On Pre-Hilbert Noncommutative Jordan Algebras Satisfying $\|x^2\| = \|x\|^2$

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Abstract

Let A be a real or complex algebra. Assuming that a vector space A is endowed with a pre-Hilbert norm $\|.\|$ satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. We prove that A is finite dimensional in the following cases :

1. A is a real weakly alternative algebra without divisors of zero.
2. A is a complex powers associative algebra.
3. A is a complex flexible algebraic algebra.
4. A is a complex Jordan algebra.

In the first case A is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ and A is isomorphic to $\mathbb{C}$ in the last three cases. These last permits us to show that if A is a complex pre-Hilbert noncommutative Jordan algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$. Moreover we give an example of an infinite dimensional real pre-Hilbert Jordan algebra with divisors of zero and satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$.

Introduction

Let A be a real or complex algebra not necessarily associative or finite dimensional. Assuming that a vector space A is endowed with a pre-Hilbert norm $\|.\|$ satisfying $\|x^2\| \leq \|x\|^2$ for all $x \in A$. B. Zalar (1995, [6]) proved that :

1. If A is a real alternative algebra containing a unit element e such that $\|e\| = 1$, then A is finite dimensional and is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.
2. If A is a real associative algebra satisfying $\|x^2\| = \|x\|^2$, then A is finite dimensional and is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.
3. If A is a complex normed algebra containing a unit element e such that $\|e\| = 1$, then A is finite dimensional and is isomorphic to $\mathbb{C}$.

These results were extended respectively to the following cases :

1. If A is a real alternative algebra containing a nonzero central element a such that $\|ax\| = \|a\|\|x\|$, then A is finite dimensional and is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. (2008, [4])
2. If A is a real alternative algebra satisfying $\|x^2\| = \|x\|^2$, then A is finite dimensional and is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. (2008, [4])
3. If A is a complex normed algebra without divisors of zero and containing an invertible element v such that $\|vx\| = \|xv\| = \|v\|\|x\|$, then A is finite dimensional and is isomorphic to $\mathbb{C}$. (2010, [3])

In the present paper we extend the above results to more general situation. Indeed, we prove that, if A is a real or complex pre-Hilbert algebra satisfying $\|x^2\| \leq \|x\|^2$ for all $x \in A$. Then A is finite dimensional in the following cases :

1. A is a real weakly alternative algebra without divisors of zero and satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$.
2. A is a real weakly alternative algebra without divisors of zero and containing a nonzero central element a such that $\|ax\| = \|a\|\|x\|$ for all $x \in A$.
3. A is a complex powers associative algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$.

In the first two cases A is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ and A is isomorphic to $\mathbb{C}$ in the last two cases. This last allows us to show that if A is a complex pre-Hilbert noncommutative Jordan algebra (resp, flexible algebraic algebra or Jordan algebra) satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$. Moreover we give an example of an infinite dimensional real pre-Hilbert Jordan algebra (weakly alternative algebra) with divisors of zero and satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$.

Part 1 : Notation and preliminary results

Definitions 0.1 *Let B be an arbitrary algebra and K is a field of characteristic not 2.*

i) B is called alternative if it is satisfied the identities $(y, x, x) = 0$ and $(x, x, y) = 0$ (where $(.,.,.)$ means associator), for all $x, y \in B$.

ii) B is called a powers associative if, for every x in B, the subalgebra $B(x)$ generated by x is associative.

iii) B is called flexible if $(x, y, x) = 0$ for all $x, y \in B$.

iv) B is called a Jordan algebra if it is commutative and satisfied the Jordan identity : (J) $(x^2, y, x) = 0$ for all $x, y \in B$.

v) B is called a noncommutative Jordan algebra if it is flexible and satisfied the Jordan identity (J).

vi) B is called weakly alternative if it is a noncommutative Jordan algebra and satisfied the identity $(x, x, [x, y]) = 0$ (where $[.,.]$ means commutator). An alternative algebra or Jordan algebra is evidently weakly alternative.

vii) We say that B is algebraic if, for every x in B, the subalgebra $B(x)$ of B generated by x is finite dimensional.

viii) B is called a pre-Hilbert algebra if it is endowed with a space norm comes from an inner product $(.|.)$.

ix) B is termed normed (resp, absolute valued) if it is endowed with a space norm : $\|.\|$ such that $\|xy\| \leq \|x\|\|y\|$ (resp, $\|xy\| = \|x\|\|y\|$), for all $x, y \in B$.

The most natural examples of absolute valued algebras are $\mathbb{R}, \mathbb{C}, \overset{*}{\mathbb{C}}, \mathbb{H}$ (the algebra of Hamilton quaternion) and $\mathbb{O}$ (the algebra of Cayley numbers), with norms equal to their usual absolute values. The algebra $\overset{*}{\mathbb{C}}$, obtained by replacing the product of $\mathbb{C}$ with the one defined by $x \circ y = x^* y^*$, where $*$ means the standard involution of $\mathbb{C}$.

We have the following very known results :

Lemma 0.2 [5] *Let A be a powers associative algebra over K and without divisors of zero. If e is a nonzero idempotent in A, then A has an identity element e.*

Proposition 0.3 [1] *If $\{x_i\}$ is a set of commuting elements in a flexible algebra A over K, then the subalgebra generated by the $\{x_i\}$ is commutative.*

Proposition 0.4 [2] *Let A be a noncommutative Jordan algebra over K, then A is a powers associative algebra.*

Theorem 0.5 [5] *The subalgebra generated by any two elements of an alternative algebra A is associative.*

Theorem 0.6 [6] *Let A a real pre-Hilbert associative algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A is finite dimensional and is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.*

Theorem 0.7 [4] *Let A be a real pre-Hilbert commutative algebra without divisors of zero and satisfying $\|x^2\| \leq \|x\|^2$ for all $x \in A$. Suppose that A containing a nonzero central element a such that $\|ax\| = \|a\|\|x\|$ for all $x \in A$. Then A is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\overset{*}{\mathbb{C}}$.*

Theorem 0.8 [6] *Let A be a real pre-Hilbert alternative algebra with identity e. Suppose that $\|x^2\| \leq \|x\|^2$ for all $x \in A$ and $\|e\| = 1$. Then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.*

Part 2 : Real pre-Hilbert weakly alternative algebras

We have the following importing results :

Theorem 0.9 *Let A be a real pre-Hilbert weakly alternative algebra with identity e and without divisors of zero. Suppose that $\|x^2\| \leq \|x\|^2$ for all $x \in A$ and $\|e\| = 1$. Then A is finite dimensional and is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.*

Proof. It is sufficient to prove that A is an alternative algebra, the result ensues then of the Theorem (0.8).

Theorem 0.10 *Let A be a real pre-Hilbert weakly alternative algebra without divisors of zero. Suppose that $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.*

Proof. A is a powers associative algebra (Proposition (0.4)) then the subalgebra $A(x)$ of A, generated by $x \in A$, is associative and verifying the conditions of the Theorem (0.6). Therefore $A(x)$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, thus there is a nonzero idempotent $e \in A$ such that $xe = ex = x$, that is, A is a unital algebra of unit e (Lemma (0.2)). So the result is a consequence of the Theorem (0.9).

Corollary 0.11 *Let A be a real pre-Hilbert Jordan algebra without divisors of zero. Suppose that $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

We give an extension of the Theorem (0.9)

Theorem 0.12 *Let A be a real pre-Hilbert weakly alternative algebra without divisors of zero and satisfying $\|x^2\| \leq \|x\|^2$ for all $x \in A$. Suppose that A containing a nonzero central element a such that $\|ax\| = \|a\|\|x\|$ for all $x \in A$. Then A is finite dimensional and is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.*

Proof. Let $x \in A$, the subalgebra $A(a, x)$ of A generated by $\{x, a\}$ is commutative. The Theorem (0.7) implies that $\|x^2\| = \|x\|^2$, thus the result is a consequence of the Theorem (0.10).

Remark 0.13 *In the previous results the hypothesis without divisors of zero is necessary. The following example proves it :*

Let H be an infinite dimensional real Hilbert space, we define the multiplication on the vector space $A = \mathbb{R} \oplus H$ by : $(\alpha + x)(\beta + y) = (\alpha\beta - (x|y)) + (\alpha y + \beta x)$. And the scalar product by : $((\alpha + x)|(\beta + y)) = \alpha\beta + (x|y)$. Then A is an infinite dimensional real pre-Hilbert Jordan (weakly alternative) algebra with identity satisfying $\|a^2\| = \|a\|^2$ and has a zero divisors.

Part 3 : Complex pre-Hilbert noncommutative Jordan algebras satisfying $\|x^2\| = \|x\|^2$

We need the following results :

Proposition 0.14 [3] *Let A be a complex pre-Hilbert commutative associative (resp, commutative algebraic) algebra and satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

Theorem 0.15 [3] *Let A be a complex pre-Hilbert algebra with identity e. Suppose that $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

We have the following importing results :

Lemma 0.16 *Let A be a complex pre-Hilbert alternative (resp, commutative) algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A has nonzero divisors.*

Theorem 0.17 *Let A be a complex pre-Hilbert alternative algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

Proof. Let $a \in A$, the subalgebra $A(a)$ of A generated by a is commutative and associative (Theorem (0.5)). The Proposition (0.14) prove that $A(a)$ is isomorphic to $\mathbb{C}$, then there exist a nonzero idempotent $f \in A$. According to the Theorem (0.15) it is sufficient to prove that f is a unit element of A. Let $b \in A$, we have $f(b - fb) = 0$ and $(b - bf)f = 0$. As A is without divisors of zero (Lemma (0.16)), then $fb = bf = b$. Thus A is finite dimensional and is isomorphic to $\mathbb{C}$.

Theorem 0.18 *Let A be a complex pre-Hilbert powers associative algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

Theorem 0.19 *Let A be a complex pre-Hilbert flexible algebraic algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

Proof. Let $a \in A$ be a nonzero element, according to Proposition (0.3) and Lemma (0.16), the subalgebra $A(a)$ of A is commutative, algebraic and without divisors of zero. Thus $A(a)$ is isomorphic to $\mathbb{C}$ (Proposition (0.14)). This implies that A is a powers associative algebra, then the result is a consequence of the Theorem (0.18).

We state now the main Theorem :

Theorem 0.20 *Let A be a complex pre-Hilbert non commutative Jordan algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

Proof. The Proposition (0.4) implies that A is a powers associative algebra, hence A is isomorphic to $\mathbb{C}$ (Theorem (0.18)).

Corollary 0.21 *Let A be a complex pre-Hilbert weakly alternative (resp, Jordan) algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

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