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Cocategory and Nilpotency
Research group subject
Course IV : Groupoid and
classifying spaces

RATIONAL HOMOTOPY THEORY SEMINAR

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I - Groupoids

Groupoids: Generalization of the concepts of groups and that of categories

Algebraic approach

$$\text{Groupoid} := \left(G, \begin{array}{l} \downarrow \\ \text{Set} \end{array}, \begin{array}{l} \text{---}^{-1}: G \rightarrow G \\ \text{unary} \\ \text{operation} \end{array}, \begin{array}{l} \text{---}^*: G \times G \rightarrow G \\ \text{not binary} \\ \text{operation} \end{array} \right)$$

[ie: a^{-1} always defined, but not $a * b$]
with the additional following axioms:

• Associativity:

$$\begin{array}{l} a * b \\ \text{and} \\ b * c \end{array} \text{ exist} \Rightarrow \begin{array}{l} (a * b) * c \\ \text{and} \\ a * (b * c) \end{array} \text{ exist and equals}$$

• Inverse

$$a^{-1} * a \text{ and } a * a^{-1} \text{ exist}$$

• Identity

$$a * b \text{ exists} \Rightarrow \begin{array}{l} a * b * b^{-1} = a \\ a^{-1} * a * b = b \end{array}$$

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Following this axioms we have

$$\begin{aligned}
 & (a^{-1})^{-1} = a \\
 & a \times b \text{ exists} \Rightarrow (a \times b)^{-1} = b^{-1} \times a^{-1} \\
 & \text{existence + equality}
 \end{aligned}$$

Informative example

$$GL(K) = \bigcup_{n \geq 0} GL_n(K)$$

$A \times B = AB$ when its defined
 @ $A, B \in GL_n(K)$

Category theoretic approach

Groupoid := Small category + All morphism are iso

small category := objects and Mor are sets not proper class

More precisely : Groupoid := $\left\{ \begin{array}{l} G_0 \text{ (set of objects)} \\ G(x,y) = \{ f : x \rightarrow y \} \\ \text{set of morphisms} \\ \& \text{composition} \end{array} \right.$

such that :

- Asso : $f(gh) = (fg)h$

- identity : $\forall x \in G_0 \exists id_x \in G(x,x)$ st $f id_x = f$
 $id_y f = f$

- Inverse : $\forall x,y \in G_0 \exists inv_{x,y} : G(x,y) \rightarrow G(y,x)$
 st $f \cdot inv_{y,x} = id_y$
 $inv_{x,y} \cdot f = id_x$

$G(x,x)$ are called vertex groups

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Informative examples

Example 1

X: top space

$$\left[\begin{array}{l} G_0 := X \\ G(x, y) := \left. \begin{array}{l} \gamma: (0, 1) \rightarrow X \\ \gamma(0) = x \\ \gamma(1) = y \end{array} \right\} / \text{homotopy} \end{array} \right]$$

$\pi_1(X)$: fundamental groupoid

$G(x, x) = \pi_1(X, x)$ fundamental group

Example 2 Action group $X \curvearrowright G$

$$G_0 := X$$

$$G(x, y) := \{ g \in G \mid gx = y \}$$

$$G(x, x) = G_x \quad \text{isotropy groups}$$

Example 3 Equivalence relation

$$G_0 := X$$

$$G(x, y) := \begin{cases} \cdot & \text{if } x \sim y \\ \emptyset & \text{if not} \end{cases}$$

$$G(x, x) = \cdot$$

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Connection between the two approaches

~~Algebraic~~
 Category theoretic approach \longrightarrow algebraic approach

$G_0 \longrightarrow G = \cup G(x,y)$
 $0 \longrightarrow *$
 $inv \longrightarrow -1$

algebraic approach \longrightarrow Category theoretic approach

$G \longrightarrow$

- $G_0 = \{ \underbrace{x * x^{-1}}_a, x \in G \}$
 (objects)
- Morphism
 $\{ f: a \rightarrow b \} = \{ f \in G \text{ st } b * f * a \text{ exists} \}$
- $id_a = a$
- $inv_{a,b}: G(a,b) \rightarrow G(b,a)$
 $f \longmapsto f^{-1}$

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Fibrations of groupoids

that is: $p: E \rightarrow B$ morphism of groupoid
(as categories)

$$\left. \begin{array}{l} \text{st } \forall x \in \text{Obj}(E) \\ \forall b \in \text{Hom}_B(p(x), -), \exists e \in \text{Hom}_E(x, -) \\ \text{tg } e(p) = b \end{array} \right\}$$

Informative example

when E and B groups

fibration \Leftrightarrow surjective morph

II - Classifying space

- G group topological
- EG weakly contractible space with $\pi_*(EG) = *$
a free G -action

$$\boxed{BG := EG / \text{free } G\text{-action}}$$

Informative examples

$$\begin{array}{l} B\mathbb{Z} / \mathbb{Z} = \mathbb{R}P^\infty \\ CP^\infty = BS^1 \end{array} \left| \begin{array}{l} G \text{ discrete group} \Rightarrow BG \cong K(G, 1) \\ S^1 = B\mathbb{Z}, \pi^1 = B\mathbb{Z} \\ VS^n = BG \quad G \text{ free group} \\ \pi_1(S) = \text{closed connected surface of genus } S \end{array} \right.$$

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Fibration (Homotopy theoretic view)

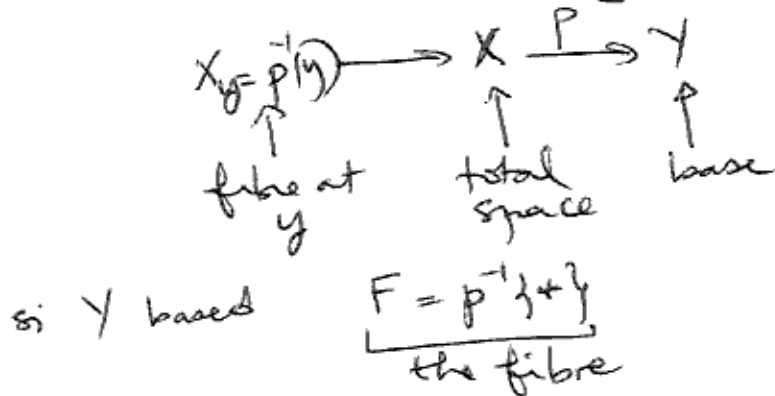
• lifting property verifying by $p: X \rightarrow Y$
continuous map

respecting to a pair (Z, A)

$$\text{if } \begin{array}{ccc} A & \xrightarrow{f} & X \\ i \downarrow & \nearrow G & \downarrow p \\ Z & \xrightarrow{g} & Y \end{array}$$

• Serre fibration: continuous map + surjective + lifting property
respect to any $(Z \times I, Z \times \{0\})$
where Z CW-complex

• Fibration = Serre-fibration + Z top space projection



• Informative examples:

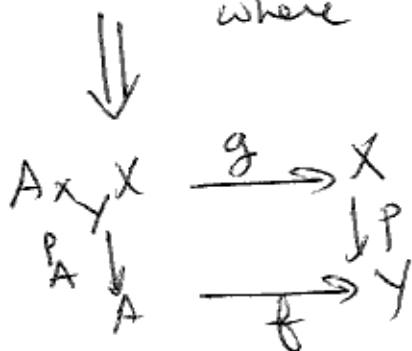
Example 1 $F \rightarrow Y \times F \rightarrow Y$ (trivial fibration)

Example 2: $A \times_Y X = A \times X / f(a) = p(x)$ (fibre product)

where $A \xrightarrow{f} Y \xleftarrow{p} X$

p fibration
 \Downarrow
 p_A iso is

$f: A \hookrightarrow Y \Rightarrow A \times_Y X = p^{-1}(A)$



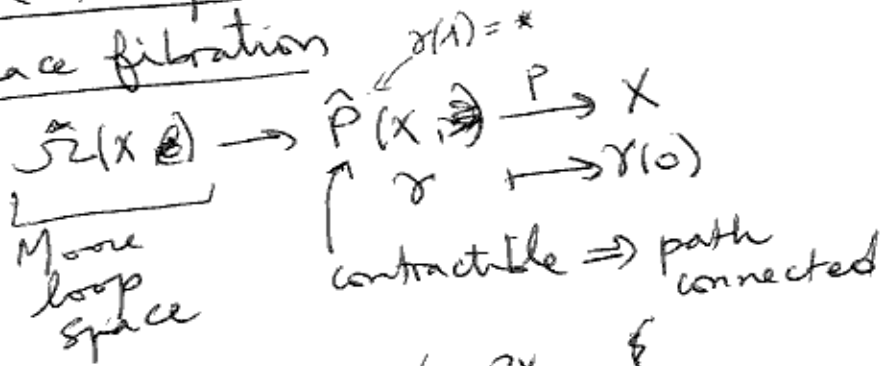
pullback diagram

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G-fibration $\left\{ \begin{array}{l} G \text{ group} \\ P \hookrightarrow G \text{ (right action)} \\ p: P \rightarrow X \text{ fibration} \\ p(z \cdot g) = p(z) \quad \forall z \in P, \forall g \in G \\ \forall z \in P \quad G \rightarrow P_{p(z)} \quad \text{weak homotopy equiv} \\ g \mapsto z \cdot g \end{array} \right.$

Rmk the pullback of G-fibration is a G-fibration
 the action is $(y, z) \cdot g = (y, z \cdot g)$

Informative example
Moore Path space fibration



$$p(\sigma \cdot w) = p(\sigma) \quad \forall \sigma \in PX, \forall w \in \Omega X$$

$$A_2: \Omega X = (PX)_{x_0} \rightarrow (PX)_{x_0} = id \Rightarrow \text{weak homotopy eq}$$

PX contractible \Downarrow
 A_2 so is

