A History of Spectral Sequences

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> *Problem 17. What are the relations connecting the homology structure of the bundle, base space, fiber and group?*
>
> Samuel Eilenberg 1949

> *It is now abundantly clear that the spectral sequence is one of the fundamental algebraic structures needed for dealing with topological problems.*
>
> William S. Massey 1955

In December 1946 Princeton University held a conference to celebrate its bicentennial. The sessions on mathematics were titled *"The Problems of Mathematics."* Princeton enjoyed a leadership role in topology and a problem, list, prepared by S. Eilenberg (1913–1998) came out of the conference, received at the *Annals of Mathematics* on July 1, 1947 and published in volume 50 (1949). The problems were chosen to give a *"birds-eye view of some of the trends of present day Topology."* This paper is concerned with Problem 17 in this list quoted above.

Eilenberg mentions briefly recent results of Jean Leray (1906–1998) announced in the 1946 *Comptes Rendus*[2] without proof and "indicating interesting new methods."

Less than seven years later, May 3–7, 1953, Cornell University hosted an international conference titled *Fiber bundles and differential geometry.* This conference led to two problem sets in algebraic topology; one was prepared by F. Hirzebruch (1927–2012) and treats questions of differential topology, especially characteristic classes; the other was prepared by W.S. Massey (1920–) and treats questions of homotopy theory. There Massey wrote the passage quoted above.

This talk describes the development of the spectral sequence and its impact on algebraic topology during these years. Because we are discussing an algebraic technique, a key role is played by the problems to which these ideas were applied. We begin by establishing the background for the development of spectral sequences. This discussion splits into three parts, divided conveniently if not naturally, by events before, during, and after the Second World War.

§1. Algebraic topology before the Second World War

Looking back from 1940, the contemporary developments in algebraic topology display a remarkable vitality. New homology theories were being developed for larger and larger classes of spaces. The simplicial theory for polyhedra had produced a set of powerful results (e.g., fixed point theorems, Poincaré duality) that set benchmarks for the newer approaches. The theories of J.W. Alexander (1888–1971), E. Čech (1893–1960), and S. Lefschetz (1884–1972) offered successful extensions of the simplicial theory that functioned as tools for new applications of topological ideas.

In 1935-36 the higher homotopy groups were introduced by W. Hurewicz (1904–1956)[3]. A central problem for algebraic topology was (and still is) the computation of these groups for well-understood spaces. Beginning with the first nontrivial example of an essential map between spheres of differing dimension, Heinz Hopf (1894–1971) extended his study of linking invariants to obtain nontrivial homotopy classes of mappings $S^{4n-1} \to S^{2n}$. Further progress was obtained by Hans Freudenthal (1905–1990) who proved his landmark suspension theorem. However, precious little more was known or even conjectured about the homotopy groups of spaces, even up to 1950 when Hopf addressed the Cambridge, Massachusetts International Congress of Mathematicians, asking *"Wie kann man einen Überblick über sie gewinnen . . . ?"*

Hurewicz's higher homotopy groups could be seen to be denumberable for a polyhedron by an application of the simplicial approximation theorem. In some cases, such as the space $S^1 \vee S^2$, the higher homotopy groups are not finitely generated. How the fundamental group influenced such cases and the general question of the higher homotopy groups of a simply-connected space being finitely generated was still open in 1950. Hurewicz had also shown many connections between the homotopy groups and other invariants—the Hurewicz Theorem for homology, the long exact sequence of homotopy groups for the homogeneous space associated to a closed subgroup of a Lie group, and a study of aspherical spaces (spaces for which the higher homotopy groups vanish) showing that their homotopy type and homology groups are determined solely by the fundamental group.

In the mid-30's the newly defined cohomology ring was also developed. Furthermore, Hassler Whitney (1907–1989) had introduced the notion of sphere spaces and had made progress toward a classification of them using characteristic classes, introduced independently by Whitney and Eduard Stiefel (1909–1978), a student of Hopf. The Stiefel-Whitney classes of manifolds are obstructions to the extension of families of vector fields. The combination of cohomology and homotopy groups (the lower dimensional homotopy groups of Stiefel manifolds in this case) lies at the heart of seminal work of Eilenberg giving a general obstruction theory for the extension of mappings. The ideas of simple spaces (where the fundamental group acts trivially on the higher homotopy groups), local coefficients (introduced by K. Reidemeister (1893–1971)), Hopf invariants, and the various classification theorems of Hopf and Whitney are all encompassed by Eilenberg's general method.

Outside the centers of Princeton and Vienna, another approach to topological questions was being developed in France and Switzerland during the 20's and 30's. Recalling the differential methods of Poincaré, Élie Cartan (1869–1951) published a series of papers in which the topology of a Lie group is used to deduce some of its global analytical properties. In his study of the linear independence of differential forms on a Lie group up to coboundary, Cartan conjectured that the resulting numbers ought to be the Betti numbers, that is, combinatorial invariants of a manifold could be deduced from the differentiable structure. In 1931, Georges de Rham (1903–1990)

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[1]Based on the author's paper, *A history of spectral sequences: Origins to 1953*, in History of Topology, 631663, edited by Ioan James, North-Holland, Amsterdam, 1999.

[2]Eilenberg reviewed these notes for the Mathematical Reviews–MR #8,49d, 8,49e, 8,166b, 8,166e.

[3]E. Čech had already defined the higher homotopy groups at the 1932 International Congress of Mathematicians in Zürich. However, these groups were treated then as a mere curiosity.

proved Cartan's conjecture establishing differential forms as a subtle tool for the study of algebraic topology. Applying these ideas, R. Brauer (1901–1977) carried out the computation of the Betti numbers of the classical groups. The Cartan program of deducing the topology of Lie groups algebraically via the Lie algebra was completed in the work of Weil, Chevalley, Koszul, and Henri Cartan to be discussed below[4].

§2. Some algebraic topology during the war

The interruption caused by the Second World War damped the vitality of research in algebraic topology but in no way did it stop it. Though communication became difficult during the war, considerable advances in topology appeared in these years. In countries at war, mathematicians were by and large involved in the war effort. Without graduate students and communication with other mathematicians, progress slowed.

Isolated from the war, Switzerland represents a notable exception. Several papers came out of the Swiss school of Hopf and his students during these years that are central to this account.

The first and most important appeared in the 1941 *Annals of Mathematics* when the war had stopped publication of *Compositio Mathematica*. Hopf introduced a generalization of compact Lie groups, his notion of a Γ-*Mannigfaltigkeit*, which today is called an *H-space*[5]. An H-space (or H-manifold in Hopf's case) is a space endowed with a continuous multiplication and a unit with respect to this multiplication. The main application of this idea is to prove a generalization of a result of L. Pontryagin (1908–1988), who had computed the rational homology of the classical Lie groups. Hopf showed that Pontryagin's result followed from the structure of an H-space and not the more subtle properties of Lie groups. He proved that the rational cohomology of the exceptional Lie groups are that of a product of spheres, a result left open by the case-by-case analysis of Pontryagin and Brauer. This paper is a landmark in algebraic topology. Hopf had shown how to reverse the flow of ideas to go from the topological to the analytic, thus demonstrating the potential of certain fundamental topological and algebraic structures.

A major step in expressing the relation between the various topological invariants of the base, fibre and total space of a sphere space is taken in the thesis of Werner Gysin (1915–??), a student of Hopf in Zürich. Gysin studied the homology structure of a sphere space composed of manifolds via a construction associated to a simplicial mapping. In modern parlance, Gysin had identified a form of the transgression homomorphism which, in the case of sphere spaces, is realized as a long exact sequence called the *Gysin sequence*:

$$\cdots \to H_p(M;\mathbb{Q}) \to H_p(B;\mathbb{Q}) \to H_{p-d-1}(B;\mathbb{Q}) \to H_{p-1}(M;\mathbb{Q}) \to \cdots .$$

Outside Switzerland, other developments were published during the war. Notably the work of Ch. Ehresmann (1905–1979) and J. Feldbau (1914–1945) extended the differential geometric notion of a connection to homotopy theory and fibre spaces.

[4] See also the third volume of the series *Connections, Curvature, and Cohomology* by Greub, Halperin and van Stone, Academic Press, New York, New York, 1976.

[5] The terminology *H-space* (for *Hopf space*) is due to Serre.

Steenrod carried out a program of research that culminated in his celebrated book, *The Topology of Fibre Bundles*.

We next take up our main story, the work of Leray during and after the Second World War.

§3. Leray

Before the war, Jean Leray (1906–1998) had made substantial contributions to the mathematical study of fluid dynamics. He had participated in the mathematical circle around Élie Cartan[6]. Among those contributions is a paper written with Julius Schauder (1899–1943) in which they extend the fixed point methods of Brouwer to establish the existence of solutions of certain classes of differential equations.

At the outbreak of the war, Leray was an officer in the French Army. After France was occupied by the Germans he was arrested by the Germans and was taken to an officers' prison camp in Edelbach, Austria, OFLAG XVIIA, where he spent the remaining war years. *Une université en captivité* was organized with Lieutenant Leray as director for which the captors provided library access from the University of Vienna. Leray feared that his specialty of applied mathematics might lead to forced support of the German war effort, and so he admitted only his experience in topology as his focus of research and teaching.

Leray's fixed point work with Schauder utilized an approximation procedure based on the classical Brouwer theorem that, in a limit, proved the desired result for a Banach space. Leray sought to avoid having to go through the simplicial theory and so chose to work solely at the level of the topological: From his course:

My initial intention was to imagin a theory of equations and of transformation applying directly to topological spaces. I had to have recourse to new procedures, avoiding the classical procedures, and it was impossible for me to expose this this theory of equations and of transformations, without, on one hand, giving a new definition of the cohomology ring, and on the other hand, adapting the cited reasoning of M. Hopf to hypotheses more general than his.[7]

Among the sources Leray acknowledges in his introduction are the papers of de Rham, Alexander, Kolmogoroff, Alexandroff, and Čech on cohomology, as well as Hopf's seminal paper on H-spaces. Leray takes the cohomology ring as the fundamental topological invariant of study and names his theory *l'anneau d'homologie*, reserving the term *groupes de Betti* for the homology groups (following Alexandroff and Hopf). The complete *Cours de Topologie Algébrique professé en captivité* was published in Liouville's Journal in 1945.

Leray's basic object is the *couverture*[8]. To define a couverture Leray begins with an

[6] Leray had written up Cartan's lectures leading to the book, *La méthode du répère mobile, la théorie des groupes continus et les espaces généralisés*, Hermann, 1935.

[7] Mon dessein initial fut d'imaginer une théorie des équations et des transformations s'appliquant directement aux espaces topologiques. J'ai dû recourir à des procédés nouveaux, renoncer à des procédés classiques, et il m'est impossible d'exposer cette théorie des équations et des transformations, sans, d'une part, donner une nouvelle définition de l'anneau d'homologie et d'autre part, adapter les raissonnements cités de M. Hopf à des hypothèses plus générales que les siennes.

[8] Borel writes in: "I do not know of any translation of *couverture* in the mathematical literature." In his reviews of these papers, Eilenberg uses *cover* for couverture, but this is obviously inadequate. I follow the notation of Borel in this survey of Leray's work.

abstract *complex*, a graded module K over a ring R, required to be finitely generated and equipped with a differential d of degree 1. An abstract complex K is made *concrete over a space* X if there is an assignment to each nonzero $k \in K$ of a nonempty subset of X, called the *support of* k, written $|k| \subset X$, and required to satisfy $|d(k)| \subset |k|$. A basic operation on concrete complexes is that of taking a *section*over a subspace of X. We write xK for a section over a point x. Given two concrete complexes K and K' over X and a point $x \in X$, consider

$$r_x \colon K \otimes_R K' \to xK \otimes_R xK'$$

given by the tensor product of the quotients. If h is in $K \otimes K'$, let $|h| = \{x \in X \mid r_x(h) \neq 0\}$. This defines a concrete complex $K \bigcirc K'$, the *intersection* of K and K'.

A *couverture* is a concrete complex K for which all supports are closed and xK is acyclic for all $x \in X$, that is $H^p(xK) = \{0\}$ for all $p > 0$ and $H^0(xK) \cong R$ with generator the *unit cocycle*, $K^0 = \sum_\alpha x^{0,\alpha}$, the sum of the generators of degree zero.

If X is normal, then the collection of all couvertures on X is a differential graded R-algebra with product given by $\bigcirc$. Its cohomology is Leray's *anneau d'homologie* of X, $H(X, R)$. If a couverture has acyclic supports, then $H(X, R)$ may be computed from that of the underlying abstract complex.

Having set up his cohomology theory, Leray turns to applications. Most are the classical theorems of fixed point sort (his *théorie des équations*), thus giving a new way to obtain his results with Schauder. Leray draws particular attention to his ability to prove the main theorems of Hopf on manifolds with multiplication in this context.

At the heart of Leray's development of his cohomology theory there is an argument, his *Lemme 2* of No. 4, in which he proves that the product of a given complex with an acyclic complex has the same homology as the given complex. The Lemma shows that the product of point sections of two couvertures $xK \bigcirc xK'$ is again acyclic. This result is key to Leray's proof of the Künneth theorem. The argument is an induction on the *weight* of a cocycle, which, in this case, is the maximal degree of an element of the acyclic complex. The same argument occurs in four places in Leray's big paper and it is the precursor of what will become the underlying structure of a spectral sequence.

Already in the 1945 paper, Leray had studied the effect of a mapping (*représentation*) $\phi \colon E' \to E$ at the level of the couvertures. Let ϕ^{-1} be the inverse transformation of ϕ, generally multivalued, and define $\phi^{-1}(k)$ when k is a *forme de* E, that is, a class of an element in a couverture on E. The mapping ϕ^{-1} effects a change of supports for a concrete complex with $|\phi^{-1}(k)| = \phi^{-1}(|k|)$, and hence, defines a new concrete complex on E'. Though one obtains a complex in this way, a couverture need not go over to a couverture. The generalization of Steenrod of homology to local coefficients offered a model for Leray that could be extended and incorporated into his cohomology theory. The result was a series of remarkably original notes appearing in *Comptes Rendus* in 1946, where Leray introduces sheaves (*faisceaux*[9]) and spectral sequences. These developments are aimed at the study of fibre spaces, though the methods apply more generally to any continuous mapping of locally compact spaces.

[9]Weyl, in his 1954 ICM address on the work of the Fields Medal winners, Kodaira and Serre, remarked that "Princeton has decreed that 'sheaf' should be the English equivalent of the French 'faisceau."'

Eilenberg[10] reviewed Leray's 1946 *Comptes Rendus* notes very briefly and somewhat cryptically. The next year, however, Leray's work was transformed at the hands of Jean-Louis Koszul (1921–) and Henri Cartan (1904–2008). In an elegant note in the *Comptes Rendus* of 1947, Koszul extracted from Leray's description of the homology ring of a mapping an algebraic construction that gives rise to all of the structure.

In a subsequent note, Koszul considers an additional grading on A by degree giving everything in sight a bigrading. The main example is the exterior algebra generated by the dual of the Lie algebra associated to a compact, connected Lie group.

June 26 to July 2, 1947, the *Colloque International de Topologie Algébrique* took place in Paris. The participants included H. Cartan, Leray, Ehresmann, Freudenthal, Hirsch, Hodge, Hopf, Koszul, de Rham, Stiefel, J.H.C. Whitehead, and Whitney. The published papers of Cartan and Leray differ considerably from their delivered talks. By this time Cartan had received a letter from Weil, then in Brazil, in which he sketched his celebrated proof of the de Rham theorem. Cartan, in his published short report to the conference, offered his criticism of the work of Eilenberg and Steenrod axiomatizing homology:

I nevertheless want to try to characterize here in a few lines the conjecture that was the object of my exposé of 27 June, 1947. ... This theory differs from that of Eilenberg and Steenrod ... on many important points. From the beginning, it only tries to axiomatize the cohomology in the sense of Čech, or in the sense of Alexander ... while, to the contrary, the theory of Eilenberg-Steenrod sets for itself the goal of including all the homology and cohomology theories. Precisely for the reason of its generality, the theory of Eilenberg-Steenrod can neither carry theorems of uniqueness for very particular spaces, nor the principal interest of our theory that resides in its theorems of uniqueness.[11]

The potential for application of the methods of algebraic topology to questions in analysis rested on having analytic means apparent and accessible at the level of cohomology, as in the de Rham theory. Furthermore, the unique properties of spaces such as manifolds, or polyhedra, leading to duality theorems, are glossed over in the global approach of Eilenberg and Steenrod. Cartan had recognized the potential of Leray's theory of *couvertures* and *faisceaux* and he began a program to clarify the foundations. Between the lecture in Paris and the publication of the proceedings, Cartan gave a course on algebraic topology at Harvard (spring 1948) in which he presented a version of Leray's complexes with supports. In the proceedings paper Cartan sketched his program, naming the relevant topological structure *carapace*.

[10] MR#8,49d, #8,49e, #8,166b, #8,166c.

[11] *Je voudrais néamoins tenter de caractériser ici en quelques lignes la tentative qui avait fait l'objet de mon exposé du 27 juin 1947. ... Cette théorie diffère de celle d'Eilenberg et Steenrod ... sur plusieurs points importants. Tout d'abord, elle ne vise à axiomatiser que la cohomologie au sens de Čech, ou au sens d'Alexander ... au contraire, la théorie d'Eilenberg-Steenrod se donnait pour but d'englober toutes les théories de l'homologie ou de la cohomologie. Précisément à cause de sa généralité, la théorie d'Eilenberg-Steenrod ne pouvait comporter de théorème d'unicité que pour des espaces de nature très particulière; tandis que l'intérêt principal de notre théorie réside dans ses théorèmes d'unicité.*

Koszul[12] recalls Leray's lecture treating the action of a discrete group on a topological space in which the Cartan-Leray spectral sequence is introduced. He writes: *"Cela était tout inattendu et a fait sensation."* He also reports that after Leray's lecture on this work, Whitney rose to say that, after this talk, he no longer understood algebraic topology, and, if homology was going to be like this, he would have to study other parts of mathematics. (In fact, he did.) A short note coauthored by Cartan and Leray sketching this application appeared in the proceedings. Leray's proceedings paper is based on his courses at the Collège de France 1947-48. Already in this paper, he has adopted the algebraic refinements of Koszul and the *"perfectionnements"* of Cartan on differential graded and filtered algebras.

The powerful use of algebra in Koszul's thesis demonstrated that purely algebraic objects, together with their apparent additional structure and associated objects like the spectral sequence, gave a detailed picture of the workings of geometric phenomena, in particular, the topology of homogeneous spaces. In the framework of Lie groups, where Lie algebras are available, the first nontrivial cases of fibre spaces, homogeneous spaces, had been analyzed. The next step in a more general direction was taken by Cartan.

Cartan had spent the spring of 1948 at Harvard where he lectured on topology. Once back in Paris, Cartan established his famous seminar, the *Séminaire Cartan* which met from 1948 to 1964. The first three years of topics deal with algebraic topology. In the first year, the topics were foundational, dealing with simplicial, singular, and Čech theories. The last few lectures of Cartan, numbered 12 through 17, were not published in the 1955 reissue of the notes by the *Secrétariat mathématique.* These dealt with the theory of sheaves and carapaces, drawing from his lectures at Harvard, but still in a preliminary form for Cartan.

The lectures of the Cartan Seminars remain among the clearest expositions of certain topics in algebraic topology. The role of this level of exposition is crucial in this account. The atmosphere of consolidation of a growing subject and the wealth of challenging problems open to the initiated made the proceedings of the *Séminaire Cartan* a window of opportunity to a maturing field.

§4. The Summer of 1950

The summer of 1950 began with a major conference event in the history of topology, the *Colloque de Topologie (espaces fibrés)*, which took place in Brussels, 5–8 June, organized by Guy Hirsch. This conference provides a glimpse of the state of progress on the problem of the homology of fibre spaces. The proceedings[13] opens with a note of appreciation to Élie Cartan "whose works had opened the way for much of the research presented in the course of the meeting." The published speakers were Hopf and Eckmann from Switzerland, H. Cartan, Leray, Ehresmann, Koszul from France, and Hirsch from Belgium.

In two landmark papers given at this conference Cartan gave his penetrating analysis of the transgression for homogeneous spaces and principal fibre spaces. A *principal fibre space* $E \to B$ with structure group G as fibre, is a G-space E with B

[12] Letter of April 30, 1997.

[13] *Colloque de Topologie (espaces fibrés)*, Bruxelles, 1950, CBRM, Liège, 1951.

as its orbit space. Based in part on Koszul's thesis and work of Chevalley and Weil[14], Cartan exposed an algebraic formalism on which the structure of the cohomology of a differentiable principal bundle may be founded.

Chevalley writes in his review[15], *"At this stage, the topology may be thrown out, leaving only the algebraic facts in evidence."*

The talks of Hopf and Eckmann in Paris emphasized the homotopy-theoretic approach to fibre spaces. In his talk Eckmann[16] reported some progress on a question of Deane Montgomery (1909–1992) and Samelson as to whether Euclidean n-space, $\mathbb{R}^n$, can be the total space of a fibre space with compact fibre. In the first issue of the *Proceedings of the American Mathematical Society*, April 1950, Gail S. Young (1915–1999) published partial results on this problem.

During the academic year, Armand Borel (1924–2003) and Jean-Pierre Serre (1926–) attended Leray's course on topology at the Collège de France. Borel stuck to the classes, but Serre flagged in enthusiasm, and learned about spectral sequences from Borel. Borel and Serre decided to apply Leray's ideas to this problem and quickly came up with a complete solution—there is no fibration of $\mathbb{R}^n$ with a compact fibre that does not reduce to a point. The proof is *"une application simple"* of the ideas of Leray.

Post-war topology was thriving. The directions of research in place before the war were being played out successfully with the addition of new questions, new methods and new researchers. The task of consolidation was undertaken by the leaders of the field (Cartan in France, Eilenberg and Steenrod in the US). This remarkable time presented a rich field of opportunity for new and able researchers. The next two years realized that opportunity and changed the field of algebraic topology dramatically.

§5. Serre's thesis

After the successful application of Leray's spectral sequence to the question of Montgomery and Samelson and a version of Leray's *anneau spectral* for cohomology of groups, Serre turned to other possible applications of these ideas. An example of a fibre space that was well-known in 1950 is given by the limit of the finite-dimensional complex projective spaces, $S^1 \hookrightarrow S^\infty \to \mathbb{C}P(\infty)$. Since S^1 is an Eilenberg-Mac Lane space, $K(\mathbb{Z},1)$, and S^∞ has trivial homotopy groups, this fibre space identifies $\mathbb{C}P(\infty)$ as a $K(\mathbb{Z},2)$. The *algèbre spectrale* of Leray for this case was clear and suggested the possibility that an induction might lead to the computation of the cohomology of the higher Eilenberg-Mac Lane spaces from this initial case. Serre writes:

I had noticed that the theory of Leray allows one to arrive at this calculation by proceeding by induction on n, because on has at one's disposal a fibre space E have the following properties: a) E is contractible, b) the base of E is a $K(\pi,n)$, which implies that c) the fibres of E are $K(\pi,n-1)$'s.[17]

[14]See and Weil's endnotes for a version of his unpublished work on this subject.

[15]MR# 13,107e,f.

[16]Letter of A. Borel, May 5, 1997.

[17] *"J'avais remarqué que la théorie de Leray permet d'aborder ce calcul en procédant par récurrence sur n, pourvu que l'on dispose d'un espace fibré E ayant les propriétés suivantes: a) E est contractile, b) la base de E est un $K(\pi,n)$, ce qui entraîne c) les fibres de E sont des $K(\pi,n-1)$."*

The identification $\Omega K(\pi, n) \simeq K(\pi, n-1)$ soon led Serre to the insight that the sequence of spaces $\Omega X \hookrightarrow PX \to X$ where $PX = \{\text{continuous maps } \lambda\colon (I, 0) \to (X, x_0)\}$ and the mapping $PX \to X$ is evaluation at 1, $\lambda \mapsto \lambda(1)$ would do.

The singular theory of Lefschetz and Eilenberg offered the best properties for the study of homotopy-theoretic constructions, especially via the Hurewicz homomorphism. However, there lacked an analogue of Leray's theory for singular homology. At a Bourbaki meeting in the fall of 1950, Serre discussed this problem with Cartan and Koszul. He writes,

"...fort heureusement, J.-L. Koszul et H. Cartan m'ont suggéré une certaine filtration du complexe singulier ... qui s'est révélée avoir toutes les vertus nécessaires."

The suggestion led to the technical part of Serre's thesis (Chap. II of) which is based on cubical singular theory for which the cubes *"lend themselves better than simplices to the study of direct products, and, a fortiori, of fibre spaces that are the generalization of them.*[18]

Having overcome the technical difficulties, the consequences were announced in a series of three *Comptes Rendus* notes. In the first note the term *spectral sequence* (*suite spectrale*) makes its appearance. It applies to the homology spectral sequence for which the terms *spectral ring* or *spectral algebra* of Leray and Koszul did not apply.

The thesis was finished in the spring of 1951 and sent to Steenrod for the *Annals of Mathematics* at the advice of Eilenberg. Steenrod gave it priority and it appeared at the end of 1951. The thesis is a remarkable mix of technical detail and simple direct argument.

The ability to compute the homology of Eilenberg-Mac Lane spaces led to new computations in homotopy theory. In particular, Cartan and Serre introduced a new homotopy-theoretic construction giving fibre spaces susceptible to analysis with the Serre spectral sequence. They introduced the method of *killing homotopy groups*. George Whitehead independently considered such a tower of spaces. At around the same time M.M. Postnikov (1927–2004) introduced a dual construction, the *Postnikov tower*, using the simplicial methods of Eilenberg in an effort to understand the degree to which the homology of a space is determined by its homotopy groups.

In the spring of 1952, Serre announced his complete computation of the mod 2 cohomology of the Eilenberg-Mac Lane spaces. The expected inductive argument was successful with the introduction of the idea of a simple system of generators, due to Borel.

The final major paper of this period takes off from the proof of the finitude of the homotopy groups of spheres.

§6. Borel's thesis

Coming from ETH, Borel spent the academic year 1949–50 in Paris at the CNRS and attended Leray's course at the Collège de France, in fact, he helped to complete the exposition in a note appearing at the end of Leray's paper. Borel's thesis begins with the substance of the course of Leray in 1949–50. Since Lie groups and homogeneous spaces are locally compact, and Leray's methods are general enough to include

[18] *"se prêtent mieux que les simplexes à l'étude des produits directs, et, a fortiori, des espaces fibrés qui en sont la généralisation."*

both de Rham cohomology and singular cohomology with coefficients in any ring, they became his tool of choice. In particular, the basic methods were topological and the analytic structure of Lie groups played a minor role. All of the strands of research up to 1950 concerning the topology of Lie groups find a place in Borel's thesis.

On March 25, 1952, Borel submitted his thesis before a committee of Leray (Président), Cartan and Lichnerowicz to obtain his doctorate from the Université de Paris. The paper appeared in the 1953 *Annals of Mathematics*.

Another test of the topological methods was the new set of invariants given by the Steenrod operations at each prime. Wu and Thom had demonstrated the importance of these operations mod 2 for characteristic classes by 1950. The computation of the mod p cohomology of compact Lie groups was amply demonstrated in Borel's thesis. After a lecture series by Steenrod in May 1951, Serre and Borel tackled the question of the mod p operations and successfully determined the action of the reduced powers on $\mathrm{U}(n)$, the unitary groups, $\mathrm{Sp}(n)$, the symplectic groups, and $\mathrm{SO}(n)$, the special orthogonal groups . This computation brought the Chern classes into the framework of Wu and Thom, and also settled many cases of the nonexistence of sections of certain fibre spaces given by homogenous spaces. In particular, Borel and Serre settled a problem of Hopf as to which spheres possessed an almost-complex structure. They show that only S^2 and S^6 can have such a structure, a surprising result at the time.

§7. Reception

Leray was unhappy with the slow acceptance of his ideas: *"Ces notions furent mal accueillies en Amérique au moment de leur publication. C'était trop difficile. Les Mathematical Reviews demandèrent 'À quoi ça peut servir?'"*

The method of spectral sequences did not spread rapidly after its initial appearance. Leray's 1946 *Comptes Rendus* notes led Koszul and Cartan in Strasbourg to extract the algebraic essence from which further constructions could be made, notably by Koszul in his thesis, by Cartan in his work on the transgression, by Cartan and Leray for finite groups acting on a space, and by Serre in his note on the cohomology of groups. Leray However, little interest outside of France was evident before Serre's thesis.

Several missed opportunities present themselves—Mac Lane (1909–2005)[19] reports visiting Paris in 1947 and discussing sheaves and spectral sequences with Leray. Lyndon's thesis under Mac Lane is founded on a filtration of the cohomology of a group $H^*(G;M)$ relating the associated graded to subquotients of $H^*(Q;H^*(K,M))$ when $1 \to K \to G \to Q \to 1$ is a group extension and M a G-module. Mac Lane admits not making the connection between Lyndon's work and Leray's work—"Leray was obscure!"

Another near miss is the work of Tatsuji Kudo in Japan. In the 1950 Osaka Journal, Kudo analyzed a fibre space with CW-complex as base by considering the preimages of the skeleta and the resulting long exact sequences of pairs. In a subsequent paper, published in 1952, Kudo began to work with Leray's ideas and recast his previous analysis in this language.

Whitehead, Massey and others in the United States did try to understand Leray's

[19] Letter of August 11, 1997.

work after its appearance, in order to get at the source of the "marvelous results he claimed... [20]". The exact sequence was a fundamental tool of expression by this time. Massey soon gave a new algebraic reformulation of spectral sequences, his *exact couples*.

With the arrival of Serre's thesis at the *Annals of Mathematics* Steenrod[21] spread the word in the United States of some "earth-shaking results on the homotopy groups of spheres." He also sent it out to George Whitehead (then at Brown) and to Henry Whitehead (at Oxford), both among the few world's experts on the homotopy groups of spheres.

In contrast to the perception of Leray's papers, Serre's work was "brilliantly clear" in exposition[32] and its effect was immediate.

From Henry Whitehead's group, Peter Hilton (1923–2010) took immediately to spreading Serre's and Borel's work, in particular, to his seminar in Cambridge where he had arrived from Manchester in 1952. Among some of the early participants were J.F. Adams, M.F. Atiyah, E.C. Zeeman, D.B.A. Epstein, and C.T.C. Wall. At Oxford, Whitehead held lectures on Serre's results and invited him to visit. I.M. James recalls this visit and the impact of Serre's thesis on his work.

In the United States Eilenberg had followed the development of spectral sequences from the beginning. In the *Séminaire Cartan* of 50/51, Eilenberg presented his version of spectral sequences in two lectures January 22 and February 5, 1951. In these reports he described another construction of spectral sequences that is featured in the classic book with Cartan *Homological Algebra*.

At Brown, George Whitehead's first Ph.D. student John C. Moore (1926–) immediately took up Serre's methods in his thesis.

In the former Soviet Union, leadership in topology was changing around the time of Serre's and Borel's work. In Moscow, a seminar was held on Serre's thesis only in 1956 led by Albert S. Schwarz (1934–), M.M. Postnikov (1927–2004) and V.G. Boltyanskiĭ (1925–). The participants of the seminar included S.P. Novikov, D.B. Fuks, A.G. Vinogradov, and others who went on to make up the next generation of Soviet topologists.

§8. Closing Remarks

In Steenrod's report of the spring 1953 conference on *Fiber bundles and differential geometry* at Cornell University he writes of the

"upheaval within topology itself resulting from the use of fiber space techniques. ... The most striking feature of the conference was the frequent use of the same apparatus in two or more widely separated disciplines, with strong suggestions of a probable unification of geometry on some higher level."

That apparatus was the spectral sequence. The landscape of homotopy theory had changed radically and central to this change was the appearance and application of spectral sequences, in particular, of Serre's and Borel's Paris theses.

How was topology different after the introduction of spectral sequences? It is certain that the algebraic content in algebraic topology increased. However, this may

[20] Letter of November 6, 1996.
[21] Letter from George Whitehead of September 6, 1997.

be understood more subtly as a kind of algebraization of the subject. Several currents support this development.

The period of development of algebraic topology examined in this brief history is remarkable. The rich atmosphere of difficult problems and untested methods around 1950 was ripe for the sudden realignment that occurred. Among the factors making these changes possible pedagogy played an unexpectedly important role—the spread of the crucial ideas was made possible by the high standard of exposition of the *Séminaire Cartan* and the subsequent clarity of the doctoral theses of Serre and Borel.

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