$$  $$S_j^1\ ,(\omega(i))_{i\in\underline{3}})=lk(K_i,K_j)$$

and

$$I($$  $$,(\omega(i))_{i\in\underline{3}})=\Theta(M,\tau)$$

for any numbering of the (plain) diagrams (exercise).

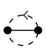
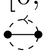
Let us now study the case of $I($  $S_j^1\ ,(\omega(i))_{i\in\underline{3}})$ that depends on the chosen propagating forms, and on the diagram numbering.

A *dilation* is a homothety with positive ratio.

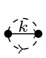
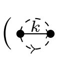
Let U^+K_j denote the fiber space over K_j made of the tangent vectors of the knot K_j of $\check{M}$ that orient K_j, up to dilation. The fiber of U^+K_j is made of one point, so that the total space of this *unit positive tangent bundle of* K_j is K_j. Let U^-K_j denote the fiber space over K_j made of the opposite tangent vectors of K_j, up to dilation.

For a knot K_j in $\check{M}$,

$$\check{C}(K_j;$$  $$S_j^1\ )=\{(K_j(z),K_j(z\exp(i\theta)));(z,\theta)\in S^1\times]0,2\pi[\}.$$

Let $C_j=C(K_j;$  $S_j^1\ )$ be the closure of $\check{C}(K_j;$  $S_j^1\ )$ in $C_2(M)$. This closure is diffeomorphic to $S^1\times[0,2\pi]$ where $S^1\times 0$ is identified with U^+K_j, $S^1\times\{2\pi\}$ is identified with U^-K_j and $\partial C(K_j;$  $S_j^1\ )=U^+K_j-U^-K_j$.

Lemma 4.4 *For any $i\in\underline{3}$, let $\omega(i)$ and $\omega'(i)=\omega(i)+d\eta(i)$ be propagating forms of $(C_2(M),\tau)$, where $\eta(i)$ is a one-form on $C_2(M)$.*

$$I($$  $$S_j^1\ ,(\omega'(i))_{i\in\underline{3}})-I($$  $$S_j^1\ ,(\omega(i))_{i\in\underline{3}})=\int_{U^+K_j}\eta(k)-\int_{U^-K_j}\eta(k).$$

Proof: Apply the Stokes theorem to $\int_{C_j}(\omega'(k)-\omega(k))=\int_{C_j}d\eta(k)$. ◇

Exercise 4.5 Find a knot K_j of $\mathbb{R}^3$ and a form $\eta(k)$ of $C_2(\mathbb{R}^3)$ such that the right-hand side of Lemma 4.4 does not vanish. (Use Lemma 2.8, hints can be found in Subsection 5.2.)

Say that a propagating form ω of $(C_2(M),\tau)$ is *homogeneous* if its restriction to $\partial C_2(M)$ is $p_\tau^*(\omega_{S^2})$ for the homogeneous volume form ω_{S^2} of S^2 of total volume 1.

Lemma 4.6 *For any $i\in\underline{3}$, let $\omega(i)$ be a homogeneous propagating form of $(C_2(M),\tau)$. Then $I($*  *$S_j^1\ ,(\omega(i))_{i\in\underline{3}})$ does not depend on the choices of the $\omega(i)$, it is denoted by $I_\theta(K_j,\tau)$.*

Proof: Apply Lemma 2.8 with $\eta_A=0$, so that $\eta(k)=0$ in Lemma 4.4. ◇

4.3 A configuration space invariant for links in $\mathbb{Q}$-spheres

Let $\mathbb{K}=\mathbb{R}$. Let $[\Gamma, o(\Gamma)]$ denote the class in $\mathcal{A}^t_n(C)$ of a numbered Jacobi diagram Γ of $\mathcal{D}^e_n(C)$ equipped with a vertex-orientation $o(\Gamma)$, then $I(\Gamma, o(\Gamma), (\omega(i))_{i\in\underline{3n}})[\Gamma, o(\Gamma)] \in \mathcal{A}^t_n(C)$ is independent of the orientation of $o(\Gamma)$, it will be simply denoted by $I(\Gamma, (\omega(i))_{i\in\underline{3n}})[\Gamma]$.

Theorem 4.7 *Let $(\check{M}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $L\colon \coprod_{j=1}^k S^1_j \hookrightarrow \check{M}$ be an embedding. For any $i \in \underline{3n}$, let $\omega(i)$ be a homogeneous propagating form of $(C_2(M), \tau)$.*

Set

$$Z_n(L, \check{M}, \tau) = \sum_{\Gamma\in\mathcal{D}^e_n(C)} \frac{(3n-\sharp E(\Gamma))!}{(3n)!2^{\sharp E(\Gamma)}} I(\Gamma, (\omega(i))_{i\in\underline{3n}})[\Gamma] \in \mathcal{A}^t_n(\coprod_{j=1}^k S^1_j).$$

Then $Z_n(L, \check{M}, \tau)$ is independent of the chosen $\omega(i)$, it only depends on the diffeomorphism class of (M, L), on $p_1(\tau)$ and on the $I_\theta(K_j, \tau)$, for the components K_j of L.

More precisely, set

$$Z(L, \check{M}, \tau) = (Z_n(L, \check{M}, \tau))_{n\in\mathbb{N}} \in \mathcal{A}^t(\coprod_{j=1}^k S^1_j).$$

There exist two constants $\alpha \in \mathcal{A}(S^1; \mathbb{Q})$ and $\xi \in \mathcal{A}(\emptyset; \mathbb{Q})$ such that

$$\exp(\frac{1}{4} p_1(\tau)\xi) \prod_{j=1}^k (\exp(-I_\theta(K_j, \tau)\alpha)\sharp_j)\, Z(L, \check{M}, \tau) = Z(L, M)$$

only depends on the diffeomorphism class of (M, L). Here $\exp(-I_\theta(K_j)\alpha)$ acts on $Z(L, \check{M}, \tau)$, on the copy S^1_j of S^1 as indicated by the subscript j.

$$Z(L, M) \in \mathcal{A}^t(\coprod_{j=1}^k S^1_j; \mathbb{Q}).$$

Furthermore, if $\check{M} = \mathbb{R}^3$, then the projection $Z^u(L, S^3)$ of $Z(L, S^3)$ on $\mathcal{A}(\coprod_{j=1}^k S^1_j)$ is a universal *finite type invariant of links in $\mathbb{R}^3$, i.e. Z^u_n satisfies the properties stated for Z^K_n in Theorem 3.8. It is the configuration space invariant studied by Altschüler, Freidel [AF97], Dylan Thurston [Thu99], Sylvain Poirier [Poi02] and others. If $k = 0$, then $Z(\emptyset, M)$ is the Kontsevich configuration space invariant $Z_{KKT}(M)$ that is a universal invariant for $\mathbb{Z}$-spheres according to a theorem of Kuperberg and Thurston [KT99, Les04b], and that was completed to a universal finite type invariant for $\mathbb{Q}$-spheres by Delphine Moussard [Mou12].*

The proof of this theorem is sketched in Section 5.

Under its assumptions, let ω_0 be a homogeneous propagating form of $(C_2(M), \tau)$, let ι be the involution of $C_2(M)$ that permutes two elements in $\check{M}^2 \setminus$ diagonal, set $\omega = \frac{1}{2}(\omega_0 - \iota_*(\omega_0))$, and set $\omega(i) = \omega$ for any i.

Let $\mathrm{Aut}(\Gamma)$ be the set of automorphisms of Γ that is the set of permutations of the half-edges that map a pair of half-edges of an edge to another such and a triple of half-edges that contain a vertex to another such. Set

$$\beta_\Gamma = \frac{(3n - \sharp E(\Gamma))!}{(3n)!2^{\sharp E(\Gamma)}}.$$

Then

$$\sum_{\Gamma \in \mathcal{D}^e_n(C)} \beta_\Gamma I(\Gamma, (\omega(i))_{i\in \underline{3n}})[\Gamma] = \sum_{\Gamma \text{ unnumbered, unoriented}} \frac{1}{\sharp\mathrm{Aut}(\Gamma)} I(\Gamma, (\omega)_{i\in\underline{3n}})[\Gamma]$$

where the sum of the right-hand side runs over the degree n Jacobi diagrams on C without looped edges.

Indeed, for a numbered graph Γ, there are $\frac{1}{\beta_\Gamma}$ ways of renumbering it, and $\sharp\mathrm{Aut}(\Gamma)$ of them will produce the same numbered graph.

4.4 On the universality proofs

Theorem 4.8 *Let y, $z \in \mathbb{N}$. Recall $\underline{y} = \{1, 2, \dots, y\}$. Let $\check{M}$ be an asymptotically standard $\mathbb{Q}$-homology $\mathbb{R}^3$. Let L be a link in $\check{M}$. Let $(B_b)_{\in\underline{y}}$ be a collection of pairwise disjoint balls in $\check{M}$ such that every B_b intersects L as a ball of a crossing change that contains a positive crossing c_b, and let $L((B_b)_{b\in\underline{y}})$ be the link obtained by changing the positive crossings c_b to negative crossings. Let $(A_a)_{a\in\underline{z}}$ be a collection of pairwise disjoint rational homology handlebodies in $\check{M}\setminus(L\cup_{b=1}^{y} B_b)$. Let (A'_a/A_a) be rational LP surgeries in $\check{M}$. Set $X = [M, L; (A'_a/A_a)_{a\in\underline{z}}, (B_b, c_b)_{b\in\underline{y}}]$ and*

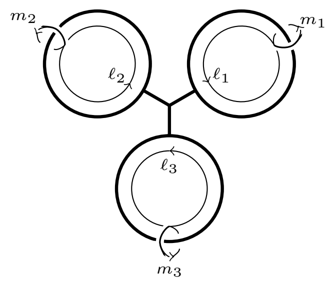
$$Z_n(X) = \sum_{I\subset \underline{z}\cup\underline{y}} (-1)^{\sharp I} Z_n\left(L((B_b)_{b\in I\cap\underline{y}}), M((A'_a/A_a)_{a\in I\cap\underline{z}})\right).$$

If $2n < 2y + z$, then $Z_n(X)$ vanishes .

SKETCH OF PROOF: Like in [KT99] or [Les04b], one can use (generalized) propagators for the $M((A'_a/A_a)_{a\in I\cap\underline{z}})$ that coincide for different I wherever it makes sense (for example, for configurations that do not involve points in surgered pieces A_a). Then contributions to the alternate sum of the integrals that do not involve at least one point in an A_a or in an A'_a, for all a cancel. Assume that every crossing change is performed by moving only one strand. Again, contributions to the alternate sum of the integrals that do not involve at least one point on a moving strand cancel. Furthermore, if the moving strand of c_b is moved very slightly, and if no other vertex is constrained to lie on the other strand in the ball of the crossing change, then the alternate sum is close to zero. Thus in order to produce a contribution to the alternate sum, a graph must have at least $(2y + z)$ vertices. See [AF97] or [Les05, Section 5.4] for more details. ◇

This implies that Z^u_n is of degree at most n for links in $\mathbb{R}^3$, and that Z_n is of degree at most $2n$ for $\mathbb{Z}$-spheres or $\mathbb{Q}$-spheres.

Now, under the hypotheses of Theorem 4.8, assume that A_a is the standard genus 3 handlebody with three handles with meridians $m_j^{(a)}$ and longitudes $l_j^{(a)}$ such that $\langle m_i^{(a)}, \ell_j^{(a)} \rangle_{\partial A_a} = \delta_{ij}$. See A_a as a thickening of the trivalent graph below.



In $A_a \cup_{\partial A_a} (-A'_a)$, there is a surface S_j such that $\partial(S_j \cap A_a) = m_j^{(a)}$. Assume that $\langle S_1, S_2, S_3 \rangle_{A_a \cup_{\partial A_a}(-A'_a)} = 1$. (For example, choose A'_a such that $A_a \cup_{\partial A_a} (-A'_a) = (S^1)^3$, like in the case of the Borromean surgery of [Mat87].) Assume that the $l_j^{(a)}$ bound surfaces $D_j^{(a)}$ in $\check{M}$.

Assume that the collection of surfaces $\{D_j^{(a)}\}_{a \in \underline{z}, j \in \underline{3}}$ reads $\{D_{p,1}\}_{p \in \underline{P}} \sqcup \{D_{p,2}\}_{p \in \underline{P}}$ so that

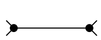
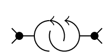
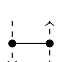
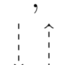
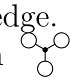
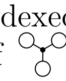
- for any $q \in \underline{P}$, for $\delta \in \underline{2}$, if $D_{q,\delta} = D_{j(q,\delta)}^{(a(q,\delta))}$, the interior of $D_{q,\delta}$ intersects

$$L \cup \bigcup_{a \in \underline{z}} \left(A_a \cup \cup_{j \in \underline{3}, D_j^{(a)} \neq D_{q,\delta}} D_j^{(a)} \right) \cup \cup_{b \in \underline{y}} (B_b)$$

  only in $A_{a(q,3-\delta)} \cup D_{j(q,3-\delta)}^{(a(q,3-\delta))}$.

  Note that $\langle D_{q,\delta}, \ell_{j(q,3-\delta)}^{(a(q,3-\delta))} \rangle_M = lk(\partial D_{q,1}, \partial D_{q,2})$.

Example 4.9 Note that these assumptions are realised in the following case. Start with an embedding of a Jacobi diagram Γ whose univalent vertices belong to chords (plain edges between two univalent vertices) on $\cup_{i=1}^k S_i^1$ in $\check{M}$. Assume that the trivalent vertices of Γ are indexed in $\underline{z}$, and assume that its chords are indexed in $\underline{y}$. Apply the following operations

replace edges  without univalent vertices by ,
replace a chord  indexed by b by a crossing change c_b  $\rightarrow$  in a ball B_b that is a neighborhood of the plain edge.
Thicken the trivalent graph  associated with the trivalent vertex indexed by a, and call it A_a. Then the surfaces $D_j^{(a)}$ are the disks bounded by the small loops of .

Conversely, under the assumptions before the example, associate the following vertex-oriented Jacobi diagram

$$\Gamma([M, L; (A'_a/A_a)_{a\in\underline{z}}, (B_b, c_b)_{b\in\underline{y}}])$$

on $\cup_{i=1}^k S_i^1$, with

- two univalent vertices joined by a chord for each crossing change ball B_b at the corresponding places on $\cup_{i=1}^k S_i^1$ (in $L^{-1}(B_b)$),
- one trivalent vertex for each A_a, where the three adjacent half-edges of the vertex correspond to the three $D_j^{(a)}$, with the fixed cyclic order,

such that any pair of half-edges corresponding to some $D_{p,1}$ and its friend $D_{p,2}$ forms an edge between two trivalent vertices.

Theorem 4.10 *Under the assumptions above, let* $X = [M, L; (A'_a/A_a)_{a\in\underline{z}}, (B_b, c_b)_{b\in\underline{y}}]$. *When* $2n = 2y + z$,

$$Z_n(X) = \prod_{p\in\underline{P}} lk(\partial D_{p,1}, \partial D_{p,2})[\Gamma(X)] \in \mathcal{A}_n^t(\coprod_{j=1}^k S_j^1).$$

Sketch of proof: When $z = 0$, the proof of Theorem 4.8 can be pushed further in order to prove the result like in [AF97] or [Les05, Section 5.4]. In general, when $y = 0$, it is a consequence of the main theorem in [Les04b] (Theorem 2.4). The general result can be obtained by mixing the arguments of [Les04b, Section 3] with the arguments of the link case. ◇

This theorem is the key to proving the universality of Z^u among Vassiliev invariants for links in $\mathbb{R}^3$ and to proving the universality of Z among finite type invariants of $\mathbb{Z}$-spheres. This universality implies that all finite type invariants factor through Z.

Remark 4.11 Theorem 4.10 with Z^{LMO} instead of Z is proved in [Le97], when $y = 0$, when the (A'_a/A_a) are Borromean surgeries and when the $D_j^{(a)}$ are disks such that $lk(\partial D_{p,1}, \partial D_{p,2}) = 1$. Then the main theorem of [AL05] implies Theorem 4.10 with Z^{LMO} instead of Z, when $y = 0$ and when the A'_a are integral homology handlebodies.

5 Compactifications, anomalies and questions

In this section, we prove Theorem 4.7 in a more general setting using the straight links introduced in Subsection 5.2. We begin with the introduction of appropriate compactifications of configuration spaces.

5.1 Compactifications of configuration spaces

Let N be a finite set. See the elements of M^N as maps $m\colon N \to M$.

For a non-empty $I \subseteq N$, let E_I be the set of maps that map I to ∞. For $I \subseteq N$ such that $\sharp I \geq 2$, let Δ_I be the set of maps that map I to a single element of M. When I is a finite set, and when V is a vector space of positive dimension, $\check{S}_I(V)$ denotes the space of injective maps from I to V up to translation and dilation. When $\sharp I \geq 2$, $\check{S}_I(V)$ embeds in the compact space $S_I(V)$ of non-constant maps from I to V up to translation and dilation.

Lemma 5.1 *The fiber of the unit normal bundle of Δ_I in M^N over a configuration m is $S_I(T_{m(I)}M)$.*

PROOF: Exercise. $\diamond$

Let $\check{C}_N(M)$ denote the space of injective maps from N to $\check{M}$.

Define a compactification $C_N(M)$ of $\check{C}_N(M)$ by generalizing the previous construction of $C_2(M) = C_{\underline{2}}(M)$ as follows.

Start with M^N. Blow up E_N that is the point $m = \infty^N$ such that $m^{-1}(\infty) = N$.
Then for $k = \sharp N, \sharp N - 1, \dots, 3, 2$, in this decreasing order, successively blow up the (closures of the preimages under the composition of the previous blow up maps of the) Δ_I such that $\sharp I = k$ (choosing an arbitrary order among them if they are not disjoint) and, next, the (closures of the preimages under the composition of the previous blow up maps of the) E_J such that $\sharp J = k-1$ (again choosing an arbitrary order if they are not disjoint).

Lemma 5.2 *The successive manifolds that are blown-up in the above process are smooth and transverse to the boundaries. The manifold $C_N(M)$ is a smooth compact $(3\sharp N)$-manifold independent of the possible order choices in the process. For $i, j \in N$, $i \neq j$, the map*

$$\begin{array}{llll} p_{i,j}\colon & \check{C}_N(M) & \to & C_2(M) \\ & m & \mapsto & (m(i), m(j)) \end{array}$$

smoothly extends to $C_N(M)$.

SKETCH OF PROOF: A configuration m_0 of M^N induces the following partition $\mathcal{P}(m_0)$ of

$$N = m_0^{-1}(\infty) \coprod_{x \in \check{M} \cap m_0(N)} m_0^{-1}(x).$$

Pick disjoint neighborhoods V_x in M of the points x of $m_0(N)$ that are furthermore in $\check{M}$ for x in $\check{M}$ and that are identified with balls of $\mathbb{R}^3$ by C^∞-charts. Consider the neighborhood $\prod_{x\in m_0(N)} V_x^{m_0^{-1}(x)}$ of m_0 in M^N. The first blow-ups that transformed this neighborhood are

- the blow-up of $E_{m_0^{-1}(\infty)}$ if $m_0^{-1}(\infty) \neq \emptyset$ that changed (a smaller neighborhood of $\infty^{m_0^{-1}(\infty)}$ in) $V_\infty^{m_0^{-1}(\infty)}$ to $[0,\varepsilon_\infty[\times S^{3\sharp m_0^{-1}(\infty)-1}$,
- and the blow-ups of the $\Delta_{m_0^{-1}(x)}$, for the $x \in \check{M}$ such that $\sharp m_0^{-1}(x) \geq 2$, that changed (a smaller neighborhood of $x^{m_0^{-1}(x)}$ in) $V_x^{m_0^{-1}(x)}$ to $[0,\varepsilon_x[\times F(U_x^{m_0^{-1}(x)})$, where $U_x \subset V_x$ and $F(U_x^{m_0^{-1}(x)})$ fibers over U_x, and the fiber over $y \in U_x$ is $S_{m_0^{-1}(x)}(T_yM)$.

When considering how the next blow-ups affect the preimage of a neighborhood of m_0, we can restrict to our new factors.

First consider a factor $[0,\varepsilon_x[\times F(U_x^{m_0^{-1}(x)})$. Picking $i \in m_0^{-1}(x)$ and fixing a Riemannian structure on TU_x identifies $S_{m_0^{-1}(x)}(T_yM)$ with the space of maps $c\colon m_0^{-1}(x) \to T_yM$ such that $c(i) = 0$ and $\sum_{j\in m_0^{-1}(x)} \parallel c(j) \parallel^2 = 1$. Then (λ, c) is identified with $y + \lambda c$ in $V_x^{m_0^{-1}(x)}$ (where V_x is identified with an open subset of $\mathbb{R}^3$), for $\lambda \neq 0$. Now, $[0,\varepsilon_x[\times F(U_x^{m_0^{-1}(x)})$ must be blown-up along its intersections with the preimage closures of the Δ_I such that $\sharp I \geq 2$, $I \subset m_0^{-1}(x)$ and I is maximal. These intersections respect the product structure by $[0,\varepsilon_x[$ and the fibration over U_x so that we only need to understand the blow-ups of the intersections of the Δ_I with a fiber of $F(U_x^{m_0^{-1}(x)})$. These are nothing but configurations in a ball of $\mathbb{R}^3$, and we can iterate our process.

Now consider the possible factor $[0,\varepsilon_\infty[\times S^{3\sharp m_0^{-1}(\infty)-1}$ and blow up its intersections with the preimage closures of the E_J for $J \subset m_0^{-1}(\infty)$ maximal and with the preimage closures of the Δ_I with $I \subset m_0^{-1}(\infty)$ in an order compatible with the algorithm. Here, $S^{3\sharp m_0^{-1}(\infty)-1}$ is the unit sphere of $(\mathbb{R}^3_\infty)^{m_0^{-1}(\infty)}$. A point $d \in (\mathbb{R}^3_\infty)^{m_0^{-1}(\infty)}$ is in the preimage closure of E_J under the previous blow-up if $d(J) = 0$. In particular, the E_J and the Δ_I again read as products by $[0,\varepsilon_\infty[$, and we study what happens near a given d of $S^{3\sharp m_0^{-1}(\infty)-1}$. For such a d, we proceed as before if $d^{-1}(0) = \emptyset$. Otherwise the factor of $d^{-1}(0)$ must be treated differently, namely by blowing up $0^{d^{-1}(0)}$ in $S^{3\sharp m_0^{-1}(\infty)-1}$. Then iterate.

This produces a compact manifold $C_N(M)$ with boundary and ridges, that is finally independent of the order of the blow-ups (when this order is compatible with the algorithm), since it is locally independent. The interior of $C_N(M)$ is $\check{C}_N(\check{M})$. Since the blow-ups separate all the pairs of points at some scale, p_e naturally extends there. The introduced local coordinates show that the extension is smooth. See [Les04a, Section 3] for more details. $\diamond$

Lemma 5.3 *The closure of* $\check{C}(L;\Gamma)$ *in* $C_{V(\Gamma)}(M)$ *is a smooth compact submanifold of* $C_{V(\Gamma)}(M)$ *that is denoted by* $C(L;\Gamma)$.

PROOF: Exercise. ◇

Proposition 4.2 is a consequence of Lemmas 5.2 and 5.3.

5.2 Straight links

A one-chain c of S^2 is *algebraically trivial* if for any two points x and y outside its support, the algebraic intersection of an arc from x to y transverse to c with c is zero, or equivalently if the integral of any one form of S^2 along c is zero.

Let $(\check{M}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Say that K_j is *straight* with respect to τ if the curve $p_\tau(U^+K_j)$ of S^2 is algebraically trivial (recall the notation from Proposition 2.3 and Subsection 4.2). A link is *straight* with respect to τ if all its components are. If K_j is straight, then $p_\tau(\partial C(K_j; \overset{\curvearrowleft}{\bullet\!\!-\!\!\bullet} S^1_j))$ is algebraically trivial.

Lemma 5.4 *Recall* $C_j = C(K_j; \overset{\curvearrowleft}{\bullet\!\!-\!\!\bullet} S^1_j)$, $C_j \subset C_2(M)$.

If $p_\tau(\partial C_j)$ *is algebraically trivial, then for any propagating chain* F *of* $(C_2(M), \tau)$ *transverse to* C_j *and for any propagating form* ω_p *of* $(C_2(M), \tau)$,

$$\int_{C_j} \omega_p = \langle C_j, F \rangle_{C_2(M)} = I_\theta(K_j, \tau)$$

where $I_\theta(K_j, \tau)$ *is defined in Lemma 4.6. In particular,* $I_\theta(K_j, \tau) \in \mathbb{Q}$ *and* $I_\theta(K_j, \tau) \in \mathbb{Z}$ *when* M *is an integral homology sphere.*

PROOF: Exercise. Recall Lemmas 2.8 and 4.4. ◇

Proposition 5.5 *Let* $\check{M}$ *be an asymptotically standard* $\mathbb{Q}$*-homology* $\mathbb{R}^3$*. For any parallel* $K_\parallel$ *of a knot* K *in* $\check{M}$*, there exists an asymptotically standard parallelization* $\tilde{\tau}$ *homotopic to* τ*, such that* K *is straight with respect to* $\tilde{\tau}$*, and* $I_\theta(K_j, \tilde{\tau}) = lk(K, K_\parallel)$ *or* $I_\theta(K_j, \tilde{\tau}) = lk(K, K_\parallel) + 1$*.*

For any embedding $K\colon S^1 \to \check{M}$ *that is straight with respect to* τ*,* $I_\theta(K, \tau)$ *is the linking number of* K *and a parallel of* K*.*

SKETCH OF PROOF: For any knot embedding K, there is an asymptotically standard parallelization $\tilde{\tau}$ homotopic to τ such that $p_{\tilde{\tau}}(U^+K)$ is one point. Thus K is straight with respect to $(M, \tilde{\tau})$. Then $\tilde{\tau}$ induces a parallelization of K, and $I_\theta(K, \tilde{\tau})$ is the linking number of K with the parallel induced by $\tilde{\tau}$. (Exercise).

In general, for two homotopic asymptotically standard parallelizations τ and $\tilde{\tau}$ such that K is straight with respect to τ and $\tilde{\tau}$, $I_\theta(K, \tau) - I_\theta(K, \tilde{\tau})$ is an even integer (exercise) so that $I_\theta(K, \tau)$ is always the linking number of K with a parallel of K.

In $\mathbb{R}^3$ equipped with τ_s, any link is represented by an embedding L that sits in a horizontal plane except when it crosses under, so that the non-horizontal arcs crossing under are in vertical planes. Then the non-horizontal arcs have an algebraically trivial contribution to $p_\tau(U^+K_j)$, while the horizontal contribution can be changed by adding kinks  so that L is straight with respect to τ_s. In this case $I_\theta(K_j, \tau_s)$ is the *writhe* of K_j that is the number of positive self-crossings of K_j minus the number of negative self-crossings of K_j. In particular, up to isotopy of L, $I_\theta(K_j, \tau_s)$ can be assumed to be ± 1 (Exercise).

Similarly, for any number ι that is congruent mod $2\mathbb{Z}$ to $I_\theta(K, \tau)$ there exists an embedding K' isotopic to K and straight such that $I_\theta(K', \tau) = \iota$ (Exercise). $\diamond$

5.3 Rationality of Z

Let us state another version of Theorem 4.7 using straight links instead of homogeneous propagating forms. Recall $\beta_\Gamma = \frac{(3n - \sharp E(\Gamma))!}{(3n)! 2^{\sharp E(\Gamma)}}$.

Theorem 5.6 *Let $(\check{M}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $L\colon \coprod_{j=1}^k S^1_j \hookrightarrow \check{M}$ be a straight embedding with respect to τ. For any $i \in \underline{3n}$, let $\omega(i)$ be a propagating form of $(C_2(M), \tau)$. Set*

$$Z^s_n(L, \check{M}, \tau) = \sum_{\Gamma \in \mathcal{D}^e_n(C)} \beta_\Gamma I(\Gamma, (\omega(i))_{i \in \underline{3n}})[\Gamma] \in \mathcal{A}^t_n(\coprod_{j=1}^k S^1_j).$$

Then $Z^s_n(L, \check{M}, \tau)$ is independent of the chosen $\omega(i)$. In particular, with the notation of Theorem 4.7,

$$Z^s_n(L, \check{M}, \tau) = Z_n(L, \check{M}, \tau).$$

This version of Theorem 4.7 allows us to replace the configuration space integrals by algebraic intersections in configuration spaces, and thus to prove the rationality of Z for straight links as follows.

For any $i \in \underline{3n}$, let $F(i)$ be a propagating chain of $(C_2(M), \tau)$. Say that a family $(F(i))_{i \in \underline{3n}}$ is *in general* $3n$ *position* with respect to L if for any $\Gamma \in \mathcal{D}^e_n(C)$, the $p_e^{-1}(F(j_E(e)))$ are pairwise transverse chains in $C(L; \Gamma)$. In this case, define $I(\Gamma, o(\Gamma), (F(i))_{i \in \underline{3n}})$ as the algebraic intersection in $(C(L; \Gamma), o(\Gamma))$ of the codimension 2 rational chains $p_e^{-1}(F(j_E(e)))$. If the $\omega(i)$ are propagating forms of $(C_2(M), \tau)$ supported in sufficiently small neighborhoods of the $F(i)$, then

$$I(\Gamma, o(\Gamma), (F(i))_{i \in \underline{3n}}) = I(\Gamma, o(\Gamma), (\omega(i))_{i \in \underline{3n}})$$

for any $\Gamma \in \mathcal{D}^e_n(C)$, and $I(\Gamma, o(\Gamma), (\omega(i))_{i \in \underline{3n}})$ is rational, in this case.

5.4 On the anomalies

The constants $\alpha = (\alpha_n)_{n \in \mathbb{N}}$ and $\xi = (\xi_n)_{n \in \mathbb{N}}$ of Theorem 4.7 are called *anomalies*. The anomaly ξ is defined in [Les04a, Section 1.6], $\xi_{2n} = 0$ for any integer n, and $\xi_1 = -\frac{1}{12}$[] according to [Les04a, Proposition 2.45]. The computation of ξ_1 can also be deduced from Corollary 2.12.

We define α below. Let $v \in S^2$. Let D_v denote the linear map

$$\begin{array}{rccc} D_v: & \mathbb{R} & \longrightarrow & \mathbb{R}^3 \\ & 1 & \mapsto & v. \end{array}$$

Let Γ be a numbered Jacobi diagram on $\mathbb{R}$. Define $\check{C}(D_v;\Gamma)$ like in Subsection 4.1 where the line D_v of $\mathbb{R}^3$ replaces the link L of $\check{M}$. Let $\check{Q}(v;\Gamma)$ be the quotient of $\check{C}(D_v;\Gamma)$ by the translations parallel to D_v and by the dilations. Then the map p_{e,S^2} associated with an edge e of Γ maps a configuration to the direction of the vector from its origin to its end in S^2. It factors through $\check{Q}(v;\Gamma)$ that has two dimensions less. Now, define $\check{Q}(\Gamma)$ as the total space of the fibration over S^2 whose fiber over v is $\check{Q}(v;\Gamma)$. The configuration space $\check{Q}(\Gamma)$ carries a natural smooth structure, it can be compactified as before, and it can be oriented as follows, when a vertex-orientation $o(\Gamma)$ is given. Orient $\check{C}(D_v;\Gamma)$ as before, orient $\check{Q}(v;\Gamma)$ so that $\check{C}(D_v;\Gamma)$ is locally homeomorphic to the oriented product (translation vector z in $\mathbb{R}v$, ratio of homothety $\lambda \in]0,\infty[$) $\times \check{Q}(v;\Gamma)$ and orient $\check{Q}(\Gamma)$ with the (base($=S^2$) $\oplus$ fiber) convention. (This can be summarized by saying that the S^2-coordinates replace (z,λ).)

Proposition 5.7 *For $i \in \underline{3n}$, let $\omega(i,S^2)$ be a two-form of S^2 such that $\int_{S^2}\omega(i,S^2)=1$. Define*

$$I(\Gamma,o(\Gamma),\omega(i,S^2)) = \int_{\check{Q}(\Gamma)} \bigwedge_{e\in E(\Gamma)} p^*_{e,S^2}(\omega(j_E(e),S^2).$$

Let $\mathcal{D}^c_n(\mathbb{R})$ denote the set of connected numbered diagrams on $\mathbb{R}$ with at least one univalent vertex, without looped edges. Set

$$2\alpha_n = \sum_{\Gamma\in\mathcal{D}^c_n(\mathbb{R})} \frac{(3n-\sharp E(\Gamma))!}{(3n)!2^{\sharp E(\Gamma)}} I(\Gamma,o(\Gamma),\omega(i,S^2))[\Gamma,o(\Gamma)] \quad \in \mathcal{A}(\mathbb{R}).$$

Then α_n does not depend on the chosen $\omega(i,S^2)$, $\alpha_1 = \frac{1}{2}\Big[$  *$\Big]$ and $\alpha_{2n}=0$ for all n. The series $\alpha = \sum_{n_i n\mathbb{N}}\alpha_n$ is called the* Bott and Taubes anomaly*.*

Proof: The independence of the choices of the $\omega(i,S^2)$ comes from Lemma 5.8 below. Let us prove that $\alpha_{2n}=0$ for all n. Let Γ be a numbered graph and let $\overline{\Gamma}$ be obtained from Γ by reversing the orientations of the ($\sharp E$) edges of Γ. Consider the map r from $\check{Q}(\overline{\Gamma})$ to $\check{Q}(\Gamma)$ that composes a configuration by the multiplication by (-1) in $\mathbb{R}^3$. It sends a configuration over $v \in S^2$ to a configuration over $(-v)$, and it is therefore a fibered map over the orientation-reversing antipode of S^2. Equip Γ and $\overline{\Gamma}$ with the same vertex-orientation. Then our map r is orientation-preserving if and only if $\sharp T(\Gamma)+1+\sharp E(\Gamma)$ is even. Furthermore for all the edges e of $\overline{\Gamma}$, $p_{e,S^2}\circ r = p_{e,S^2}$, then since $\sharp E = n + \sharp T$,

$$I(\overline{\Gamma},o(\Gamma),\omega(i,S^2)) = (-1)^{n+1} I(\Gamma,o(\Gamma),\omega(i,S^2)).$$

◇

It is known that $\alpha_3 = 0$ and $\alpha_5 = 0$ [Poi02]. Furthermore, according to [Les02], α_{2n+1} is a combination of diagrams with two univalent vertices, and $Z^u(S^3, L)$ is obtained from the Kontsevich integral by inserting d times the plain part of 2α on each degree d connected component of a diagram.

5.5 The dependence on the forms in the invariance proofs

The variation of $I(\Gamma, o(\Gamma), (\omega(j))_{j\in\underline{3n}})$ when some $\omega(i = j_E(f \in E(\Gamma)))$ is changed to $\omega(i) + d\eta$ for a one-form η on $C_2(M)$ reads

$$\int_{(C(L;\Gamma),o(\Gamma))} \left(p_f^*(d\eta) \wedge \bigwedge_{e\in(E(\Gamma)\setminus\{f\})} p_e^*(\omega(j_E(e))) \right).$$

According to the Stokes theorem, it reads $\int_{\partial(C(L;\Gamma),o(\Gamma))} \left(p_f^*(\eta) \wedge \bigwedge_{e\in(E(\Gamma)\setminus\{f\})} p_e^*(\omega(j_E(e))) \right)$ where the integral along $\partial(C(L;\Gamma), o(\Gamma))$ is actually the integral along the codimension one faces of $C(L;\Gamma)$ that are considered as open. Such a codimension one face only involves one blow-up.

For any non-empty subset B of $V(\Gamma)$, the codimension one face associated with the blow-up of E_B in $M^{V(\Gamma)}$ is denoted by $F(\Gamma, \infty, B)$, it lies in the preimage of $\infty^B \times \check{M}^{V(\Gamma)\setminus B}$, in $C(L;\Gamma)$.

The other codimension one faces are associated with the blow-ups of the Δ_B in $M^{V(\Gamma)}$, for subsets B of $V(\Gamma)$ of cardinality at least 2. The face of $C(L;\Gamma)$ associated with Δ_B is denoted by $F(\Gamma, B)$. Let $b \in B$. Assume that $b \in U(\Gamma)$ if $U(\Gamma) \cap B \neq \emptyset$. The image of $F(\Gamma, B)$ in $M^{V(\Gamma)}$ is in the set of maps m of Δ_B that define an injection from $(V(\Gamma) \setminus B) \cup \{b \in B\}$ to $\check{M}$, that factors through an injection isotopic to the restriction of i_Γ on $U(\Gamma) \cap ((V(\Gamma) \setminus B) \cup \{b\})$. This set of maps $\check{C}_{(V(\Gamma)\setminus B)\cup\{b\}}(\check{M}, i_\Gamma)$ is a submanifold of $\check{C}_{(V(\Gamma)\setminus B)\cup\{b\}}(\check{M})$. Thus, $F(\Gamma, B)$ is a bundle over $\check{C}_{(V(\Gamma)\setminus B)\cup\{b\}}(\check{M}, i_\Gamma)$.

When B has no univalent vertices, the fiber over a map m is the space $\check{S}_B(T_{m(b)})$ of injective maps from B to $T_{m(b)}$ up to translations and dilations.

When B contains univalent vertices of a component K_j, the fiber over m is the submanifold $\check{S}_B(T_{m(b)}M, \Gamma)$ of $\check{S}_B(T_{m(b)}M)$, made of the configurations that map the univalent vertices of B to a line of $T_{m(b)}M$ directed by U^+K_j at $m(b)$, in an order prescribed by Γ. If B does not contain all the univalent vertices of Γ on S^1_j, this order is unique. Otherwise, $F(\Gamma, B)$ has $\sharp(B \cap U(\Gamma))$ connected components corresponding to the total orders that induce the cyclic order of $B \cap U(\Gamma) \cap i_\Gamma^{-1}(S^1_j)$.

When B is a subset of the set of vertices $V(\Gamma)$ of a numbered graph Γ, $E(\Gamma_B)$ denotes the set of edges of Γ between two elements of B (edges of Γ are plain), and Γ_B is the subgraph of Γ made of the vertices of B and the edges of $E(\Gamma_B)$.

Lemma 5.8 *Let $(\check{M}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $C = \coprod_{j=1}^k S^1_j$.*

For $i \in \underline{3n}$, let $\omega(i)$ be a closed 2-form on $[0,1]\times C_2(M)$ whose restriction to $\{t\}\times C_2(M)$ is denoted by $\omega(i,t)$, for any $t\in[0,1]$. Assume that for $t\in[0,1]$, $\omega(i,t)$ restricts to $(\partial C_2(M)\setminus UB_M)$ as $p_\tau^*(\omega(i,t)(S^2))$, for some two-form $\omega(i,t)(S^2)$ of S^2 such that $\int_{S^2}\omega(i,t)(S^2)=1$. Set

$$Z_n(t)=\sum_{\Gamma\in\mathcal{D}_n^e(C)}\beta_\Gamma I(\Gamma,(\omega(i,t))_{i\in\underline{3n}})[\Gamma]\in\mathcal{A}_n^t(\coprod_{j=1}^k S_j^1).$$

Then

$$Z_n(1)-Z_n(0)=\sum_{\left\{\begin{array}{c}(\Gamma,B);\Gamma\in\mathcal{D}_n^e(C),B\subset V(\Gamma),\sharp B\geq 2;\\ \Gamma_B \text{ is a connected component of } \Gamma\end{array}\right\}} I(\Gamma,B)$$

where

$$I(\Gamma,B)=\beta_\Gamma\int_{[0,1]\times F(\Gamma,B)}\bigwedge_{e\in E(\Gamma)}p_e^*(\omega(j_E(e)))[\Gamma].$$

Under the assumptions of Theorem 4.7 (where the $\omega(i)$ are homogeneous) or Theorem 5.6 (where L is straight with respect to τ), when (M,L,τ) is fixed, $Z_n(L,\check{M},\tau)$ is independent of the chosen $\omega(i)$.

In particular, when $k=0$, $Z(\check{M},\tau)$ coincides with the Kontsevich configuration space integral invariant described in [Les04a].

Furthermore, the α_n of Proposition 5.7 are also independent of the forms $\omega(i,S^2)$.

Sketch of proof: According to the Stokes theorem, for any $\Gamma\in\mathcal{D}_n^e(C)$,

$$I(\Gamma,(\omega(i,1))_{i\in\underline{3n}})-I(\Gamma,(\omega(i,0))_{i\in\underline{3n}})=\sum_F\int_{[0,1]\times F}\bigwedge_{e\in E(\Gamma)}p_e^*(\omega(j_E(e)))$$

where the sum runs over the codimension one faces F of $C(L;\Gamma)$. Below, we sketch the proof that the only contributing faces are the faces $F(\Gamma,B)$ such that $\sharp B\geq 2$ and Γ_B is a connected component of Γ, or equivalently, that the other faces do not contribute.

Like in [Les04a, Lemma 2.17] faces $F(\Gamma,\infty,B)$ do not contribute. When the product of all the p_e factors through a quotient of $[0,1]\times F(\Gamma,B)$ of smaller dimension, the face $F(\Gamma,B)$ does not contribute. This allows us to get rid of

- the faces $F(\Gamma,B)$ such that B is not a pair of univalent vertices of Γ, and Γ_B is not connected (see [Les04a, Lemma 2.18]),
- the faces $F(\Gamma,B)$ such that $\sharp B\geq 3$ where Γ_B has a univalent vertex that was trivalent in Γ (see [Les04a, Lemma 2.19]).

We also have faces that cancel each other, for graphs that are identical outside their Γ_B part.

- The faces $F(\Gamma, B)$ (that are not already listed) such that Γ_B has at least a bivalent vertex cancel (mostly by pairs) by the parallelogram identification (see [Les04a, Lemma 2.20]).

- The faces $F(\Gamma, B)$ where Γ_B is an edge between two trivalent vertices cancel by triples, thanks to the Jacobi (or IHX) relation (see [Les04a, Lemma 2.21]).

- Similarly, two faces where B is made of two (necessarily consecutive in C) univalent vertices of Γ cancel $(3n - \sharp E(\Gamma))$ faces $F(\Gamma', B')$ where $\Gamma'_{B'}$ is an edge between a univalent vertex of Γ and a trivalent vertex of Γ, thanks to the STU relation.

Thus, we are left with the faces $F(\Gamma, B)$ such that Γ_B is a (plain) connected component of Γ, and we get the wanted formula for $(Z_n(1) - Z_n(0))$.

In the anomaly case, the same analysis of faces leaves no contributing faces, so that the α_n are independent of the forms $\omega(i, S^2)$ in Proposition 5.7.

Back to the behaviour of $Z(L, \check{M}, \tau)$ under the assumptions of Theorem 4.7 or Theorem 5.6, assume that (M, L, τ) is fixed and apply the formula of the lemma to compute the variation of $Z_n(L, \check{M}, \tau)$ when some propagating chain $\omega(i, 0)$ of $(C_2(M), \tau)$ is changed to some other propagating chain $\omega(i, 1) = \omega(i, 0) + d\eta$. According to Lemma 2.8, under our assumptions, η can be chosen so that $\eta = p_\tau^*(\eta_{S^2})$ on $\partial C_2(M)$ and $\eta_{S^2} = 0$ if $\omega(i, 0)$ and $\omega(i, 1)$ are homogeneous. Define $\omega(i) = \omega(i, 0) + d(t\eta)$ on $[0, 1] \times C_2(M)$ $(t \in [0, 1])$, and extend the other $\omega(j)$ trivially.

Then $(Z_n(1) - Z_n(0))$ vanishes if $\omega(i, 0)$ and $\omega(i, 1)$ are homogeneous, like all the involved $I(\Gamma, B)$, so that $Z_n(L, \check{M}, \tau)$ is independent from the chosen homogeneous propagating forms $\omega(i)$ of $C_2(M, \tau)$ in Theorem 4.7. Now, assume that L is straight.

When $i \notin j_E(E(\Gamma))$, the integrand of $I(\Gamma, B)$ factors through the natural projection of $[0, 1] \times F(\Gamma, B)$ onto $F(\Gamma, B)$, and $I(\Gamma, B) = 0$, consequently. Assume $i = j_E(e_i \in E(\Gamma))$, then

$$I(\Gamma, B) = \beta_\Gamma \int_{[0,1] \times F(\Gamma, B)} p_{e_i}^*(d(t\eta)) \wedge \bigwedge_{e \in E(\Gamma) \setminus e_i} p_e^*(\omega(j_E(e)).$$

The form $\bigwedge_{e \in E(\Gamma_B)} p_e^*(\omega(j_E(e))$ pulls back through $[0, 1] \times F(\Gamma_B, B)$, and through $F(\Gamma_B, B)$ when $e_i \notin E(\Gamma_B)$, so that, for dimension reasons, $I(\Gamma, B)$ vanishes unless $e_i \in E(\Gamma_B)$. Therefore, we assume $e_i \in E(\Gamma_B)$.

When B contains no univalent vertices, $I(\Gamma, B)$ factors through

$$\int_{[0,1] \times \cup_{m(b) \in \check{M}} \check{S}_B(T_{m(b)} M)} p_{e_i}^*(d(t\eta)) \bigwedge_{e \in E(\Gamma_B)} p_e^*(\omega(j_E(e))).$$

Here the parallelization τ identifies the bundle $\cup_{m(b) \in \check{M}} \check{S}_B(T_{m(b)} M)$ with $\check{M} \times \check{S}_B(\mathbb{R}^3)$, and the integrand factors through the projection of $[0, 1] \times \check{M} \times \check{S}_B(\mathbb{R}^3)$ onto $[0, 1] \times \check{S}_B(\mathbb{R}^3)$ whose dimension is smaller (by 3). In particular, $I(\Gamma, B) = 0$ in this case, the independence of the choice of the $\omega(i)$ is proved when $k = 0$ (when the link is empty), and $Z(\check{M}, \tau)$ coincides with the Kontsevich configuration space integral invariant described in [Les04a].

Let us now study the sum of the $I(\Gamma, B)$, where $(\Gamma \setminus \Gamma_B)$ is a fixed labeled graph and Γ_B is a fixed numbered connected diagram with at least one univalent vertex on S^1_j.

This sum factors through $\int_{[0,1]\times\cup_{m(b)\in K_j}\check{S}_B(T_{m(b)}M,\Gamma)} p^*_{e_i}(d(t\eta))\bigwedge_{e\in E(\Gamma_B)\setminus e_i} p^*_e(\omega(j_E(e)))$.

At a collapse, the univalent vertices of Γ_B are equipped with a linear order that make it a numbered graph $\tilde{\Gamma}_B$ on $\mathbb{R}$. The map p_τ induces a map $p_{a,\tau}$ from a connected component of $[0,1] \times \cup_{m(b)\in K_j}\check{S}_B(T_{m(b)}M,\Gamma)$ that corresponds to the $\tilde{\Gamma}_B$-linear order to $[0,1] \times \check{Q}(\tilde{\Gamma}_B)$ through which the integrand factors ($\check{Q}(\tilde{\Gamma}_B)$ was defined in Subsection 5.4). Furthermore, $p_{a,\tau}\left(\cup_{m(b)\in K_j}\check{S}_B(T_{m(b)}M,\Gamma)\right)$ bounds the restriction of the bundle $\check{Q}(\tilde{\Gamma}_B)$ over a region bounded by $p_\tau(U^+K_j)$. When K_j is straight, there is such a region with empty interior. Thus, the Stokes theorem ensures that $I(\Gamma, B) = 0$, in this case, too. $\diamond$

Now, Theorem 5.6 is a corollary of Theorem 4.7 (that is not yet completely proved).

5.6 The dependence on the parallelizations in the invariance proofs

Recall that $\mathcal{A}^t_n(C)$ splits according to the number of connected components without univalent vertices of the graphs. Then it is easy to observe that

$$Z(L,\check{M},\tau) = \sum_{n\in\mathbb{N}} Z_n(L,\check{M},\tau) = Z^u(L,\check{M},\tau)Z(M;\tau)$$

where Z^u is obtained from Z by sending the graphs with components that have no univalent vertices to 0, and $Z(M;\tau) = Z(\emptyset,\check{M},\tau)$. According to [Les04a, Theorem 1.9], $Z(M) = Z(M;\tau)\exp(\frac{1}{4}p_1(\tau)\xi)$ is a topological invariant of M. Here, we shall now focus on $Z^u(L,\check{M},\tau)$, and define it with a given homogeneous propagating form, $\omega = \omega(i)$ for all i, so that $Z^u(L,\check{M},\tau)$ is an invariant of the diffeomorphism class of $(L,\check{M},\tau)$. We study its variation under a continuous deformation of τ and we prove the following lemma.

Lemma 5.9 *Let $(\tau(t))_{t\in[0,1]}$ define a smooth homotopy of asymptotically standard parallelizations of $\check{M}$.*

$$\frac{\partial}{\partial t}Z^u(L,\check{M},\tau(t)) = (\sum_{j=1}^k \frac{\partial}{\partial t}I_\theta(K_j,\tau(t))\alpha\sharp_j)Z^u(L,\check{M},\tau(t)).$$

PROOF: Set $Z_n(t) = Z^u_n(L,\check{M},\tau(t))$, observe that Z_n (that is valued in a finite-dimensional vector space) is differentiable thanks to the expression of $Z_n(t) - Z_n(0)$ in Lemma 5.8 (any function $\int_{[0,t]\times C}\omega$ for a smooth compact manifold C and a smooth form ω on $[0,1]\times C$ is differentiable with respect to t). Now, the forms associated to edges of Γ_B do not depend on the configuration of $(V(\Gamma)\setminus B)$. They will be integrated along $[0,1]\times(\cup_{m(b)\in K_j}\check{S}_B(T_{m(b)}M,\Gamma_B))$, while the other ones will be integrated along $\check{C}(L;\Gamma\setminus\Gamma_B)$ at $u\in[0,1]$.

Therefore, the global variation $(Z(t) - Z(0))$ reads

$$\sum_{j=1}^k \int_0^t \left(\sum_{\Gamma_B\in\mathcal{D}^c(\mathbb{R})} \beta_{\Gamma_B}\int_{c\in\cup_{m(b)\in K_j}\check{S}_B(T_{m(b)}M,\Gamma_B)}\left(\bigwedge_{e\in E(\Gamma_B)} p^*_e(\omega_{S^2})\right)(u,c)[\Gamma_B]\sharp_j\right)Z(u)du$$

where $\mathcal{D}^c(\mathbb{R}) = \cup_{n\in\mathbb{N}}\mathcal{D}^c_n(\mathbb{R})$. Define

$$I(\Gamma_B, K_j)(t) = \int_{(u,c);u\in[0,t],c\in\cup_{m(b)\in K_j}\check{S}_B(T_{m(b)}M,\Gamma_B)} \bigwedge_{e\in E(\Gamma_B)} p_e^*(\omega_{S^2})(u,c).$$

Since $I(\Gamma_B, K_j)(t) = \int_0^t \frac{\partial}{\partial u} I(\Gamma_B, K_j)(u)du$, $\frac{\partial}{\partial u} I(\Gamma_B, K_j)(u)du$ is the integral of $\bigwedge_{e\in E(\Gamma_B)} p_e^*(\omega)$ along $u \times (\cup_{m(b)\in K_j}\check{S}_B(T_{m(b)}M,\Gamma_B))$ and

$$Z(t) - Z(0) = \int_0^t \left(\sum_{j=1}^k \left(\sum_{\Gamma_B\in\mathcal{D}^c(\mathbb{R})} \beta_{\Gamma_B}\frac{\partial}{\partial u} I(\Gamma_B, K_j)(u)[\Gamma_B]\sharp_j\right) Z(u)\right) du.$$

Therefore,

$$\frac{\partial}{\partial t} Z(t) = \sum_{j=1}^k \left(\sum_{\Gamma_B\in\mathcal{D}^c(\mathbb{R})} \beta_{\Gamma_B}\frac{\partial}{\partial t} I(\Gamma_B, K_j)(t)[\Gamma_B]\sharp_j\right) Z(t)$$

and we are left with the computation of $\frac{\partial}{\partial t} I(\Gamma_B, K_j)(t)$.

The restriction of $p_{\tau(.)}$ from $[0,1] \times U^+K_j$ to S^2 induces a map

$$p_{a,\tau,\Gamma_B} \colon [0,1] \times \cup_{m(b)\in K_j}\check{S}_B(T_{m(b)}M,\Gamma_B) \to \check{Q}(\Gamma_B)$$

for any Γ_B.

$$I(\Gamma_B, K_j)(t) = \int_{\mathrm{Im}(p_{a,\tau,\Gamma_B})} \bigwedge_{e\in E(\Gamma_B)} p_e^*(\omega_{S^2}).$$

Integrating $\bigwedge_{e\in E(\Gamma_B)} p_e^*(\omega_{S^2})[\Gamma_B]$ along the fiber in $\check{Q}(\Gamma_B)$ yields a two form on S^2 that is homogeneous, because everything is. Thus this form reads $2\alpha(\Gamma_B)\omega_{S^2}[\Gamma_B]$ where $\alpha(\Gamma_B) \in \mathbb{R}$, and where $\sum_{\Gamma_B\in\mathcal{D}^c(\mathbb{R})} \beta_{\Gamma_B}\alpha(\Gamma_B)[\Gamma_B] = \alpha$. Therefore

$$I(\Gamma_B, K_j)(t) = 2\beta_{\Gamma_B}\alpha(\Gamma_B)\int_{[0,t]\times U^+K_j} p_{\tau(.)}^*(\omega_{S^2}).$$

Since $\frac{\partial}{\partial t}\int_{[0,t]\times U^+K_j} p_{\tau(.)}^*(\omega_{S^2}) = \frac{1}{2}\frac{\partial}{\partial t} I_\theta(K_j, \tau(t))$, we conclude easily. $\diamond$

Then the derivative of $\prod_{j=1}^k \exp(-I_\theta(K_j,\tau(t))\alpha)\sharp_j Z(L,\check{M},\tau(t))$ vanishes so that

$$\prod_{j=1}^k \exp(-I_\theta(K_j,\tau(t))\alpha)\sharp_j Z(L,\check{M},\tau(t))$$

does not change when τ smoothly varies.

5.7 End of the proof of Theorem 4.7

Now, see what happens when the homotopy class of τ changes. When it changes in a ball that does not meet the link, the forms can be changed only in the neighborhoods of the unit tangent bundle of this ball. Using Lemma 5.8 again, the variation will be seen on the faces $F(\Gamma, B)$ where the forms associated to the edges of Γ_B do not depend on the parameter in $[0,1]$ so that their product vanishes. In particular, the expression

$$\exp(\frac{1}{4}p_1(\tau)\xi)\prod_{j=1}^{k}(\exp(-I_\theta(K_j,\tau)\alpha)\sharp_j)\, Z(L,\check{M},\tau)$$

of Theorem 4.7 is invariant under the natural action of $\pi_3(SO(3))$ on the homotopy classes of parallelizations.

We now examine the effect of the twist of the parallelization by a map $g\colon(B_M,1)\to(SO(3),1)$. Without loss, assume that $p_\tau(U^+K_j)=v$ for some v of S^2 and that g maps K_j to rotations with axis v. We want to compute $Z(L,\check{M},\tau\circ\psi(g))-Z(L,\check{M},\tau)$. Identify UB_M with $B_M\times S^2$ via τ. There exists a form ω on $[0,1]\times B_M\times S^2$ that reads $p_\tau^*(\omega_{S^2})$ on $\partial([0,1]\times B_M\times S^2)\setminus(1\times B_M\times S^2))$ and that reads $p^*_{\tau\circ\psi(g)}(\omega_{S^2})$ on $1\times B_M\times S^2$. Extend this form to a form Ω on $[0,1]\times C_2(M)$, that restricts to $0\times\partial C_2(M)$ as $p_\tau^*(\omega_{S^2})$, and to $1\times\partial C_2(M)$ as $p^*_{\tau\circ\psi(g)}(\omega_{S^2})$. Define

$$Z_n(t)=\sum_{\Gamma\in\mathcal{D}^e_n(C)}\beta_\Gamma I(\Gamma,(\Omega_{|t\times C_2(M)})_{i\in\underline{3n}})[\Gamma]\in\mathcal{A}^t_n(\coprod_{j=1}^{k}S^1_j).$$

Set

$$I(\Gamma_B,K_j,\Omega)(t)=\int_{(u,c);u\in[0,t],c\in\cup_{m(b)\in K_j}\check{S}_B(T_{m(b)}M,\Gamma_B)}\bigwedge_{e\in E(\Gamma_B)}p_e^*(\Omega)[\Gamma_B],$$

$\beta_j(t)=\sum_{\Gamma_B\in\mathcal{D}^c(\mathbb{R})}\beta_{\Gamma_B}I(\Gamma_B,K_j,\Omega)(t)$ and $\gamma_j(t)=\frac{\partial}{\partial t}\beta_j(t)$. Thanks to Lemma 5.8, like in the proof of Lemma 5.9, $Z(t)$ is differentiable, and $Z'(t)=(\sum_{j=1}^k\gamma_j(t)\sharp_j)Z(t)$.

By induction on the degree, it is easy to see that this equation determines $Z(t)$ as a function of the $\beta_j(t)$ and $Z(0)$ whose degree 0 part is 1, and that $Z(t)=\prod_{j=1}^k\exp(\beta_j(t))\sharp_j Z(0)$.

Extend Ω over $[0,2]\times C_2(M)$ so that its restriction to $[1,2]\times B_M\times S^2$ is obtained by applying $\psi(g)^*$ to the Ω translated, and extend all the introduced maps, then $\gamma_j(t+1)=\gamma_j(t)$ because everything is carried by $\psi(g)^*$. In particular $\beta_j(2)=2\beta_j(1)$.

Now, $Z(2)=Z(M,L,\tau\circ\psi(g)^2)=\prod_{j=1}^k\exp((I_\theta(K_j,\tau\circ\psi(g)^2)-I_\theta(K_j,\tau))\alpha)\sharp_j Z(L,\check{M},\tau)$, since g^2 is homotopic to the trivial map outside a ball (see Lemma 6.1, 2). Again, by induction on the degree, this shows $\beta_j(2)=(I_\theta(K_j,\tau\circ\psi(g)^2)-I_\theta(K_j,\tau))\alpha$. Conclude by observing that under our assumptions,

$$I_\theta(K_j,\tau\circ\psi(g)^2)-I_\theta(K_j,\tau)=2(I_\theta(K_j,\tau\circ\psi(g))-I_\theta(K_j,\tau)).$$

This finishes the proof of Theorem 4.7 in general.

5.8 Some open questions

1. A Vassiliev invariant is *odd* if it distinguishes some knot from the same knot with the opposite orientation. Are there odd Vassiliev invariants ?

2. More generally, do Vassiliev invariants distinguish knots in S^3 ?

3. According to a theorem of Bar-Natan and Lawrence [BNL04], the LMO invariant fails to distinguish rational homology spheres with isomorphic H_1, so that, according to a Moussard theorem [Mou12], rational finite type invariants fail to distinguish $\mathbb{Q}$-spheres. Do finite type invariants distinguish $\mathbb{Z}$-spheres ?

4. Find relationships between Z or other finite type invariants and Heegaard Floer homologies. See [Les12] to get propagators associated with Heegaard diagrams.

5. Compare Z with the LMO invariant Z_{LMO}.

6. Compute the anomalies α and ξ.

7. Find surgery formulae for Z.

8. Kricker defined a lift $\tilde{Z}^K$ of the Kontsevich integral Z^K (or the LMO invariant) for null-homologous knots in $\mathbb{Q}$-spheres [Kri00, GK04]. The Kricker lift is valued in a space $\tilde{A}$ that is mapped to $\mathcal{A}_n(S^1)$ by a map H. It satisfies $Z^K = H \circ \tilde{Z}^K$. The space $\tilde{A}$ is a space of trivalent diagrams whose edges are decorated by rational functions whose denominators divide the Alexander polynomial. Compare the Kricker lift $\tilde{Z}^K$ with the equivariant configuration space invariant $\tilde{Z}^c$ of [Les11] valued in the same diagram space $\tilde{A}$.

9. Does $Z = H \circ \tilde{Z}^c$ also hold ?

6 More on parallelizations of 3-manifolds and Pontrjagin classes

In this section, M is a smooth oriented connected 3-manifold with possible boundary.

6.1 $[(M,\partial M),(SO(3),1)]$ is an abelian group.

Again, see S^3 as $B^3/\partial B^3$ and see B^3 as $([0,2\pi]\times S^2)/(0\sim\{0\}\times S^2)$. Recall that $\rho\colon B^3\to SO(3)$ maps $(\theta\in[0,2\pi], v\in S^2)$ to the rotation $\rho(\theta,v)$ with axis directed by v and with angle θ.

Also recall that the group structure of $[(M,\partial M),(SO(3),1)]$ is induced by the multiplication of maps, using the multiplication of $SO(3)$.

Any $g\in[(M,\partial M),(SO(3),1)]_m$ induces a map

$$H_1(g;\mathbb{Z})\colon H_1(M,\partial M;\mathbb{Z})\longrightarrow (H_1(SO(3),1)=\mathbb{Z}/2\mathbb{Z}).$$

Since

$$H_1(M,\partial M;\mathbb{Z}/2\mathbb{Z})=H_1(M,\partial M;\mathbb{Z})/2H_1(M,\partial M;\mathbb{Z}))=H_1(M,\partial M;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z},$$

$\mathrm{Hom}(H_1(M,\partial M;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})=\mathrm{Hom}(H_1(M,\partial M;\mathbb{Z}/2\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})=H^1(M,\partial M;\mathbb{Z}/2\mathbb{Z})$, and the image of $H_1(g;\mathbb{Z})$ under the above isomorphisms is denoted by $H^1(g;\mathbb{Z}/2\mathbb{Z})$. (Formally, this $H^1(g;\mathbb{Z}/2\mathbb{Z})$ denotes the image of the generator of $H^1(SO(3),1;\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}$ under $H^1(g;\mathbb{Z}/2\mathbb{Z})$ in $H^1(M,\partial M;\mathbb{Z}/2\mathbb{Z})$.)

Lemma 6.1 *Let M be an oriented connected 3-manifold with possible boundary. Recall that $\rho_M(B^3)\in[(M,\partial M),(SO(3),1)]_m$ is a map that coincides with ρ on a ball B^3 embedded in M and that maps the complement of B^3 to the unit of $SO(3)$.*

1. *Any homotopy class of a map g from $(M,\partial M)$ to $(SO(3),1)$, such that $H^1(g;\mathbb{Z}/2\mathbb{Z})$ is trivial, belongs to the subgroup $<[\rho_M(B^3)]>$ of $[(M,\partial M),(SO(3),1)]$ generated by $[\rho_M(B^3)]$.*
2. *For any $[g]\in[(M,\partial M),(SO(3),1)]$, $[g]^2\in<[\rho_M(B^3)]>$.*
3. *The group $[(M,\partial M),(SO(3),1)]$ is abelian.*

Proof: Let $g\in[(M,\partial M),(SO(3),1)]_m$. Assume that $H^1(g;\mathbb{Z}/2\mathbb{Z})$ is trivial. Choose a cell decomposition of M with respect to its boundary, with only one three-cell, no zero-cell if $\partial M\neq\emptyset$, one zero-cell if $\partial M=\emptyset$, one-cells, and two-cells. Then after a homotopy relative to ∂M, we may assume that g maps the one-skeleton of M to 1. Next, since $\pi_2(SO(3))=0$, we may assume that g maps the two-skeleton of M to 1, and therefore that g maps the exterior of some 3-ball to 1. Now g becomes a map from $B^3/\partial B^3=S^3$ to $SO(3)$, and its homotopy class is $k[\tilde\rho]$ in $\pi_3(SO(3))=\mathbb{Z}[\tilde\rho]$. Therefore g is homotopic to $\rho_M(B^3)^k$. This proves the first assertion.

Since $H^1(g^2;\mathbb{Z}/2\mathbb{Z}) = 2H^1(g;\mathbb{Z}/2\mathbb{Z})$ is trivial, the second assertion follows.

For the third assertion, first note that $[\rho_M(B^3)]$ belongs to the center of $[(M,\partial M),(SO(3),1)]$ because it can be supported in a small ball disjoint from the support (preimage of $SO(3)\setminus\{1\}$) of a representative of any other element. Therefore, according to the second assertion any square will be in the center. Furthermore, since any commutator induces the trivial map on $\pi_1(M)$, any commutator is in $< [\rho_M(B^3)] >$. In particular, if f and g are elements of $[(M,\partial M),(SO(3),1)]$,

$$(gf)^2 = (fg)^2 = (f^{-1}f^2g^2f)(f^{-1}g^{-1}fg)$$

where the first factor equals $f^2g^2 = g^2f^2$. Exchanging f and g yields $f^{-1}g^{-1}fg = g^{-1}f^{-1}gf$. Then the commutator that is a power of $[\rho_M(B^3)]$ has a vanishing square, and thus a vanishing degree. Then it must be trivial. $\diamond$

6.2 Any oriented 3-manifold is parallelizable.

In this subsection, we prove the following standard theorem. The spirit of our proof is the same as the Kirby proof in [Kir89, p.46]. But instead of assuming familiarity with the obstruction theory described by Steenrod in [Ste51, Part III], we use this proof as an introduction to this theory.

Theorem 6.2 (Stiefel) *Any oriented 3-manifold is parallelizable.*

Lemma 6.3 *The restriction of the tangent bundle TM of an oriented 3-manifold M to any closed (non-necessarily orientable) surface S immersed in M is trivialisable.*

PROOF: Let us first prove that this bundle is independent of the immersion. It is the direct sum of the tangent bundle of the surface and of its normal one-dimensional bundle. This normal bundle is trivial when S is orientable, and its unit bundle is the 2-fold orientation cover of the surface, otherwise. (The orientation cover of S is its 2-fold orientable cover that is trivial over annuli embedded in the surface). Then since any surface S can be immersed in $\mathbb{R}^3$, the restriction $TM_{|S}$ is the pull-back of the trivial bundle of $\mathbb{R}^3$ by such an immersion, and it is trivial. $\diamond$

Then using Stiefel-Whitney classes, the proof of Theorem 6.2 quickly goes as follows. Let M be an orientable smooth 3-manifold, equipped with a smooth triangulation. (A theorem of Whitehead proved in the Munkres book [Mun66] ensures the existence of such a triangulation.) By definition, the *first Stiefel-Whitney class* $w_1(TM) \in H^1(M;\mathbb{Z}/2\mathbb{Z} = \pi_0(GL(\mathbb{R}^3)))$ seen as a map from $\pi_1(M)$ to $\mathbb{Z}/2\mathbb{Z}$ maps the class of a loop c embedded in M to 0 if $TM_{|c}$ is orientable and to 1 otherwise. It is the obstruction to the existence of a trivialisation of TM over the one-skeleton of M. Since M is orientable, the first Stiefel-Whitney class $w_1(TM)$ vanishes and TM can be trivialised over the one-skeleton of M. The *second Stiefel-Whitney class* $w_2(TM) \in H^2(M;\mathbb{Z}/2\mathbb{Z} = \pi_1(GL^+(\mathbb{R}^3)))$ seen as a map from $H_2(M;\mathbb{Z}/2\mathbb{Z})$ to $\mathbb{Z}/2\mathbb{Z}$ maps the class of a connected closed surface S to 0 if $TM_{|S}$ is trivialisable and to 1 otherwise. The second Stiefel-Whitney class $w_2(TM)$ is the obstruction to the existence of a trivialisation of TM over the two-skeleton of M, when $w_1(TM) = 0$. According to the above lemma, $w_2(TM) = 0$, and TM can be trivialised over the two-skeleton of M. Then since $\pi_2(GL^+(\mathbb{R}^3)) = 0$, any parallelization over the two-skeleton of M can be extended as a parallelization of M. $\diamond$

We detail the involved arguments below without mentioning Stiefel-Whitney classes, (actually by almost defining $w_2(TM)$). The elementary proof below can be thought of as an introduction to the obstruction theory used above.

Elementary proof of Theorem 6.2: Let M be an oriented 3-manifold. Choose a triangulation of M. For any cell c of the triangulation, define an arbitrary trivialisation $\tau_c\colon c\times\mathbb{R}^3 \to TM_{|c}$ such that τ_c induces the orientation of M. This defines a trivialisation $\tau^{(0)}\colon M^{(0)}\times\mathbb{R}^3 \to TM_{|M^{(0)}}$ of M over the 0-skeleton $M^{(0)}$ of M. Let $C_k(M)$ be the set of k–cells of the triangulation. Every cell is equipped with an arbitrary orientation. For an edge $e \in C_1(M)$ of the triangulation, on ∂e, $\tau^{(0)}$ reads $\tau^{(0)} = \tau_e \circ \psi_{\mathbb{R}}(g_e)$ for a map $g_e\colon \partial e \to GL^+(\mathbb{R}^3)$. Since $GL^+(\mathbb{R}^3)$ is connected, g_e extends to e, and $\tau^{(1)} = \tau_e \circ \psi_{\mathbb{R}}(g_e)$ extends $\tau^{(0)}$ to e. Doing so for all the edges extends $\tau^{(0)}$ to a trivialisation $\tau^{(1)}$ of the one-skeleton $M^{(1)}$ of M.

For an oriented triangle t of the triangulation, on ∂t, $\tau^{(1)}$ reads $\tau^{(1)} = \tau_t \circ \psi_{\mathbb{R}}(g_t)$ for a map $g_t\colon \partial t \to GL^+(\mathbb{R}^3)$. Let $E(t, \tau^{(1)})$ be the homotopy class of g_t in $(\pi_1(GL^+(\mathbb{R}^3)) = \pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z})$, $E(t, \tau^{(1)})$ is independent of τ_t. Then $E(., \tau^{(1)})\colon C_2(M) \to \mathbb{Z}/2\mathbb{Z}$ is a cochain. When $E(., \tau^{(1)}) = 0$, $\tau^{(1)}$ may be extended to a trivialisation $\tau^{(2)}$ over the two-skeleton of M, as before.

Since $\pi_2(GL^+(\mathbb{R}^3)) = 0$, $\tau^{(2)}$ can next be extended over the three-skeleton of M, that is over M.

Let us now study the obstruction cochain $E(., \tau^{(1)})$ whose vanishing guarantees the existence of a parallelization of M.

If the map g_e associated to e is changed to $d(e)g_e$ for some $d(e)\colon (e, \partial e) \to (GL^+(\mathbb{R}^3), 1)$ for every edge e, define the associated trivialisation $\tau^{(1)\prime}$, and the cochain $D(\tau^{(1)}, \tau^{(1)\prime})\colon C_1(M) \to \mathbb{Z}/2\mathbb{Z}$ that maps e to the homotopy class of $d(e)$. Then $(E(., \tau^{(1)\prime}) - E(., \tau^{(1)}))$ is the coboundary of $D(\tau^{(1)}, \tau^{(1)\prime})$.

Let us show that $E(., \tau^{(1)})$ is a cocycle. Consider a 3-simplex T, then $\tau^{(0)}$ extends to T. Without loss of generality, assume that τ_T coincides with this extension, that for any face t of T, τ_t is the restriction of τ_T to t, and that the above $\tau^{(1)\prime}$ coincides with τ_T on the edges of ∂T. Then $E(., \tau^{(1)\prime})(\partial T) = 0$. Since a coboundary also maps ∂T to 0, $E(., \tau^{(1)})(\partial T) = 0$.

Now, it suffices to prove that the cohomology class of $E(., \tau^{(1)})$ (that is actually $w_2(TM)$) vanishes in order to prove that there is an extension $\tau^{(1)\prime}$ of $\tau^{(0)}$ on $M^{(1)}$ that extends on M.

Since $H^2(M; \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(H_2(M; \mathbb{Z}/2\mathbb{Z})); \mathbb{Z}/2\mathbb{Z})$, it suffices to prove that $E(., \tau^{(1)})$ maps any 2–dimensional $\mathbb{Z}/2\mathbb{Z}$-cycle C to 0.

We represent the class of such a cycle C by a non-necessarily orientable closed surface S as follows. Let $N(M^{(0)})$ and $N(M^{(1)})$ be small regular neighborhoods of $M^{(0)}$ and $M^{(1)}$ in M, respectively, such that $N(M^{(1)}) \cap (M \setminus N(M^{(0)}))$ is a disjoint union, running over the edges

e, of solid cylinders B_e identified with $]0,1[\times D^2$. The core $]0,1[\times\{0\}$ of $B_e =]0,1[\times D^2$ is a connected part of the interior of the edge e. ($N(M^{(1)})$ is thinner than $N(M^{(0)})$.)

Construct S in the complement of $N(M^{(0)}) \cup N(M^{(1)})$ as the intersection of the support of C with this complement. Then the closure of S meets the part $[0,1] \times S^1$ of every $\overline{B_e}$ as an even number of parallel intervals from $\{0\} \times S^1$ to $\{1\} \times S^1$. Complete S in $M \setminus N(M^{(0)})$ by connecting the intervals pairwise in $\overline{B_e}$ by disjoint bands. After this operation, the boundary of the closure of S is a disjoint union of circles in the boundary of $N(M^{(0)})$, where $N(M^{(0)})$ is a disjoint union of balls around the vertices. Glue disjoint disks of $N(M^{(0)})$ along these circles to finish the construction of S.

Extend $\tau^{(0)}$ to $N(M^{(0)})$, assume that $\tau^{(1)}$ coincides with this extension over $M^{(1)} \cap N(M^{(0)})$, and extend $\tau^{(1)}$ to $N(M^{(1)})$. Then $TM_{|S}$ is trivial, and we may choose a trivialisation τ_S of TM over S that coincides with our extension of $\tau^{(0)}$ over $N(M^{(0)})$, over $S \cap N(M^{(0)})$. We have a cell decomposition of $(S, S \cap N(M^{(0)}))$ with only 1-cells and 2-cells, where the 2-cells of S are in one-to-one canonical correspondence with the 2-cells of C, and one-cells bijectively correspond to bands connecting two-cells in the cylinders B_e. These one-cells are equipped with the trivialisation of TM induced by $\tau^{(1)}$. Then we can define 2–dimensional cocycles $E_S(.,\tau^{(1)})$ and $E_S(.,\tau_S)$ as before, with respect to this cellular decomposition of S, where $(E_S(.,\tau^{(1)}) - E_S(.,\tau_S))$ is again a coboundary and $E_S(.,\tau_S) = 0$ so that $E_S(C,\tau^{(1)}) = 0$, and since $E(C,\tau^{(1)}) = E_S(C,\tau^{(1)})$, $E(C,\tau^{(1)}) = 0$ and we are done. $\diamond$

6.3 The homomorphism induced by the degree

Let S be a non-necessarily orientable closed surface embedded in the interior of M, and let τ be a parallelization of M. We define a twist $g(S,\tau) \in [(M,\partial M),(SO(3),1)]_m$ below.

The surface S has a tubular neighborhood $N(S)$ that is a $[-1,1]$–bundle over S that admits (orientation-preserving) bundle charts with domains $[-1,1] \times D$ for disks D of S so that the changes of coordinates restrict to the fibers as $\pm$Identity. Then

$$g(S,\tau)\colon (M,\partial M) \longrightarrow (GL^+(\mathbb{R}^3),1)$$

is the continuous map that maps $M \setminus N(S)$ to 1 such that $g(S,\tau)((t,s) \in [-1,1] \times D)$ is the rotation with angle $\pi(t+1)$ and with axis $p_2(\tau^{-1}(N_s) = (s, p_2(\tau^{-1}(N_s))))$ where $N_s = T_{(0,s)}([-1,1] \times s)$ is the tangent vector to the fiber $[-1,1] \times s$ at $(0,s)$. Since this rotation coincides with the rotation with opposite axis and with opposite angle $\pi(1-t)$, our map $g(S,\tau)$ is a well-defined continuous map.

Clearly, the homotopy class of $g(S,\tau)$ only depends on the homotopy class of τ and on the isotopy class of S. When $M = B^3$, when τ is the standard parallelization of $\mathbb{R}^3$, and when $\frac{1}{2}S^2$ denotes the sphere $\frac{1}{2}\partial B^3$ inside B^3, the homotopy class of $g(\frac{1}{2}S^2,\tau)$ coincides with the homotopy class of ρ.

Lemma 6.4 *$H^1(g(S,\tau);\mathbb{Z}/2\mathbb{Z})$ is the mod 2 intersection with S.*
$H^1(.;\mathbb{Z}/2\mathbb{Z})\colon [(M,\partial M),(SO(3),1)] \to H^1(M,\partial M;\mathbb{Z}/2\mathbb{Z})$ is onto.

PROOF: The first assertion is obvious, and the second one follows since $H^1(M,\partial M;\mathbb{Z}/2\mathbb{Z})$ is the Poincaré dual of $H_2(M;\mathbb{Z}/2\mathbb{Z})$ and since any element of $H_2(M;\mathbb{Z}/2\mathbb{Z})$ is the class of a closed surface. ◇

Lemma 6.5 *The degree is a group homomorphism*

$$deg\colon [(M,\partial M),(SO(3),1)] \longrightarrow \mathbb{Z}$$

and $deg(\rho_M(B^3)^k) = 2k$.

PROOF: It is easy to see that $\deg(fg) = \deg(f)+\deg(g)$ when f or g is a power of $[\rho_M(B^3)]$. Let us prove that $\deg(f^2) = 2\deg(f)$ for any f. According to Lemma 6.4, there is an unoriented embedded surface S_f of the interior of C such that $H^1(f;\mathbb{Z}/2\mathbb{Z}) = H^1(g(S_f,\tau);\mathbb{Z}/2\mathbb{Z})$ for some trivialisation τ of TM. Then, according to Lemma 6.1, $fg(S_f,\tau)^{-1}$ is homotopic to some power of $\rho_M(B^3)$, and we are left with the proof that the degree of g^2 is $2\deg(g)$ for $g = g(S_f,\tau)$. This can easily be done by noticing that g^2 is homotopic to $g(S_f^{(2)},\tau)$ where $S_f^{(2)}$ is the boundary of the tubular neighborhood of S_f. In general, $\deg(fg) = \frac{1}{2}\deg((fg)^2) = \frac{1}{2}\deg(f^2g^2) = \frac{1}{2}\left(\deg(f^2)+\deg(g^2)\right)$, and the lemma is proved. ◇

Lemmas 6.1 and 6.5 imply the following lemma.

Lemma 6.6 *The degree induces an isomorphism*

$$deg\colon [(M,\partial M),(SO(3),1)] \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}.$$

Any group homomorphism $\psi\colon [(M,\partial M),(SO(3),1)] \longrightarrow \mathbb{Q}$ *reads* $\frac{1}{2}\psi(\rho_M(B^3))\,deg$.

◇

6.4 On the groups $SU(n)$

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $n \in \mathbb{N}$. The stabilisation maps induced by the inclusions

$$\begin{array}{rccl} i: & GL(\mathbb{K}^n) & \longrightarrow & GL(\mathbb{K}\oplus\mathbb{K}^n) \\ & g & \mapsto & (i(g):(x,y)\mapsto(x,g(y))) \end{array}$$

will be denoted by i. Elements of $GL(\mathbb{K}^n)$ are represented by matrices whose columns contain the coordinates of the images of the basis elements, with respect to the standard basis of $\mathbb{K}^n$

See S^3 as the unit sphere of $\mathbb{C}^2$ so that its elements are the pairs (z_1,z_2) of complex numbers such that $|z_1|^2+|z_2|^2 = 1$.

The group $SU(2)$ is identified with S^3 by the homeomorphism

$$\begin{array}{rccl} m_r^{\mathbb{C}}\colon & S^3 & \to & SU(2) \\ & (z_1,z_2) & \mapsto & \left[\begin{array}{cc} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{array}\right] \end{array}$$

so that the first non trivial homotopy group of $SU(2)$ is

$$\pi_3(SU(2)) = \mathbb{Z}[m_r^{\mathbb{C}}].$$

The long exact sequence associated to the fibration

$$SU(n-1) \overset{i}{\hookrightarrow} SU(n) \rightarrow S^{2n-1}$$

shows that $i_*^n : \pi_j(SU(2)) \longrightarrow \pi_j(SU(n+2))$ is an isomorphism for $j \leq 3$ and $n \geq 0$, and in particular, that $\pi_j(SU(4)) = \{1\}$ for $j \leq 2$ and

$$\pi_3(SU(4)) = \mathbb{Z}[i^2(m_r^{\mathbb{C}})]$$

where $i^2(m_r^{\mathbb{C}})$ is the following map

$$\begin{array}{cccc} i^2(m_r^{\mathbb{C}}) : & (S^3 \subset \mathbb{C}^2) & \longrightarrow & SU(4) \\ & (z_1, z_2) & \mapsto & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z_1 & -\overline{z}_2 \\ 0 & 0 & z_2 & \overline{z}_1 \end{bmatrix} \end{array}.$$

6.5 Definition of relative Pontrjagin numbers

Let M_0 and M_1 be two compact connected oriented 3-manifolds whose boundaries have collars that are identified by a diffeomorphism. Let $\tau_0 \colon M_0 \times \mathbb{R}^3 \rightarrow TM_0$ and $\tau_1 \colon M_1 \times \mathbb{R}^3 \rightarrow TM_1$ be two parallelizations (that respect the orientations) that agree on the collar neighborhoods of $\partial M_0 = \partial M_1$. Then the *relative Pontrjagin number* $p_1(\tau_0, \tau_1)$ is the Pontrjagin obstruction to extending the trivialisation of $TW \otimes \mathbb{C}$ induced by τ_0 and τ_1 across the interior of a signature 0 cobordism W from M_0 to M_1. Details follow.

Let M be a compact connected oriented 3-manifold. A *special complex trivialisation* of TM is a trivialisation of $TM \otimes \mathbb{C}$ that is obtained from a trivialisation $\tau_M \colon M \times \mathbb{R}^3 \rightarrow TM$ by composing $(\tau_M^{\mathbb{C}} = \tau_M \otimes \mathbb{C}) \colon M \times \mathbb{C}^3 \rightarrow TM \otimes \mathbb{C}$ by

$$\begin{array}{cccc} \psi(G) : & M \times \mathbb{C}^3 & \longrightarrow & M \times \mathbb{C}^3 \\ & (x, y) & \mapsto & (x, G(x)(y)) \end{array}$$

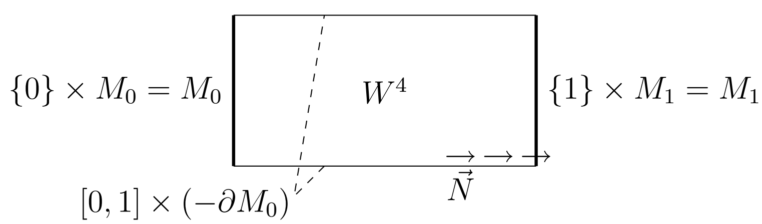
for a map $G \colon M \rightarrow SL(3, \mathbb{C})$. The definition and properties of relative Pontrjagin numbers that are given with more details below are valid for pairs of special complex trivialisations.

The *signature* of a 4-manifold is the signature of the intersection form on its $H_2(.;\mathbb{R})$ (number of positive entries minus number of negative entries in a diagonalised version of this form). Also recall that any closed oriented three-manifold bounds a compact oriented 4-dimensional manifold whose signature may be arbitrarily changed by connected sums with copies of $\mathbb{C}P^2$

or $-\mathbb{C}P^2$. A *cobordism from M_0 to M_1* is a compact oriented 4-dimensional manifold W with corners such that

$$\partial W = -M_0 \cup_{\partial M_0 \sim 0 \times \partial M_0} (-[0,1] \times \partial M_0) \cup_{\partial M_1 \sim 1 \times \partial M_0} M_1,$$

that is identified with an open subspace of one of the products $[0,1[\times M_0$ or $]0,1] \times M_1$ near ∂W, as the following picture suggests.



Let $W = W^4$ be such a cobordism from M_0 to M_1, with signature 0. Consider the complex 4-bundle $TW \otimes \mathbb{C}$ over W. Let $\vec{N}$ be the tangent vector to $[0,1] \times \{\mathrm{pt}\}$ over ∂W (under the identifications above), and let $\tau(\tau_0, \tau_1)$ denote the trivialisation of $TW \otimes \mathbb{C}$ over ∂W that is obtained by stabilizing either τ_0 or τ_1 into $\vec{N} \oplus \tau_0$ or $\vec{N} \oplus \tau_1$. Then the obstruction to extending this trivialization to W is the relative first *Pontrjagin class*

$$p_1(W; \tau(\tau_0, \tau_1))[W, \partial W] \in H^4(W, \partial W; \mathbb{Z} = \pi_3(SU(4))) = \mathbb{Z}[W, \partial W]$$

of the trivialisation.

Now, we specify our sign conventions for this Pontrjagin class. They are the same as in [MS74]. In particular, p_1 is the opposite of the second Chern class c_2 of the complexified tangent bundle. See [MS74, p. 174]. More precisely, equip M_0 and M_1 with Riemannian metrics that coincide near ∂M_0, and equip W with a Riemannian metric that coincides with the orthogonal product metric of one of the products $[0,1] \times M_0$ or $[0,1] \times M_1$ near ∂W. Equip $TW \otimes \mathbb{C}$ with the associated hermitian structure. The determinant bundle of TW is trivial because W is oriented, and $\det(TW \otimes \mathbb{C})$ is also trivial. Our parallelization $\tau(\tau_0, \tau_1)$ over ∂W is special with respect to the trivialisation of $\det(TW \otimes \mathbb{C})$. Up to homotopy, assume that $\tau(\tau_0, \tau_1)$ is unitary with respect to the hermitian structure of $TW \otimes \mathbb{C}$ and the standard hermitian form of $\mathbb{C}^4$. Since $\pi_i(SU(4)) = \{0\}$ when $i < 3$, the trivialisation $\tau(\tau_0, \tau_1)$ extends to a special unitary trivialisation τ outside the interior of a 4-ball B^4 and defines

$$\tau : S^3 \times \mathbb{C}^4 \longrightarrow (TW \otimes \mathbb{C})_{|S^3}$$

over the boundary $S^3 = \partial B^4$ of this 4-ball B^4. Over this 4-ball B^4, the bundle $TW \otimes \mathbb{C}$ is trivial and admits a trivialisation

$$\tau_B : B^4 \times \mathbb{C}^4 \longrightarrow (TW \otimes \mathbb{C})_{|B^4}.$$

Then $\tau_B^{-1}\circ\tau(v\in S^3, w\in\mathbb{C}^4)=(v,\phi(v)(w))$ where $\phi(v)\in SU(4)$, and the homotopy class of $\phi: S^3\longrightarrow SU(4)$ satisfies

$$[\phi]=-p_1(W;\tau(\tau_0,\tau_1))[i^2(m_r^{\mathbb{C}})]\in\pi_3(SU(4)).$$

Define $p_1(\tau_0,\tau_1)=p_1(W;\tau(\tau_0,\tau_1))$.

Proposition 6.7 *The first* Pontrjagin number $p_1(\tau_0,\tau_1)$ *is well-defined by the above conditions.*

Proof: According to the Nokivov additivity theorem, if a closed (compact, without boundary) 4-manifold Y reads $Y=Y^+\cup_X Y^-$ where Y^+ and Y^- are two 4-manifolds with boundary, embedded in Y that intersect along a closed 3–manifold X (their common boundary, up to orientation) then

$$\text{signature}(Y)=\text{signature}(Y^+)+\text{signature}(Y^-).$$

According to a Rohlin theorem (see [Roh52] or [GM86, p. 18]), when Y is a compact oriented 4–manifold without boundary, $p_1(Y)=3\ \text{signature}(Y)$.

We only need to prove that $p_1(\tau_0,\tau_1)$ is independent of the signature 0 cobordism W. Let W_E be a 4-manifold of signature 0 bounded by $(-\partial W)$. Then $W\cup_{\partial W}W_E$ is a 4-dimensional manifold without boundary whose signature is (signature(W_E) + signature$(W)=0$) by the Novikov additivity theorem. According to the Rohlin theorem, the first Pontrjagin class of $W\cup_{\partial W}W_E$ is also zero. On the other hand, this first Pontrjagin class is the sum of the relative first Pontrjagin classes of W and W_E with respect to $\tau(\tau_0,\tau_1)$. These two relative Pontrjagin classes are opposite and therefore the relative first Pontrjagin class of W with respect to $\tau(\tau_0,\tau_1)$ does not depend on W. ◇

Similarly, it is easy to prove the following proposition.

Proposition 6.8 *Under the above assumptions except for the assumption on the signature of the cobordism* W*,*

$$p_1(\tau_0,\tau_1)=p_1(W;\tau(\tau_0,\tau_1))-3\ \text{signature}(W).$$

6.6 On the groups $SO(3)$ and $SO(4)$

The *quaternion field* $\mathbb{H}$ is the vector space $\mathbb{C}\oplus\mathbb{C}j$ equipped with the multiplication that maps $(z_1+z_2j, z_1'+z_2'j)$ to $(z_1z_1'-z_2\overline{z_2'})+(z_2\overline{z_1'}+z_1z_2')j$, and with the conjugation that maps (z_1+z_2j) to $\overline{z_1+z_2j}=\overline{z_1}-z_2j$. The norm of (z_1+z_2j) is the square root of $|z_1|^2+|z_2|^2=(z_1+z_2j)\overline{z_1+z_2j}$, it is multiplicative. Setting $k=ij$, $(1,i,j,k)$ is an orthogonal basis of $\mathbb{H}$ with respect to the scalar product associated with the norm. The unit sphere of $\mathbb{H}$ is the sphere S^3 that is equipped with the corresponding group structure. There are two group morphisms from S^3 to $SO(4)$ induced by the multiplication in $\mathbb{H}$.

$$\begin{array}{rccl} m_\ell\colon & S^3 & \rightarrow & (SO(\mathbb{H})=SO(4))\\ & x & \mapsto & m_\ell(x)\colon v\mapsto x.v\end{array}$$

$$\begin{array}{rccl} \overline{m_r}\colon & S^3 & \longrightarrow & SO(\mathbb{H}) \\ & y & \longmapsto & (\overline{m_r}(y)\colon v \mapsto v.\overline{y}). \end{array}$$

Together, they induce the group morphism

$$\begin{array}{ccc} S^3 \times S^3 & \longrightarrow & SO(4) \\ (x,y) & \longmapsto & (v \mapsto x.v.\overline{y}). \end{array}$$

The kernel of this group morphism is $\mathbb{Z}/2\mathbb{Z}(-1,-1)$ so that this morphism is a two-fold covering. In particular,

$$\pi_3(SO(4)) = \mathbb{Z}[m_\ell] \oplus \mathbb{Z}[\overline{m_r}].$$

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $n \in \mathbb{N}$, the $\mathbb{K}$ (euclidean or hermitian) oriented vector space with the direct orthonormal basis $(v_1, \dots, v_n)$ is denoted by $\mathbb{K} < v_1, \dots, v_n >$. There is also the following group morphism

$$\begin{array}{rccl} \tilde{\rho}\colon & S^3 & \longrightarrow & SO(\mathbb{R} < i,j,k >) = SO(3) \\ & x & \longmapsto & (v \mapsto (v \mapsto x.v.\overline{x})) \end{array}$$

whose kernel is $\mathbb{Z}/2\mathbb{Z}(-1)$. This morphism $\tilde{\rho}$ is also a two-fold covering.

Lemma 6.9 *This definition of $\tilde{\rho}$ coincides with the previous one, up to homotopy.*

Proof: It is clear that the two maps coincide up to homotopy, up to orientation since both classes generate $\pi_3(SO(3)) = \mathbb{Z}$. We take care of the orientation using the outward normal first convention to orient boundaries, as usual. An element of S^3 reads $\cos(\theta)+\sin\theta v$ for a unique $\theta \in [0,\pi]$ and a unit quaternion v with real part zero, that is unique when $\theta \notin \{0,\pi\}$. In particular, this defines a diffeomorphism ψ from $]0,\pi[\times S^2$ to $S^3 \setminus \{-1,1\}$. We compute its degree at $(\pi/2, i)$. There $\mathbb{H}$ is oriented as $\mathbb{R} \oplus \mathbb{R} < i,j,k >$, where $\mathbb{R} < i,j,k >$ is oriented by the outward normal of S^2, that coincides with the outward normal of S^3 in $\mathbb{R}^4$, followed by the orientation of S^2. In particular since cos is an orientation-reversing diffeomorphism at $\pi/2$, the degree of ψ is 1 and ψ preserves the orientation. Now $(\cos(\theta)+\sin(\theta)v)w\overline{(\cos(\theta)+\sin\theta v)} = R(\theta,v)(w)$ where $R(\theta,v)$ is a rotation with axis v for any v. Since $R(\theta,i)(j) = \cos(2\theta)j + \sin(2\theta)k$, the two maps $\tilde{\rho}$ are homotopic. One can check that they are actually conjugate. ◇

Define

$$\begin{array}{rccl} m_r\colon & S^3 & \longrightarrow & (SO(\mathbb{H}) = SO(4)) \\ & y & \longmapsto & (m_r(y)\colon v \mapsto v.y). \end{array}$$

Lemma 6.10 *In*

$$\pi_3(SO(4)) = \mathbb{Z}[m_\ell] \oplus \mathbb{Z}[\overline{m_r}],$$

$$i_*([\tilde{\rho}]) = [m_\ell] + [\overline{m}_r] = [m_\ell] - [m_r].$$

Proof: The π_3-product in $\pi_3(SO(4))$ coincides with the product induced by the group structure of $SO(4)$. ◇

Lemma 6.11 *Recall that m_r denotes the map from the unit sphere S^3 of $\mathbb{H}$ to $SO(\mathbb{H})$ induced by the right-multiplication. Denote the inclusions $SO(n) \subset SU(n)$ by c. Then in $\pi_3(SU(4))$,*

$$c_*([m_r]) = 2[i^2(m_r^{\mathbb{C}})].$$

PROOF: Let $\mathbb{H}+I\mathbb{H}$ denote the complexification of $\mathbb{R}^4 = \mathbb{H} = \mathbb{R} < 1, i, j, k >$. Here, $\mathbb{C} = \mathbb{R} \oplus I\mathbb{R}$. When $x \in \mathbb{H}$ and $v \in S^3$, $c(m_r)(v)(Ix) = Ix.v$, and $I^2 = -1$. Let $\varepsilon = \pm 1$, define

$$\mathbb{C}^2(\varepsilon) = \mathbb{C} < \frac{\sqrt{2}}{2}(1+\varepsilon Ii), \frac{\sqrt{2}}{2}(j+\varepsilon Ik) > .$$

Consider the quotient $\mathbb{C}^4/\mathbb{C}^2(\varepsilon)$. In this quotient, $Ii = -\varepsilon 1$, $Ik = -\varepsilon j$, and since $I^2 = -1$, $I1 = \varepsilon i$ and $Ij = \varepsilon k$. Therefore this quotient is isomorphic to $\mathbb{H}$ as a real vector space with its complex structure $I = \varepsilon i$. Then it is easy to see that $c(m_r)$ maps $\mathbb{C}^2(\varepsilon)$ to 0 in this quotient. Thus $c(m_r)(\mathbb{C}^2(\varepsilon)) = \mathbb{C}^2(\varepsilon)$. Now, observe that $\mathbb{H} + I\mathbb{H}$ is the orthogonal sum of $\mathbb{C}^2(-1)$ and $\mathbb{C}^2(1)$. In particular, $\mathbb{C}^2(\varepsilon)$ is isomorphic to the quotient $\mathbb{C}^4/\mathbb{C}^2(-\varepsilon)$ that is isomorphic to $(\mathbb{H}; I = -\varepsilon i)$ and $c(m_r)$ acts on it by the right multiplication. Therefore, with respect to the orthonormal basis $\frac{\sqrt{2}}{2}(1 - Ii, j - Ik, 1 + Ii, j + Ik)$, $c(m_r)$ reads

$$c(m_r)(z_1 + z_2 j) = \begin{bmatrix} z_1 & -\overline{z}_2 & 0 & 0 \\ z_2 & \overline{z}_1 & 0 & 0 \\ 0 & 0 & \overline{z}_1 = x_1 - Iy_1 & -z_2 \\ 0 & 0 & \overline{z}_2 & z_1 = x_1 + Iy_1 \end{bmatrix}.$$

Therefore, the homotopy class of $c(m_r)$ is the sum of the homotopy classes of

$$(z_1 + z_2 j) \mapsto \begin{bmatrix} m_r^{\mathbb{C}}(z_1, z_2) & 0 \\ 0 & 1 \end{bmatrix} \text{ and } (z_1 + z_2 j) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & m_r^{\mathbb{C}} \circ \iota(z_1, z_2) \end{bmatrix}$$

where $\iota(z_1, z_2) = (\overline{z_1}, \overline{z_2})$. Since the first map is conjugate by a fixed element of $SU(4)$ to $i^2_*(m_r^{\mathbb{C}})$, it is homotopic to $i^2_*(m_r^{\mathbb{C}})$, and since ι induces the identity on $\pi_3(S^3)$, the second map is homotopic to $i^2_*(m_r^{\mathbb{C}})$, too. ◇

The following lemma finishes to determine the maps

$$c_* : \pi_3(SO(4)) \longrightarrow \pi_3(SU(4))$$

and

$$c_* i_* : \pi_3(SO(3)) \longrightarrow \pi_3(SU(4))$$

Lemma 6.12

$$c_*([\overline{m_r}]) = c_*([m_\ell]) = -2[i^2(m_r^{\mathbb{C}})].$$
$$c_*(i_*([\tilde{\rho}])) = -4[i^2(m_r^{\mathbb{C}})].$$

PROOF: According to Lemma 6.10, $i_*([\tilde{\rho}]) = [m_\ell] + [\overline{m}_r] = [m_\ell] - [m_r]$. Using the conjugacy of quaternions, $m_\ell(v)(x) = v.x = \overline{\overline{x}.\overline{v}} = \overline{m}_r(v)(\overline{x})$. Therefore m_ℓ is conjugated to $\overline{m}_r$ via the conjugacy of quaternions that lies in $(O(4) \subset U(4))$.

Since $U(4)$ is connected, the conjugacy by an element of $U(4)$ induces the identity on $\pi_3(SU(4))$. Thus,

$$c_*([m_\ell]) = c_*([\overline{m}_r]) = -c_*([m_r]),$$

and

$$c_*(i_*([\tilde{\rho}])) = -2c_*([m_r]).$$

◇

6.7 Relating the relative Pontrjagin number to the degree

We finish proving Theorem 2.9 by proving the following proposition.

Proposition 6.13 *Let M_0 and M be two compact connected oriented 3-manifolds whose boundaries have collars that are identified by a diffeomorphism. Let $\tau_0 \colon M_0 \times \mathbb{C}^3 \to TM_0 \otimes \mathbb{C}$ and $\tau \colon M \times \mathbb{C}^3 \to TM \otimes \mathbb{C}$ be two special complex trivialisations (that respect the orientations) that coincide on the collar neighborhoods of $\partial M_0 = \partial M$. Let $[(M, \partial M), (SU(3), 1)]$ denote the group of homotopy classes of maps from M to $SU(3)$ that map ∂M to 1. For any*

$$g : (M, \partial M) \longrightarrow (SU(3), 1),$$

define

$$\begin{array}{rccc} \psi(g): & M \times \mathbb{C}^3 & \longrightarrow & M \times \mathbb{C}^3 \\ & (x,y) & \mapsto & (x, g(x)(y)) \end{array}$$

then

$$p_1(\tau_0, \tau \circ \psi(g)) - p_1(\tau_0, \tau) = p_1(\tau, \tau \circ \psi(g)) = -p_1(\tau \circ \psi(g), \tau) = p_1'(g)$$

is independent of τ_0 and τ, p_1' induces an isomorphism from the group $[(M, \partial M), (SU(3), 1)]$ to $\mathbb{Z}$, and if g is valued in $SO(3)$ then

$$p_1'(g) = 2deg(g).$$

PROOF:

Lemma 6.14 *Under the hypotheses of Proposition 6.13, $(p_1(\tau_0, \tau \circ \psi(g)) - p_1(\tau_0, \tau))$ is independent of τ_0 and τ.*

PROOF: Indeed, $(p_1(\tau_0, \tau \circ \psi(g)) - p_1(\tau_0, \tau))$ can be defined as the obstruction to extending the following trivialisation of the complexified tangent bundle of $[0,1] \times M$ restricted to the boundary. This trivialisation is $T[0,1] \oplus \tau$ on $(\{0\} \times M) \cup ([0,1] \times \partial M)$ and $T[0,1] \oplus \tau \circ \psi(g)$ on $\{1\} \times M$. But this obstruction is the obstruction to extending the map $\tilde{g}$ from $\partial([0,1] \times M)$

to $SU(4)$ that maps $(\{0\} \times M) \cup ([0,1] \times \partial M)$ to 1 and that coincides with $i(g)$ on $\{1\} \times M$, viewed as a map from $\partial([0,1] \times M)$ to $SU(4)$, over $([0,1] \times M)$. This obstruction that lies in $\pi_3(SU(4))$ since $\pi_i(SU(4)) = 0$, for $i < 3$, is independent of τ_0 and τ. ◇

Lemma 6.14 guarantees that p_1' defines two group homomorphisms to $\mathbb{Z}$ from $[(M, \partial M), (SU(3), 1)]$ and from $[(M, \partial M), (S0(3), 1)]$. Since $\pi_i(SU(3))$ is trivial for $i < 3$ and since $\pi_3(SU(3)) = \mathbb{Z}$, the group of homotopy classes $[(M, \partial M), (SU(3), 1)]$ is generated by the class of a map that maps the complement of a 3-ball B to 1 and that factors through a map that generates $\pi_3(SU(3))$. By definition of the Pontrjagin classes, p_1' sends such a generator to ± 1 and it induces an isomorphism from $[(M, \partial M), (SU(3), 1)]$ to $\mathbb{Z}$.

According to Lemma 6.1 and to Lemma 6.6, p_1' must read $p_1'(\rho_M(B^3))\frac{\deg}{2}$, and we are left with the proof of the following lemma.

Lemma 6.15

$$p_1'(\rho_M(B^3)) = 4.$$

Let $g = \rho_M(B^3)$, we can extend $\tilde{g}$ (defined in the proof of Lemma 6.14) by the constant map with value 1 outside $[\varepsilon, 1] \times B^3 \cong B^4$ and, in $\pi_3(SU(4))$

$$[c(\tilde{g}_{|\partial B^4})] = -p_1(\tau, \tau \circ \psi(g))[i^2(m_r^{\mathbb{C}})].$$

Since $\tilde{g}_{|\partial B^4}$ is homotopic to $c \circ i(\tilde{\rho})$, Lemma 6.12 allows us to conclude. ◇

7 Other complements

7.1 More on low-dimensional manifolds

Piecewise linear (or PL) n–manifolds can be defined as follows. An n-dimensional simplex is the convex hull of $(n+1)$ points that are not contained in an affine subspace of dimension $(n-1)$. For example, a 3-dimensional simplex is a tetrahedron. A topological space X has a *triangulation*, if it reads as a locally finite union of k–simplices (closed in X) that are the simplices of the triangulation so that (1) every face of a simplex of the triangulation is a simplex of the triangulation (2) when two simplices of the triangulation are not disjoint, their intersection is a simplex of the triangulation. A *subdivision* T' of a triangulation T of X is a triangulation of X such that each simplex of T' is included in a simplex of T. Say that two triangulations are *equivalent* if they have isomorphic subdivisions. A *PL manifold* is a topological manifold equipped with an equivalence class of triangulations.

When $n \leq 3$, the above notion of PL-manifold coincides with the notions of smooth and topological manifold, according to the following theorem. This is not true anymore when $n > 3$. See [Kui99].

Theorem 7.1 *When $n \leq 3$, the category of topological n–manifolds is isomorphic to the category of PL n–manifolds and to the category of C^r n–manifolds, for $r = 1, \dots, \infty$.*

For example, according to this statement that contains several theorems (see [Kui99]), any topological 3–manifold has a unique C^∞–structure. Below $n = 3$.

The equivalence between the $C^i, i = 1, 2, \dots, \infty$-categories follows from work of Whitney in 1936 [Whi36]. In 1934, Cairns [Cai35] provided a map from the C^1–category to the PL category, that is the existence of a triangulation for C^1–manifolds, and he proved that this map is onto [Cai40, Theorem III] in 1940. Moïse [Moi52] proved the equivalence between the topological category and the PL category in 1952. This diagram was completed by Munkres [Mun60, Theorem 6.3] and Whitehead [Whi61] in 1960 by their independent proofs of the injectivity of the natural map from the C^1–category to the topological category.

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