Hidden Infinities[1]

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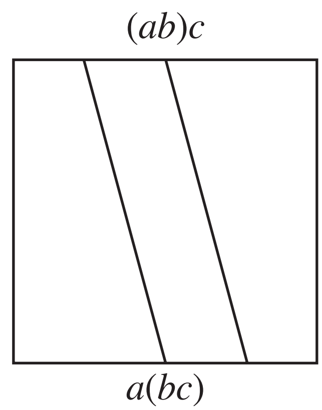
Perhaps it is a bit of a shock, but for homotopy theory notions of an algebraic sort, associativity, commutativity, etc., are **not** homotopy invariant notions. The simplest example is provided by Poincaré. The based loops on a space X is given by

$$\Omega X = \{\lambda\colon [0,1] \to X \mid \lambda(0) = \lambda(1) = *\}.$$

Loops may be multiplied together by running round the first twice as fast, and then the second twice as fast. When you multiply three loops, it matters how you associate the multiplication. The relation of homotopy allows you to fill in the data between both schemes, as shown in the picture:



Of course, we all know that loops modulo homotopy constitute a group, the *fundamental group* $\pi_1(X)$.

There is a version of based loops on a space for which the multiplication is associative, the so-called *Moore loops*:

$$\Omega^M X = \{(\lambda, r) \mid r \geq 0, \lambda\colon [0,r] \to X, \lambda(0) = \lambda(r) = *\}.$$

So, somehow, there is a multiplication that is associative, but with a homotopy equivalent replacement that is not associative. Hence, associativity is not a homotopy invariant notion.

If you come across a space Y with a multiplication, how can you know that it is homotopy equivalent to a based loop space? Jim Stasheff and Sugawara considered this question. The answer is provided by Milnor: If $Y \simeq \Omega X$, then Y has a classifying space, so that $Y \simeq \Omega BY$.

But what does it mean to have a classifying space? Is homotopy associativity enough?

[1]Based on the author's paper, *An appreciation of the work of Jim Stasheff*, in Higher homotopy structures in topology and mathematical physics (Poughkeepsie, NY, 1996), 116, Contemp. Math., 227, Amer. Math. Soc., Providence, RI, 1999.

Example: S^7, the unit octonions, has a multiplication, but there is no possible multiplication on S^7 that will be homotopy associative. It is the case that S^7 cannot have a projective space. A classical condition on multiplication has a topological analogue that obstructs associativity.

We proceed to build associativity one stage at a time.

Definition. *Let K_i denote the CW-complex constructed inductively as follows: $K_2 = *$, a point. Let K_i be the cone CL_i where L_i is the union of copies $(K_r \times K_s)_k$ of $K_r \times K_s$, where $r + s = i + 1$, and k corresponds to inserting a pair of parentheses into i symbols*

$$(1 \quad 2 \quad \cdots \quad k-1 \quad (k \quad k+1 \quad \cdots \quad k+s-1) \quad k+s \quad \cdots \quad i).$$

The intersection of copies corresponds to inserting two pairs of parentheses with no overlap or with one as subset of the other. Define $\partial_p(r,s)\, K_r \times K_s \to K_i$ to be the inclusion of the copy indexed by $(1\ 2\ \cdots\ (p\ p+1\ \cdots\ p+s-1)\ \cdots\ i)$.

Proposition. *K_i is an $(i-2)$-cell.*

Definition. *An A_n***-space** *$(X; M_1, \ldots, M_n)$ consists of a space X along with a family of maps $M_i \colon K_i \times X^{\times i} \to X$, $i \leq n$ defined such that*

1. *M_2 is a multiplication with unit.*
2. *For $\rho \in K_r$ and $\sigma \in K_s$,*

$$M_i(\partial_k(r,s)(\rho,\sigma), x_1, \ldots, x_i) = \\ M_r(\rho, x_1, \ldots, x_{k-1}, M_s(\sigma, x_k, \ldots, x_{k+s-1}), x_{k+s}, \ldots, x_i).$$

3. *For $\tau \in K_i$, $i > 2$, we have*

$$M_i(\tau, x_1, \ldots, x_{j-1}, e, x_j, \ldots, x_i) = M_{i-1}(s_j(\tau), x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_i)$$

where the maps $s_j \colon K_i \to K_{i-1}$ are degeneracies.

If the M_i exist and satisfy these conditions for all $i \geq 2$ we speak of $(X; M_i)$ as an A_∞-space.

I. An A_n-space Y has a projective n-space $YP(n)$. An A_∞-space Y has the homotopy type of a loop space, that is, $Y \simeq \Omega X$, for some X.

II. There are H-spaces that are A_n, but not A_{n+1} for certain n. This opened up a new schema that contributed to an on-going discussion of how much H-spaces were like Lie groups up to homotopy.

III. When X is an A_n-space, then $C_*(X)$ enjoys extra algebraic structure.

Definition. *Let k be a field. An $n+1$-tuple $(A, m_1, m_2, \ldots, m_n)$ constitutes an* $A(n)$**-algebra** *if A is a graded k-module, $A = \bigoplus_i A_i$, and the k-linear maps $m_i\ A^{\otimes i} \to A$ satisfy the following properties:*

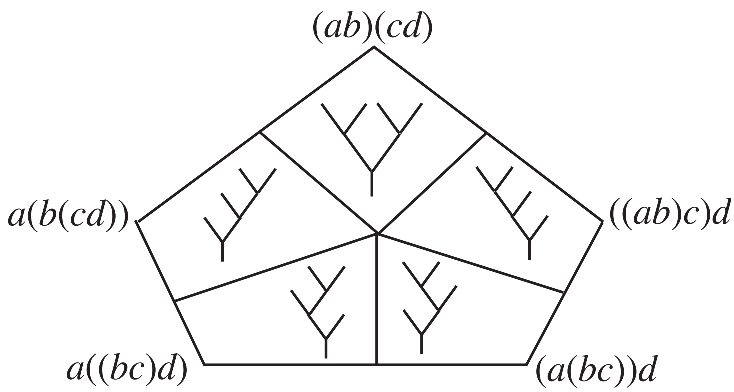
1) m_i raises degree by $i-2$, that is, $m_i([A^{\otimes i}]_q) \subset A_{q+i-2}$, for all q.

2) If $u = u_1 \otimes \cdots \otimes u_i \in A^{\otimes i}$, then

$$\sum_{r+s=i+1, 1\le p\le r} \pm m_r(u_1 \otimes \cdots \otimes m_s(u_p \otimes \cdots \otimes u_{p+s-1}) \otimes \cdots \otimes u_i) = 0,$$

where $\pm$ is determined by $(-1)^\epsilon$ where $\epsilon = (s+1)p + s\left(i + \sum_{j=1}^{p-1} \dim u_j\right)$.

*An $A(\infty)$***-algebra** *consists of an augmented k-module A and maps $m_i\colon A^{\otimes i} \to A$ satisfying the conditions above for all $i \ge 1$.*



Development of these ideas took a computational turn when Boardman and Vogt analyzed A_∞-structures. The change from complexes to tree diagrams offered a much more general framework to analyze algebraic structures, also developed by Peter May. It was Jean-Louis Loday who help revive these ideas in the 1990's, and operad structures are now seen to be hidden everywhere. The best place to learn about these hidden infinities is the book of Loday and Bruno Vallette.