Dressing-Modular vector field of Poisson-Lie group

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Abstract

We show that on any Poisson-Lie group, the effect of the dressing vector fields on volume forms generates a class on Poisson cohomology. The comparison with the modular class give rise to a complet Poisson vector field related to the modular group.

Keywords : Poisson-Lie groups ; dressing action ; modular vector field.
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I Introduction

Recall that [1] a Poisson-Lie group (G, P) is a Lie group endowed with a Poisson structure such that the multiplication $G \times G \longrightarrow G$ is a Poisson map, where $G \times G$ carries the Poisson product structure. The Poisson bivector field is said to be multiplicative and satisfies, for any $x, y \in G$

$$P_{xy} = l_{x_*} P_y + r_{y_*} P_x,$$

where l_{x_*} (resp.r_{y_*}) denotes the tangent map of the left translation by x in G (resp. the right translation by y in G). Let g be a lie algebra of G and g^* its dual. Denote by $\lambda : g^* \longrightarrow X(G)$ the infinitesimal dressing action, i.e for any $\xi \in g^*$, $\lambda(\xi) = i_{\widetilde{\xi}} P$ (see [3] and [4]), where $\widetilde{\xi}$ is the left-invariant 1-form on G with value ξ at e and $i_{\widetilde{\xi}} P$ denotes the interior product of P by $\widetilde{\xi}$. Thus $\mathcal{C}^\infty(G)$ has the structure of g^*-module, defined by $\xi \cdot f = \lambda(\xi)(f)$, $f \in \mathcal{C}^\infty(G)$, $\xi \in g^*$.
In this work we study the effect of the dressing vector fields on the volume form, by analogy with the construction of the moduler vector field. Precisely if we assume that G is oriented, one choose any volume form Ω and computes its Lie derivative along dressing vector fields. This leads to a unique 1-cocycle μ_Ω of g^* with values in g^*-module $C^\infty(G)$, such that :

$$\mathcal{L}_{\lambda(\alpha)}\Omega = \mu_\Omega(\alpha)\Omega.$$

One calls $\mathrm{S}_\Omega(x) = l_{x_*}\mu^*_\Omega(x)$ for $x \in G$ the dressing modular vector field of the Poisson Lie group (G, P) relative to Ω. The field S_Ω is Poisson and defines a class $[S_P] \in H^1_P(G)$ independent of Ω. We remark that the difference between the modular field and the dressing modular field is independent of the chosen volume form and defines a Poisson vector field X_P. We show that this field is multiplicative and then generates a one parameter subgroup of $\mathrm{Aut}(G, P)$. We ending with an example where the class $[X_P]$ is not trivial and then the modular class and the dressing modular class are different.

II The dressing modular field

Let Ω be a volume form on G. For any $\alpha \in g^*$, $\mathcal{L}_{\lambda(\alpha)}\Omega$ is a maximal form, then there exists a map μ_Ω such that :

$$\begin{aligned} \mu_\Omega &: g^* \longrightarrow C^\infty(G) \\ &\alpha \longmapsto div_\Omega(\lambda(\alpha)) = \frac{\mathcal{L}_{\lambda(\alpha)}\Omega}{\Omega}. \end{aligned}$$

Let $\mu^*_\Omega : G \longrightarrow g$ the dual map of μ_Ω defined by

$$\forall \xi \in g^*, < \mu^*_\Omega(x), \xi >= \mu_\Omega(\xi)(x).$$

definition

We call ***dressing modular field***, the vector field S_Ω associated to Ω, defined by

$$\mathrm{S}_\Omega(x) = l_{x_*}\mu^*_\Omega(x), \quad x \in G.$$

This vector field has the following propriety :

Proposition I

1. The map μ_Ω is a 1-cocycle of g^* with values in $C^\infty(G)$.i.e.

$$\mu_\Omega([\xi, \eta]_*) = \xi.\mu_\Omega(\eta) - \eta.\mu_\Omega(\xi), \quad \xi, \eta \in g^*.$$

   In particular the field S_Ω is Poisson. i.e. $\mathcal{L}_{\mathrm{S}_\Omega} P = 0$

2. The field S_Ω with respect to different volume forms differ for a hamiltonian vector field. i.e.

$$\mathrm{S}_{f\Omega} = \mathrm{S}_\Omega - \mathcal{X}_{\ln(|f|)} \quad \forall f \in C^\infty(G). \qquad (1)$$

Remark and Fact

– The vector field S_Ω defines a class $[\mathrm{S}_P] \in H^1_P(G)$, This class is an obstruction to the existence of an invariant volume form by dressing field. Indeed it is clear that if $\mathcal{L}_{\lambda(\alpha_i)}\Omega_0 = 0$ of $i = 1, \cdots, n$, the class $[\mathrm{S}_P]$ is zero. Suppose that $[\mathrm{S}_P] = 0$ one chooses any volume form Ω. There is thus a function f such that $\mathrm{S}_\Omega = X_f$. From (1), we have

$$\mathrm{S}_{e^{-f}\Omega} = \mathrm{S}_\Omega + \mathcal{X}_{-f} = o,$$

the volume form $e^{-f}\Omega$ verifies the required property.

– As shown in [4], the operator $R_\Omega : f \longmapsto div_\Omega \mathcal{X}_f$ is a derivation and hence a vector field called the modular vector field of G with respect to the volume form, i.e $\mathrm{R}_{f\Omega} = \mathrm{R}_\Omega - \mathcal{X}_{\ln(|f|)}$. From (1) we have

$$\mathrm{R}_{f\Omega} - \mathrm{S}_{f\Omega} = \mathrm{R}_\Omega - \mathrm{S}_\Omega. \qquad (2)$$

III Relation between $[\mathrm{S}_P]$ and the modular class

Theorem I

Let (G, P) be a connected Poisson-Lie group. Let $(X_i)_{1 \le i \le n}$ be a basis of g and denote by $(\alpha_i)_{1 \le i \le n}$ its dual basis, the Poisson tensor P can be written $P = \sum_{i<j} P_{ij}\widetilde{X}_i \wedge \widetilde{X}_j$, here $\widetilde{X}$ is the left-invariant vector field associated in X. We have

$$\mathrm{R}_\omega - \mathrm{S}_\omega = \sum_{i<j} P_{ij}[\widetilde{X}_j, \widetilde{X}_i],$$

for any volume form ω.

Let $\nu : g^* \longrightarrow C^\infty(G)$ be the linear map given by the pairing between P and $\delta(\widetilde{\xi})$, i.e

$$\nu(\xi) =< P, \delta(\widetilde{\xi}) >,$$

where δ is the transpose map of the Lie bracket on g. Let $\nu^* : G \longrightarrow g$ be the dual map of ν. More precisely, ν^* is defined by

$$< \nu^*(g), \alpha >= \nu(\alpha)(g) =< l_{g^{-1}}P(g), \delta(\alpha) > \quad \alpha \in g^*.$$

Proposition II

1. Denote by X_P the vector fiel defined by $X_P(g) = l_g\nu^*(g)$. Then

$$X_P = \mathrm{R}_\omega - \mathrm{S}_\omega,$$

   for any volume form ω on G.

2. The vector field X_P is multiplicative. In particular, it is a complet vector field.

Remarks

1. Let (G, P) be an exact Poisson-Lie, i.e

$$P = \widetilde{r} - \widehat{r}$$

   for some $r \in \wedge^2 g$, solution of the Yang-Baxter equation. It follows

$$\mathrm{R}_\omega - \mathrm{S}_\omega = \widetilde{L(r)} - \widehat{L(r)},$$

   for any volume form ω on G, where $L : \wedge^2 g \longrightarrow g$ is the linear operator defined by $L(X \wedge Y) = [X, Y]$ and, as usual, $\widetilde{L(r)}$ (resp. $\widehat{L(r)}$) denotes the left-invariant (resp right-invariant) tensor field associated to $L(r)$.

2. If $[\mathrm{S}_P] = 0$. The modular class is represented by the complete vector filed X_P. Then (G, P) admits a modular automorphisme group (see [5]).

IV Sketch of proof of the Theorem I

The Hamiltonian vector field associated to $f \in \mathcal{C}^\infty(G)$ is given by

$$\mathcal{X}_f = \sum_{j=1}^n (\sum_{i=1}^n P_{ij}\widetilde{X}_i(f))\widetilde{X}_j,$$

then

$$\begin{aligned} \mathrm{R}_\omega(f) &= div_\omega \sum_{j=1}^n (\sum_{i=1}^n P_{ij}\widetilde{X}_i(f))\widetilde{X}_j \\ &= \sum_{j=1}^n (\sum_{i=1}^n \widetilde{X}_j(P_{ij}\widetilde{X}_i(f)) + \sum_{j=1}^n (\sum_{i=1}^n P_{ij}\widetilde{X}_i(f))div_\omega \widetilde{X}_j. \end{aligned}$$

The dressing vector fields corresponding to α_i is given by $\lambda(\alpha_i) = \sum_{j=1}^n P_{ij}\widetilde{X}_j$.
We have

$$\begin{aligned} \mathrm{S}_\omega(f) &= \sum_{i=1}^n div_\omega(\lambda(\alpha_i))\widetilde{X}_i(f) \\ &= \sum_{i=1}^n (\sum_{j=1}^n div_\omega P_{ij}\widetilde{X}_j)\widetilde{X}_i(f) \\ &= \sum_{i=1}^n (\sum_{j=1}^n \widetilde{X}_j(P_{ij}))\widetilde{X}_i(f) + \sum_{j=1}^n (\sum_{i=1}^n P_{ij}\widetilde{X}_i(f))div_\omega \widetilde{X}_j, \end{aligned}$$

and hence

$$\begin{aligned} \mathrm{R}_\omega(f) - \mathrm{S}_\omega(f) &= \sum_{j=1}^n (\sum_{i=1}^n \widetilde{X}_j(P_{ij}\widetilde{X}_i(f)) - \sum_{i=1}^n (\sum_{j=1}^n \widetilde{X}_j(P_{ij}))\widetilde{X}_i(f) \\ &= \sum_{j=1}^n \sum_{i=1}^n P_{ij}\widetilde{X}_j\widetilde{X}_i(f) \\ &= \sum_{i<j} P_{ij}(\widetilde{X}_j\widetilde{X}_i - \widetilde{X}_i\widetilde{X}_j)(f), \end{aligned}$$

which completes the proof.

V Example

In the following example, the class $[X_P]$ is not trivial and then the modular class is different from the dressing modular class in general.
Denote by $SU(2)$, the special unitary group defined by :

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \ / \ \alpha, \beta \in C, \ \alpha\overline{\alpha} + \beta\overline{\beta} = 1 \right\}.$$

Let $\alpha = x + iy$ and $\beta = z + it$, $SU(2)$ can be identified with the unit sphere S^3 in R^4.
The Lie bracket on the Lie algebra $su(2)$ is defined by :

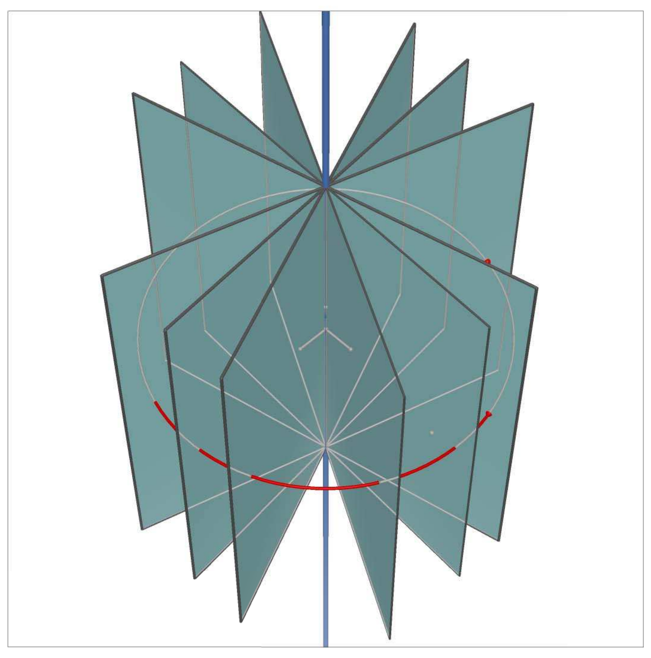
$$[Z, X] = 2Y \ ; [Z, Y] = -2X \, ; [X, Y] = 2Z.$$

The left-invariant vector fields associated with this basis are :

$$\begin{aligned} \widetilde{X} &= -y\partial_x + x\partial_y + t\partial_z - z\partial_t \\ \widetilde{Y} &= -z\partial_x - t\partial_y + x\partial z + y\partial_t \\ \widetilde{Z} &= -t\partial_x + z\partial_y - y\partial_z + x\partial_t, \end{aligned}$$

and the Poisson Lie structures on $SU(2)$ are given by

$$P(x, y, z, t) = 2k(xz - yt)\widetilde{Y} \wedge \widetilde{Z} - 2k(xy + zt)\widetilde{Z} \wedge \widetilde{X} + 2k(y^2 + z^2)\widetilde{X} \wedge \widetilde{Y}, \quad k \in R^*_+.$$



The symplectic leaves on $SU(2)$ are spheres given by

$$\lambda y + \mu z = 0 \ (\lambda, \mu \in R \, ; \, (\lambda, \mu) \neq (0, 0)).$$

We obtain

$$X_p = 4ky\partial_z - 4kz\partial_y.$$

The orbits of X_P are circles with center located on the singular locus : $x^2 + t^2 = 1$ of P and transverse to symplectic leaves (see picture), so X_P defines a nontrivial class in Poisson cohomology (see [2]).

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