Classification of Rational Homotopy Type for 8-Cohomological Dimension Elliptic Spaces -Case $\chi_\pi = 0$

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Abstract :

The classification of the rational homotopy type of elliptic spaces with homotopy Euler characteristic zero for $\dim H^*(X;\mathbb{Q}) = 8$, *this seems similar to cases well known in the Hilali thesis, but in this case, we have three generators and then we have more results about it.*

We try to give the different cases of this classification in the case $\chi_\pi = 0$, *for this we adapt the M.Mimura and H.Shiga method to our situation and by almost the same technics we get the theorem of classification of cohomology dimension 8 in the case* $\chi_\pi = 0$.

Part 1 : Recall the Hilali classification in the cases $\dim H^*(X;\mathbb{Q}) \leq 7$

Since 1988, the classification of rational homotopy type elliptic paces was given by Hilali in his preprint [Hi] and later in his thesis, he gave definitively the classification well known in the litterature, recently, in their paper,[MS], M.Mimura and H.Shiga give by another technics the same results modulo a few modifications, and more recently M.R.Hilali, H.Lamane, and M.I.Mamouni give in their paper the classification for 8-Cohomological Dimension Elliptic Spaces in the case $\chi_c(X) = 0$, we also try to complete this classification in the same manner by combining all of this methods and essentially in having grounds of the Hilali work.

Remind some definitions and results about rational elliptic spaces, we say then that a space X is calles elliptic if $\dim \pi_*(X)\otimes\mathbb{Q} < \infty$. we define also :
$\chi_\pi(X) = \sum_k (-1)^k \dim \pi_*(X)\otimes\mathbb{Q}$ the homotopy Euler characteristic
$\chi_c(X) = \sum_k (-1)^k \dim H^*(X;\mathbb{Q})$ the cohomology Euler characteristic.
We have $\chi_\pi(X) \leq 0$ and $\chi_c(X) \geq 0$
Moreover, in [Ha], Halperin showed that the following conditions are equivalent :
(1) $\chi_\pi(X) = 0$, (2) $\chi_c(X) > 0$, (3) $H^*(X;\mathbb{Q})$ is evenly graded, and that $H^*(X;\mathbb{Q})$ is a polynomial algebra truncated by a Borel ideal in this case.

Recalling the dimension formula proved in [FHT ;(32.14),p.446]
Let $\{f_1, ..., f_n\}$ be a regular sequence of graded elements in a polynomial ring $\mathbb{Q}[x_1, ..., x_n]$. We can assume that each f_i $(i = 1, ..., n)$ has no constant or linear terms and that

$$|x_1| \leq ... \leq |x_n|, |f_1| \leq ... \leq |f_n|$$

Put $A = \mathbb{Q}[x_1, ..., x_n]/(f_1, ..., f_n)$. Then we have the following dimension formula

$$\dim_{\mathbb{Q}} A = |f_1|...|f_n|/|x_1|...|x_n|.$$

By this formula we show that the cohomology algebra is generated by three generators.
On the other hand, we have the following proposition due to Halperin :

Proposition[Halperin]
If $(\wedge V, d)$ is elliptic, simply connected then :
(1) $H^*(\wedge V, d)$ satisfy the Poincaré duality
(2) $\mathrm{fd}(\wedge V, d) = \sum_{|a_j| odd} |a_j| - \sum_{|a_j| even}(|a_j| - 1)$
And by this proposition we use the Poincaré duality to construct a basis in different cases of the classification.

Part 2 : Conjecture H and Classification oh rational homotopy type of elliptic spaces

The Conjecture H gives a new technic to classify this spaces, in fact it is due to M.R.Hilali and it gives a basis in terms of the generators of $(\wedge V, d)$ which are also generators of $(H^*(X;\mathbb{Q}), 0)$

Hilali method :
The Hilali conjecture says that for an elliptic space simply connected we have $\dim V \leq \dim H^*(\wedge V, d)$ hence we can conclude that in this case that $2 \leq \dim V \leq 8$, If $(\wedge V, d)$ is generated by two generators a and b, it consists to eliminate the product ab, we give here just an example to illustrate this method, if we consider then the basis $\{1, a, b, b^2, b^3, ab, ab^2, ab^3\}$ and we assume that

$$\begin{cases} dx = a^2 + \lambda b^2 \\ dy = P(a, b) \end{cases}$$

with P is is a polynomial of coefficients in $\mathbb{Q}$ in two variables a and b, and $a^n = 0, a^{n-1} \neq 0$; $b^m = 0, b^{m-1} \neq 0$
we get then $a^n = d(P_1x + Q_1y)$ and $b^m = d(P_2x + Q_2y)$ with P_1, P_2, Q_1 and Q_2 are a polynomials in variables a and b. The idea of this methos is to eliminate the product ab.

Mimura-Shiga method :
The Mimura-Shiga method of classification is based on the dimension formula in the case of two generatorsand they showed in their paper [MS] the following inequality (with the same notations of part1)
Lemma : $2|x_i| \leq |f_i|$ for $i = 1, ..., n$.
During their argument, they use this inequality to classify, and the fact that the number of generators of the cohomology is the same that of ideal, and we have $H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1, ..., x_n]/(f_1, ..., f_n)$ where $n = 1, 2$ or 3.

Thus, in combining this two methods, we prove the theorem giving the classification classification of cohomology dimension 8 in the case $\chi_\pi = 0$.

In his thesis, and having for goal the toric rank conjecture, M.R.Hilali started the classification of elliptic spaces, he gave thus this classification when the Betti numbers sum is less than 7. In fact, by this classification one can check the validity of the conjecture for the low dimensional elliptic spaces.

Part 3 : Theorem in the case $\chi_\pi = 0, \dim H^*(X;\mathbb{Q}) = 8$

In following the same arguments and considering the fact that the cohomology algebra can be generated by three generators, we give the theorem summarizing the differents cases in the following theorem :

Theorem
Let X be an elliptic space and simply connected, if
$\dim H^*(X;\mathbb{Q}) = 8$ and $\chi_\pi = 0$, the X is formal and it is one of the following
(1) $X \sim_{\mathbb{Q}} \mathbb{S}^n_{(7)}$ with $7n = \mathrm{fd}(X)$.
(2) $X \sim_{\mathbb{Q}} \mathbb{S}^n_{(4)} \times \mathbb{S}^m_{(2)}$, with $4n + 2m = \mathrm{fd}(X)$.
(3) $X \sim_{\mathbb{Q}} \mathbb{S}^n_{(4)} \# \mathbb{S}^n_{(4)}$.
(4) $X \sim_{\mathbb{Q}} \mathbb{S}^n_{(5)} \# \mathbb{S}^m_{(3)}$.
(5) $X \sim_{\mathbb{Q}} \mathbb{S}^n_{(2)} \times \mathbb{S}^m_{(2)} \times \mathbb{S}^k_{(2)}$, with $2(n + m + k) = \mathrm{fd}(X)$
(6) $X \sim_{\mathbb{Q}} P_{\alpha,\beta,\gamma}$, where $H(P_{\alpha,\beta,\gamma};\mathbb{Q}) = \mathbb{Q}[a, b, c]/(a^2 - \alpha bc, b^2 - \beta ac, c^2 - \gamma ab)$

Références

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