

Bourbaki and Algebraic Topology

by John McCleary

a talk¹ given at the University of Casablanca, 4.VI.2013

Introduction

It has been almost 80 years since the founders of *Le Comité de rédaction du traité d'analyse* met in Paris at the *Café A. Capoulade*, 63 boulevard Saint-Michel, to discuss the drafting of a textbook on analysis. This meeting included HENRI CARTAN (1904–2008), CLAUDE CHEVALLEY (1909–1984), JEAN DELSARTE (1903–1968), JEAN DIEUDONNÉ (1906–1992), RENÉ DE POSSEL (1905–1974), and ANDRÉ WEIL (1906–1998). The fate of this project is the story of the *Bourbaki*, or should I say, the character NICOLAS BOURBAKI, author of *Éléments de mathématique*, a series of influential expositions of the basic notions of modern mathematics.

In early 2000, I learned at a meeting in Oberwohlfach that an archive of papers and internal documents of the Bourbaki was soon to be opened in Paris and the Beck fund at Vassar College provided me the means to visit the archive. The managers of this archive, Liliane Beaulieu and Christian Houzel, showed me great hospitality during my visit to Paris in July 2003, and made it possible for me to study the Bourbaki papers. These papers are now available on the web².

Historical research poses questions, to which various methods may be applied. My interests include the history of algebraic topology, a subject whose development during the twentieth century influenced a great deal of that century's mathematics. The years following the Second World War represent a high point in this story, and several important members of Bourbaki contributed to this development. However, the subject does not appear among the topics treated in *Éléments*—admittedly with many other important topics. Why?

The rumor

While I was a graduate student, I heard a rumor that there was a manuscript, 200 pages long, prepared for *Éléments* by Cartan, Koszul, Eilenberg, and Chevalley, treating algebraic topology. Furthermore, this document was based on the use of differential forms, that is, algebraic topology chez ELIE CARTAN (1869–1951) (*le pere d'Henri*). According to the story I heard, the manuscript was abandoned when the doctoral theses of JEAN-PIERRE SERRE (1926–) and ARMAND BOREL (1923–2003) were published. Serre's and Borel's subsequent papers did change the focus in topology, away from differential geometric methods to more algebraic methods, principally the spectral sequence and the Steenrod algebra, making the manuscript obsolete. My questions: So what was in this manuscript? Could I get a look at it? For the historian such a manuscript offers a look at the manner in which researchers viewed a field of study before and after a key event.

Well, *the manuscript wasn't there*, if, in fact, it exists. Two fiches can be found in the Bourbaki archive entitled *Topologie algébrique*. The first is 75 pages on algebraic

¹This talk is based on a project, supported by the Gabriel Snyder Beck Fund at Vassar College that funds research on anything French.

²<http://math-doc.ujf-grenoble.fr/archives-bourbaki/>

limits, direct, inverse, and on duality. This document reads as *topological algebra*. The second chapter is titled

POUR LE CHAPITRE I DU BLOC HOMOLOGIQUE.

It treats the homological algebra of graded modules with differential, up to cohomology, the Künneth and Universal Coefficient Theorems, and citing the example of the de Rham complex as an instance of the algebra presented. Once again, the manuscript was particularly algebraic, and barely topological in nature.

The rest of the archival work I was able to do, however, offered many insights into the workings and spirit of Bourbaki and I will relate some findings in this report. As my story unfolds, I want to consider the attraction of the axiomatic method before and after Bourbaki, one of the features of their exposition that has inspired discussion and criticism.

Who is Bourbaki?

The meeting of 10.XII.1934 in Paris was organized by André Weil who was on the faculty at the University of Strasbourg at the time, together with Henri Cartan. They were responsible for the course on the differential and integral calculus, one of three standard courses for the *license de mathématiques*, along with general physics and rational mechanics. The standard text was *Cours d'Analyse mathématique* by ÉDUARD GOURSAT (1858–1936), written before the First World War. Cartan found it wanting, incomplete where generalizations were known, and simply not the best way to present these topics. An explicit example (one with a story of its own) is the formulation of *Stokes's Theorem*:

$$\int_{\partial X} \omega = \int_X d\omega,$$

where ω is a differential form, $d\omega$ its exterior derivative, X the domain of integration and ∂X the boundary of X . When everything in sight is smooth, the proof is clear, but the importance of this formula in the case of more general domains of integration is the content of the celebrated theorem of GEORGES DE RHAM (1903–1990), proved in 1931, to answer a question of Elie Cartan relating invariant integrals on Lie groups to the topology of such manifolds.

Persistent complaining by Cartan led Weil to suggest that they write a textbook that they could be satisfied with. Weil writes that he told Cartan, “Why don’t we get together and settle such matters once and for all, and you won’t plague me with your questions any more?”

The first meetings in Paris to plan the book came after the regular meeting of *Seminaire Julia*, another of Weil’s and Cartan’s efforts to fill the gap left in French mathematics after the “hectatomb of 1914–1918 which had slaughtered virtually an entire generation” of French mathematicians, in Weil’s words. The seminar, organized by these young turcs in imitation of the seminars in Germany, needed a sponsor in order to get a room at the Sorbonne. GASTON JULIA (1893–1978) had been the youngest of their teachers at the *École Normale Supérieure* and he stepped up to sponsor them. The seminar treated a topic a year, beginning in 1933-34 with groups and algebras, going on to Hilbert spaces, then topology. The seminar continued until 1939 when it was superseded by the *Seminaire Bourbaki*.

The committee's original plan was a text in analysis, that would, according to Weil, "fix the curriculum for 25 years for differential and integral calculus." This text should be *aussi moderne que possible, un traité utile à tous*, and finally, *aussi robustes et aussi universels que possible*. Weil already knew a potential publisher in his friend Enriques Freymann who was chief editor and manager of *Maison Hermann*.

Among the innovations was the suggestion, insisted on by Delsarte, that the text be written collectively without *expert leadership*. The initial expectation was that the text would comprise 1000–1200 pages and be done in about six months. The initial group of six was expanded to nine members in January 1935, with PAUL DUBREIL (1904–1994), JEAN LERAY (1906–1998) and SZOLEM MANDELBROJT (1899–1983) added. Dubreil and Leray were replaced by JEAN COULOMB (1904–1999) and CHARLES EHRESMANN (1905–1979) before the first summer workshop in July, 1935.

The first Bourbaki congress was held in Besse-en-Chandesse in the Vosges mountains. At this workshop, the proposal was made to expand the project to add a *paquet abstrait*, treating abstract (new and modern) notions that would support analysis. These included abstract set theory, algebra, especially differential forms, and topology, with particular emphasis on existence theorems (Leray).

The *paquet* eventually became the *Fascicule de Résultats*, a summary of useful results presented in such a way that a competent mathematician could see where a desired result might be found, and provide the result themselves if they needed it. In fact, the last publication, *Fascicule XXXVI*, part two of *Variétés différentielles et analytiques*, is such a summary. It is here that the statement of Stokes's Theorem finds its place (finally).

During one of the first conferences, a new result on measures on a topological space was proved and a note was written up to submit to *Comptes-Rendus*. The name of Bourbaki for the group came from a story out of school: In 1923, Delsarte, Cartan, Weil were members of the newly matriculated class at *École Normale Supérieure*, when they received a lecture notice by a professor of vaguely Scandinavian name, for which attendance was strongly recommended. The speaker was a prankster, RAOUL HUSSON, wearing a false beard and speaking with an undefinable accent. Taking off from classical function theory, the talk had its climax in *Bourbaki's Theorem* leaving the audience "speechless with amazement." (This Bourbaki was the general who traveled with Napoleon.) Weil recalled this story and the name adopted. But why Nicolas? For the submission of the paper, the author needed a *prenom*. It was Weil's wife Eveline who christened Bourbaki Nicolas. The note was handled at the *Académie des Sciences* by Elie Cartan who stood up for the unfortunate Poldevian mathematician. The note was accepted and published.

The method of editing adopted by the Bourbaki grew out of the desire to maintain communal involvement. A text was brought before a meeting and presented, page by page, line by line, to the group who then expressed any and all criticism. A revision was handed over to another member of the group and the process repeated when a new draft was available. After enough iterations to obtain unanimous approval, either for the strength of the text or the fatigue of the group with the topic, the text would be finalized (usually by Dieudonné) and sent to the publisher.

Digression: The Axiomatic Method

During his ‘apprenticeship,’ Weil traveled extensively, spending time in Germany while the rise of National Socialism to power took place. As he was interested in number theory, he admired the mathematics of the German schools, especially the axiomatic approach led by the work of DAVID HILBERT (1862–1943) and the Göttingen school. French mathematics through the nineteenth century and into the twentieth was dominated by analysis. Even results of a number-theoretic nature were proved through analytic means. The success of Hilbert’s ideas in many fields attracted mathematicians everywhere and so, when looking for a model to shape their project, the members of Bourbaki turned to the axiomatic method. (•)

This phenomenon was not without precedent. When E.H. MOORE (1862–1932) came to lead the University of Chicago mathematics department around 1900, he consciously adopted the style of Hilbert’s *Grundlagen der Geometrie* as modern, precise, and a model to be imitated.

Roughly speaking, the *axiomatic method* is an approach to producing mathematics that presents, after some analysis, a set of axioms from which a collection of theorems may be deduced. The goal in presenting the *right set of axioms* is to avoid deception by intuition. Hilbert’s experience with algebraic number theory (the *Zahlbericht*) and invariant theory led him to tread a path leading to more abstract generalization.

When he turned to elementary geometry in his lectures of 1898–99, students in Göttingen were surprised. His goal in the *Grundlagen* was “to attempt to choose for geometry a *simple* and *complete* set of *independent* axioms and to deduce from these the most important geometrical theorems in such a manner as to bring out as clearly as possible the significance of the different groups of axioms and the scope of the conclusions to be derived from the individual axioms.”

The *Grundlagen* was an immediate success, drawing the following reaction from HENRI POINCARÉ (1858–1912): “The logical point of view alone appears to interest Professor Hilbert. Being given a sequence of propositions, he finds that all follow logically from the first. With the foundation of this first proposition, with its psychological origin, he does not concern himself The axioms are postulated; we do not know from whence they come; it is then as easy to postulate *A* as *C* His work is thus incomplete, but this is not a criticism I make against him. Incomplete one must indeed resign oneself to be. It is enough that he has made the philosophy of mathematics take a step forward”

The philosophical and foundational aspects of Hilbert’s efforts are clear. However, the mathematical aspects are not the focus of most discussions of the *Grundlagen*. Among the exercises in independence he has introduced new objects—in particular, non-Archimedean geometries. By isolating the relations among axiom groups, one can discover how the failure of one or more of the assumptions produces new results—the model of this activity being non-Euclidean geometry. His experience in algebra and number theory also supported this view, that the axiomatic method sharpened one’s tools with which to craft new arguments, discover new phenomena, and retain the past in a tidy manner in the bargain.

Another Göttingen product of importance to Bourbaki was the textbook *Moderne Algebra* by B.L. VAN DER WAERDEN (1903–1996) that appeared in 1930, giving an (•)

organized account of algebra based on axioms that revealed the similarity in approaches to certain results. The notion of isomorphism plays an important role in algebra and later surfaces as a leitmotif for Bourbaki.

It is important to see that Hilbert and van der Waerden, though formal in presentation, really sought mathematical goals that were not about the past, recovering a complete description of a theory, but were forward-looking, providing the reader with a slim scaffolding on which many new results could be built. The degree to which this view became part of the manner in which modern mathematics was done can be measured by the natural feel we have for this sort of presentation. It was not always so.

Algebraic Topology chez Bourbaki

The goal of producing a modern, robust, and universal text led to the most characteristic quality of Bourbaki—a topic was discussed repeatedly in an effort to “digest mathematics, to go to the essential points, and reformulate the math in a more comprehensive and conceptual way [Borel].” The sessions were animated to achieve this goal; after the war, there is a record in the *Journal de Bourbaki*, later known as *La Tribu*, of the rebirth of what were considered classic duels between Cartan and Dieudonné. With their work style and clear goal, “whatever was accepted would be incorporated without any credit to the author. Altogether, a truly unselfish, anonymous, demanding work by people striving to give the best possible exposition of basic mathematics, moved by their belief in its unity and ultimate simplicity [Borel].”

The earliest list of topics dates from the 1935 summer meeting:

Abstract sets (HC)
Algebra (Delsarte)
Real numbers (Dieudonné)
Topology. Theorems of existence (AW, deP)
Integration
Real functions, series, infinite products
Inequalities: O and o
Calculus of differential forms
Geometry
Analytic functions: general part

The subject of topology appears in the list and there was a discussion in the spring of 1935 of possible texts that would support their presentation. The classic books by Kerekjarto, Seifert and Threlfall, and Kuratowski were mentioned (none in French). In the first issues of the *Journal de Bourbaki*, edited by Delsarte, it was reported that Weil was reading the newly published *Topologie I* of Alexandroff and Hopf, and this text was expected to help them avoid any errors in their presentations. The team writing the topology section, Weil, de Possel, and Henri Cartan are reported in 1936 to be reading (Weil), sleeping (de Possel), or to have written nothing but still thinking about it (Cartan).

In the 1930's the essential points of combinatorial topology was discussed among the Bourbaki: already at the summer conference of 1935, an outline by Weil includes dimension, intersection, linking, and degree of mappings with the index of fixed points

among the combinatorial topics. The fundamental group (*groupe de Poincaré*) and covering surfaces were also included.

By 1937 there was a plan for the first volumes together with a target date—completion of the first volume by 1.I.1938. The *paquet abstrait* had grown to include the topics of set theory, algebra, set-theoretic topology and abstract integration. In fact, in keeping with the goal of producing a toolbox for mathematicians, the first publication was not a textbook but a list of results (*un fascicule de résultats sans démonstrations*) on set theory. Beginning the march toward analysis, it was agreed that set theory served as a basis for future volumes.

The *Journal de Bourbaki* was replaced in 1940 by *La Tribu* (*Bullétin, oecuménique, apériodique et bourbachique*). By the time of *La Tribu* the use of the notion of *structure* dominated the formulation of the publishing project. As described later in Bourbaki's entry in Le Lionnais's *Les grands courants de la pensée mathématique*, there were 'mother-structures,' simplest and shared by many mathematical activities; beyond this, one finds 'multiple structures' which blend some number of the mother-structures, for example, topological groups blend the group structure with continuity, while order structures together with algebraic structures give rise to the study of ideals and to integration.

Based on the hierarchy of structures, the *Éléments de Mathématique* presented themselves in parts. Part I dealt with the fundamental structures of analysis. In *La Tribu* of 3–15.IX.1940, Part II treated linear analysis, Part III algebraic analysis (elliptic functions, the theory of numbers), and Part IV differential topology. We find algebraic topology (that is, combinatorial topology) in this scheme in Part I.

- Book 1. Set theory
- Book 2. Algebra
- Book 3. General topology
- Book 4. Topological vector spaces
- Book 5. Elementary techniques
of infinitesimal calculus
- Book 6. Integration
- Book 7. Combinatorial topology
- Book 8. Differentials
- Book 9. Calculus of variation
- Book 10. Analytic functions

A 25 page report on the shape of books 3 and 7 was titled *Topologia Bourbachica* in which the main topics were 1. general topology, 2. topological degree, 3. covering spaces and the Poincaré group, and 4. combinatorial topology (surfaces, Betti groups, Euler-Poincaré formula, indices of vector fields).

Weil was reported to be 'meditating' on the subject of Books 7 and 8, while Ehresmann was working on parts 3 and 4 of Book 7. In late 1941, these books were listed as urgently in need of work, "*la rédaction a le regret . . . que ces livres brillent toujours par leur inexistence.*"³

The summer meeting of 1942 (in Clermont) presented a new organization of Part I:

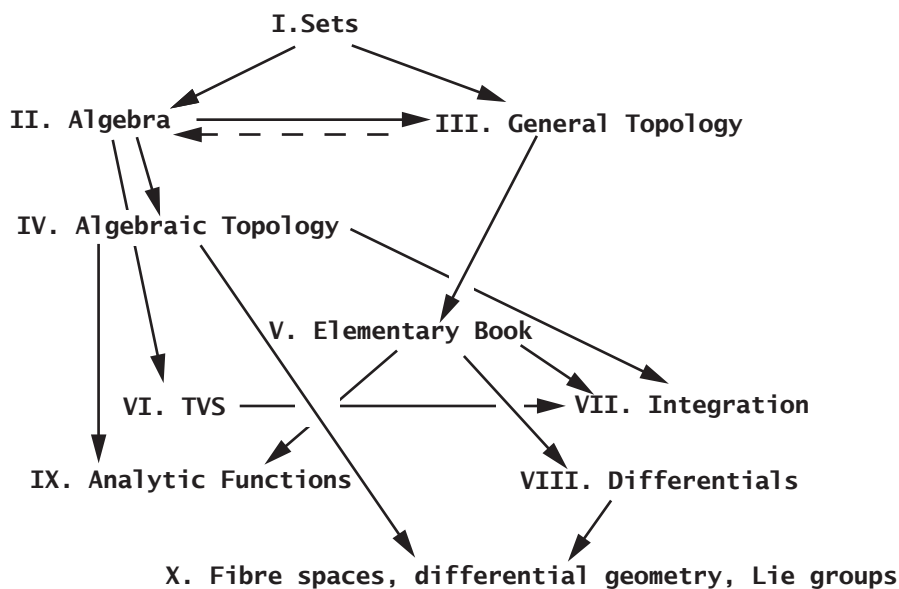
³"the editors regret . . . that these books are conspicuous by their nonexistence."

1. Sets
2. Algebra
3. General Topology
4. Functions of a real variable (elementary theory)
5. Combinatorial topology
6. Topological vector spaces
7. Differential calculus and manifolds
8. Integral calculus and differential forms
9. Analytic functions

On this plan little progress on algebraic topology took place. In *La Tribu* no. 10 of 10–15.IV.1944, it is reported that “*le récent Congrès Bourbaki que s’est tenu à Paris du 6 au 8 Avril 1944 n’ont pas moins réalisé au progrès important et depuis longtemps souhaité par la rédaction: le démarrage de la Topologie algébrique.*”⁴

A description of the core of the subject at the time was given, however: a) there should be no Menger theory of curves, no graphs, no Peano continua, no continua; b) a chapter on knots; c) higher homotopy groups and fibre spaces, which they deemed interesting, having a future, but at present in a state “*trop larvaire.*” The development of this topic took place during the war with the work of Ehresmann, Cartan, and Leray in France, Steenrod and Whitney in the US, and Hopf and Eckmann in Switzerland.

La Tribu of 11–15.1945 contains a picture of the dependencies among topics in Part I, once again featuring algebraic topology near the foundations.



11-15.IV.1945 Congress in Paris, from *La Tribu* no. 8

⁴“the recent Bourbaki Congress that was held in Paris from the 6th to the 8th of April 1944 nevertheless realized important progress, long wished for by the editors: the beginning of algebraic topology.”

This issue also contained some of the poetry of Bourbaki, which can be found in other issues of *La Tribu* as well. This poem praises the much maligned idea of filters.

The 1947 organization of the general plan changed again—the basics broke up into blocs:

General Plan

I. Sets, II. Algebra, III. General Topology

Linear bloc: IV. Functions of a real variable,

V. Topological vector spaces, VI. Integration, VII. Local differentials

Topologico-differential bloc: VIII. Algebraic topology, IX. Manifolds,

X. Lie groups

In 1946, with the end of World War II, and travel easier, SAMUEL EILENBERG (1913–1998) was drafted as a member, explicitly to prepare a report on algebraic topology. By 1949 there was an 82-page document, *Rapport SEAW sur la topologie préhomologique*, by Eilenberg and Weil, treating the important aspects of the topology of fibre spaces. This densely written report developed point-set properties of fibre spaces, including some new ideas. For example, they defined the *épiderme* of a space (with the parenthetical remark, *pourquoi pas*); this “skin” is a covering a the space with good properties of extension.

It is the 1950 Grand Plan that gives the familiar list of topics to be treated:

Part I.

1. Sets
2. Algebra
3. General topology
- 3^{bis}. Geometric topology
4. Functions of a real variable
5. Topological vector spaces
6. Integration
7. Manifolds
8. Analytic functions
9. Lie groups

Part II treated Commutative Algebra, Part III Algebraic Topology and its applications, and Part IV Functional Analysis.

The new topic, Geometric topology, was named by Serre to treat topics like coverings, fibre spaces, homotopy, polyhedra, retracts, and the fundamental group. This term went on in the literature, but it did not sit well with the Bourbaki who coined other terms to mock it.

So What Happened?

Another French enterprise was born about this time that affected the efforts to bring a text on algebraic topology together. In 1948/49, the *Séminaire Henri Cartan* began in Paris. Cartan had just come from Harvard in 1948, having spoken on topological notions, especially what later became sheaves. From its inception the seminar treated

topological themes, beginning with basic notions in 48/49 and going on to treat fibre spaces, spectral sequences, sheaves, homology of groups and Eilenberg-Mac Lane spaces, in later years. The level of exposition of these lectures was consistent with the expectations of the Bourbaki, and many of the lectures were given by then current members of Bourbaki.

The discussions of algebraic topology in the earliest plans for *Éléments de mathématique* and its appearance among the basic tools for the intended audience of Bourbaki make it clear what status the topic had for the group. However, the development of the subject was so rapid in the post-war years that it could not be understood in the manner that the Bourbaki set as a standard for their published work—that the essential concepts be identified, and the axiomatic basis presented in such a way that the main theorems would be smoothly proven from first principles. The collateral development of homological algebra, which would provide a tool for algebraic topology was finally taken up by Bourbaki, but only in recent times (1980). It is significant that some of this development was carried out by members of Bourbaki itself—Cartan, Eilenberg, Serre, Borel, and others—and so the subject was too new to be handled in the same manner as other contributions of Bourbaki.

The published work of Bourbaki does not make for easy reading. The austere style is associated with a monolithic view of the unity of mathematics that is precisely and properly presented in their work. The philosophical cadre of “structure” as guidepost and goal makes for a good explanation of the finished product. However, the record of the archives tells a different story. The austerity is a result of group editing. The course of a document was almost chaotic from first presentation to final publication, spiced by the lively interchanges of mathematicians of the first order, committed to an extraordinary standard.

From the point of view of an enterprise, Bourbaki’s *Éléments* stands out as an effort to rebuild a mathematical culture, based on a method (the axiomatic method) that was seen to be fruitful, by a collective of gifted mathematicians whose anonymity in their work was offset by the *joie de vivre* the process involved. We should all be so moved to do the same. (And I wonder what kind of report on algebraic topology we would produce today.)

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