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Cocategory and Nilpotency
Research ~~group~~ subjet
Course IV : Groupoid and
classifying spaces

Rational Homotopy Theory Seminar

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I - Groupoids

Groupoids: Generalization of the concepts of groups and that of categories

Alegbraic approach

$$\text{Groupoid} := \Big(\underbrace{G}_{\text{set}}, \ \underbrace{{}^{-1} : G \to G}_{\text{unary operation}}, \ \underbrace{* : G \times G \to G}_{\text{not binary operation}} \Big)$$

[ie: a^{-1} always defined, but not $a * b$]

with the additional following axioms:

- **Associativity**:
  $a*b$ and $b*c$ exist $\Rightarrow$ $(a*b)*c$ and $a*(b*c)$ exist and equals

- **Inverse**
  $a^{-1} * a$ and $a * a^{-1}$ exist

- **Identity**
  $a*b$ exists $\Rightarrow$ $a*b*b^{-1} = a$, $a^{-1}*a*b = b$

③

Following this axioms we have

$$\boxed{\begin{array}{l} (a^{-1})^{-1} = a \\ a*b \text{ exists} \Rightarrow \underbrace{(a*b)^{-1} = b^{-1}*a^{-1}}_{\text{existence + equality}} \end{array}}$$

Informative example

$$GL(K) = \bigcup_{n \geq 0} Gl_n(K)$$

$A*B = AB$ when its defined

i.e. $A, B \in Gl_n(K)$

Category theoretic approach

$$\boxed{\text{Groupoid} := \text{Small category} + \text{All morphism are iso}}$$

small category := objects and Mor are sets not proper class

More precisely : Groupoid := $\begin{cases} G_0 \text{ (set of objects } x) \\ G(x,y) = \{ f : x \to y \} \text{ set of morphisms} \\ \text{composition} \end{cases}$

such that :

- Asso : $f(gh) = (fg)h$
- identity : $\forall x \in G_0 \; \exists id_x \in G(x,x)$ st $f\,id_x = f$, $id_x f = f$
- Inverse : $\forall x, y \in G_0 \; \exists \, inv_{x,y} : G(x,y) \to G(y,x)$ st $f \cdot inv_{y,x} = id_y$, $inv_{y,x} f = id_x$

$\boxed{G(x,x) \text{ are called vertex groups}}$

Informative examples

Example 1

X : top space

$$\underbrace{\left[\begin{array}{l} G_0 := X \\ G(x,y) := \left\{ \begin{array}{l} \gamma : [0,1] \longrightarrow X \\ \text{tq } \gamma(0) = x \\ \gamma(1) = y \end{array} \right\} \Big/ \text{homotopy} \end{array}\right]}_{\pi_1(X) \text{: fundamental groupoid}}$$

$G(x,x) = \pi_1(X, x)$ fundamental group

Example 2 Action group $X \circlearrowleft G$

$G_0 := X$

$G(x,y) := \{ g \in G \text{ tq } gx = y \}$

$G(x,x) = G_x$ isotropy groups

Example 3 Equivalence relation

$G_0 := X$

$G(x,y) := \begin{cases} \{\cdot\} & \text{if } x \sim y \\ \phi & \text{if not} \end{cases}$

$G(x,x) = \{\cdot\}$

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Connection between the two approachs

- ~~A~~ Category theoretic approach $\longrightarrow$ algebraic approach

$$G_0 \longrightarrow G = \bigcup G(x,y)$$

$$\circ \longrightarrow *$$

$$\mathrm{inv} \longrightarrow {}^{-1}$$

- algebraic approach $\longrightarrow$ Category theoretic approach

$G \longrightarrow$

- $G_0 = \{\underbrace{x * x^{-1}}_{a \text{ (objects)}},\ x \in G\}$

Morphism

- $\{f : a \to b\} = \{f \in G \text{ st } b * f * a\}$ exists

- $id_a = a$

- $\mathrm{inv}_{a,b} : G(a,b) \longrightarrow G(b,a)$
  $f \longmapsto f^{-1}$

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Fibrations of groupoids

that's : $p : E \to B$ morphism of groupoid (as categories)

st $\forall x \in Obj(E)$
$\forall b \in Hom_B(p(x), -)$, $\exists e \in Hom_E(x, -)$
tq $e(p) = b$

Informative example
when E and B groups
fibration $\iff$ surjective morph

II - Classifying space

- G group topological
- EG weakly contractible space with $\pi_*(EG) = *$
a free G-action

$$\boxed{BG := EG / \text{free } G\text{-action}}$$

Informative examples

$B\mathbb{Z}/2\mathbb{Z} = \mathbb{R}P^\infty$
$\mathbb{C}P^\infty = BS^1$

G discret group $\Rightarrow BG \simeq K(G,1)$
$S^1 = B\mathbb{Z}$, $\mathbb{T}^n = B\mathbb{Z}^n$
$\vee S^n = BG$ G free group
$\cancel{B}\pi_1(S) =$ closed connected surface of genus 1

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Fibration (Homotopy theoretic view)

- lifting property verifying by $\underbrace{p : X \to Y}_{\text{continuous map}}$

respecting to a pair (Z, A)

if
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Z \arrow[r, "g"'] \arrow[ur, "G", "\exists k ?"'] & Y \end{tikzcd}$$

- Serre fibration : continuous map + surjective + lifting property respect to any $(Z \times I, Z \times \{0\})$ where Z CW-complex

- Fibration = Serre-Fibration + Z top space projection

$$X_y = p^{-1}(y) \longrightarrow X \xrightarrow{p} Y$$

fibre at y — total space — base

si Y based $\underbrace{F = p^{-1}\{*\}}_{\text{the fibre}}$

- Informative examples:

Example 1 $F \longrightarrow Y \times F \longrightarrow Y$ (trivial fibration)

Example 2 : $A \times_Y X = A \times X / f(a) = p(x)$ (fibre product)

where $A \xrightarrow{f} Y \xleftarrow{p} X$

$\Downarrow$

$$\begin{tikzcd} A \times_Y X \arrow[r, "g"] \arrow[d, "p_A"'] & X \arrow[d, "p"] \\ A \arrow[r, "f"'] & Y \end{tikzcd}$$

pullback diagram

$\boxed{\begin{array}{l} p \text{ fibration} \\ \Downarrow \\ p_A \text{ so is} \end{array}}$

$f : A \hookrightarrow Y \Rightarrow A \times_Y X = p^{-1}(A)$

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G-fibration $\begin{cases} G \text{ group} \\ P \circlearrowleft G \text{ (right action)} \\ p: P \to X \text{ fibration} \\ p(z\cdot g) = p(z) \quad \forall z \in P, \forall g \in G \\ \forall z \in P \quad G \to P_{p(z)} \quad \text{weak homotopy equiv} \\ \qquad g \mapsto z\cdot g \end{cases}$

Rmk the pullback of G fibration is a G-fibration

the action is $(y,z)\cdot g = (y, z\cdot g)$

Informative example

Moore Path space fibration

$$\underbrace{\hat{\Omega}(X, *)}_{\text{Moore loop space}} \to \hat{P}(X, *) \xrightarrow{p} X$$

$\gamma(1) = *$

$\gamma \mapsto \gamma(0)$

contractible $\Rightarrow$ path connected

&

$p(\gamma * w) = p(\gamma) \quad \forall \gamma \in PX, \ \forall w \in \Omega X$

$A_z : \Omega X = (PX)_{x_0} \to (PX)_{x_0} = id \Rightarrow$ weak homotopy eq

PX contractible $\Downarrow$

A_z so is