Hidden Infinities

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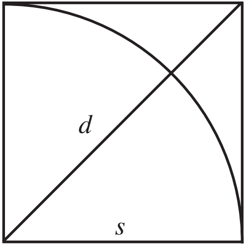
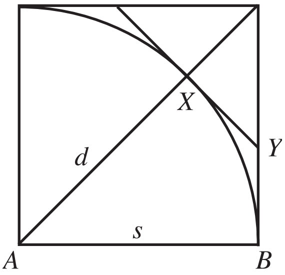
The ratio between the side and the diagonal of a square



The
ratio between d and s begins with $d = s + r$
with $0 < r < s$. Notice that this implies

$$\frac{d}{s} = 1 + \frac{r}{s}.$$

The ratio between the side and the diagonal of a square



Notice that $XY \cong BY$.

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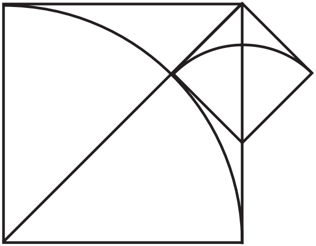
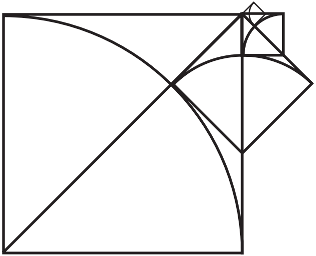


The ratio between s and r begins with $s = 2r + r'$ with $0 < r' < r$. Notice that this implies

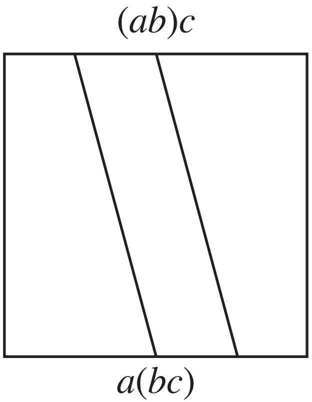
$$\frac{s}{r} = 2 + \frac{r'}{r}.$$

The ratio between the side and the diagonal of a square

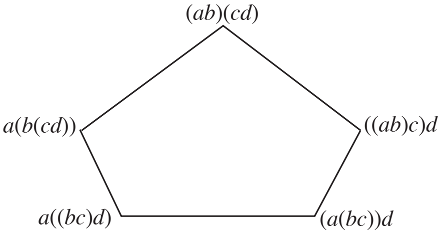


But the picture indicates that the process will repeat, and hence will repeat forever. A hidden infinity given by

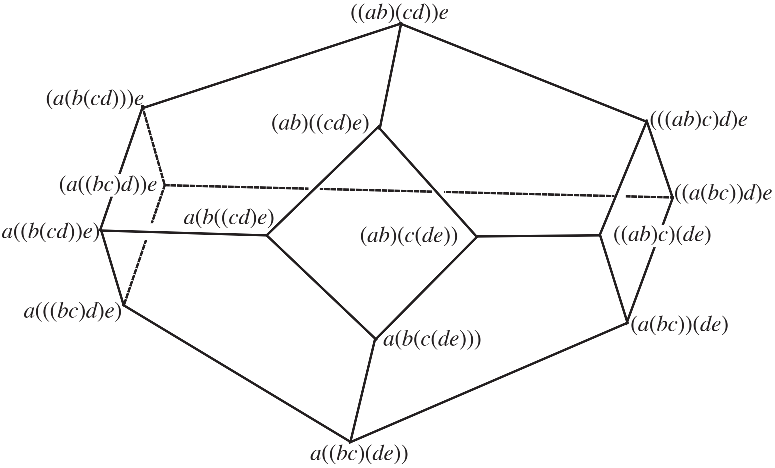
$$\sqrt{2} = \frac{d}{s} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$


(ab)c
a(bc)

The spaces K_3 and K_4



The space K_5


((ab)(cd))e
(a(b(cd)))e
(ab)((cd)e)
(((ab)c)d)e
(a((bc)d))e
((a(bc))d)e
a(b((cd)e)
a((b(cd))e)
(ab)(c(de))
((ab)c)(de)
a(((bc)d)e)
a(b(c(de)))
(a(bc))(de)
a((bc)(de))

Definition

Let K_i denote the CW-complex constructed inductively as follows: $K_2 = *$, a point. Let K_i be the cone CL_i where L_i is the union of copies $(K_r \times K_s)_k$ of $K_r \times K_s$, where $r + s = i + 1$, and k corresponds to inserting a pair of parentheses into i symbols

$$(1 \quad 2 \quad \cdots \quad k-1 \quad (k \quad k+1 \quad \cdots \quad k+s-1) \quad k+s \quad \cdots \quad i).$$

The intersection of copies corresponds to inserting two pairs of parentheses with no overlap or with one as subset of the other. Define $\partial_p(r,s)\; K_r \times K_s \to K_i$ to be the inclusion of the copy indexed by $(1\; 2\; \cdots\; (p\; p+1\; \cdots\; p+s-1)\; \cdots\; i)$.

Definition

An A_n**-space** $(X; M_1, \dots, M_n)$ consists of a space X along with a family of maps $M_i \colon K_i \times X^{\times i} \to X$, $i \leq n$ defined such that

1) M_2 is a multiplication with unit.
2) For $\rho \in K_r$ and $\sigma \in K_s$,

$$M_i(\partial_k(r,s)(\rho,\sigma), x_1, \dots, x_i) = \\ M_r(\rho, x_1, \dots, x_{k-1}, M_s(\sigma, x_k, \dots, x_{k+s-1}), x_{k+s}, \dots, x_i).$$

3) For $\tau \in K_i$, $i > 2$, we have

$$M_i(\tau, x_1, \dots, x_{j-1}, e, x_j, \dots, x_i) \\ = M_{i-1}(s_j(\tau), x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_i)$$

where the maps $s_j \colon K_i \to K_{i-1}$ are degeneracies.

If the M_i exist and satisfy these conditions for all $i \geq 2$ we speak of $(X; M_i)$ as an A_∞-space.

When X is an A_n-space, then $C_*(X)$ enjoys extra algebraic structure.

Definition

Let k be a field. An $n+1$-tuple $(A, m_1, m_2, \ldots, m_n)$ constitutes an $A(n)$**-algebra** if A is a graded k-module, $A = \bigoplus_i A_i$, and the k-linear maps $m_i\, A^{\otimes i} \to A$ satisfy the following properties:

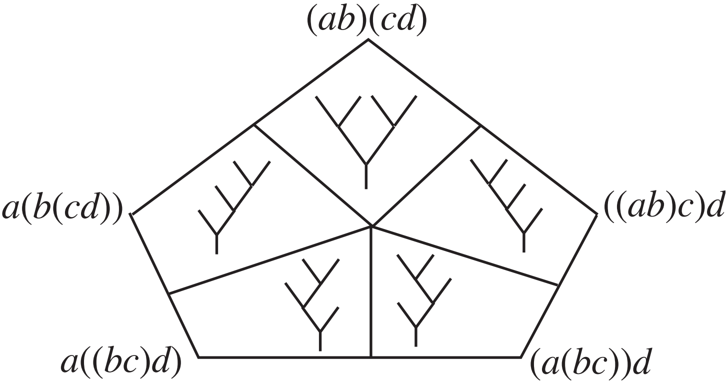
1) m_i raises degree by $i-2$, that is, $m_i([A^{\otimes i}]_q) \subset A_{q+i-2}$, for all q.
2) If $u = u_1 \otimes \cdots \otimes u_i \in A^{\otimes i}$, then

$$\sum_{r+s=i+1, 1\leq p\leq r} \pm m_r(u_1 \otimes \cdots \otimes m_s(u_p \otimes \cdots \otimes u_{p+s-1}) \otimes \cdots \otimes u_i) = 0,$$

where $\pm$ is determined by $(-1)^\epsilon$ where
$\epsilon = (s+1)p + s\left(i + \sum_{j=1}^{p-1} \dim u_j\right)$.

An $A(\infty)$**-algebra** consists of an augmented k-module A and maps $m_i\colon A^{\otimes i} \to A$ satisfying the conditions above for all $i \geq 1$.

Boardman-Vogt



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