# New Approch to compute Solution of singular continuous-time fractional linear systems

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## Abstract

In this work, we give new method to compute solution of singular fractional continuous-time linear systems. Based on the Caputo fractional derivative, the solution is derived using decomposition method.
**Keywords :** singular, fractional, linear systems, regular pencil, decomposition.

## Introduction

Dynamic systems described using fractional derivatives have been studied by many in the engineering area [5] [6] [7] [8] . These last years, many efforts have been done to devellops fractional systems in the field of reaserch. Note that fractional calculus is a generalization of differentiation and integration to non integer order. Many phenomena can be modeled by fractional transfer functions.

## Partie 3 : Singular Fractional Linear Continuous Systems

we will consider singular continuous-time systems described by the following representaion

$$Ed^{\alpha}x(t) = Ax(t) + Bu(t), 0 < \alpha \leq 1 \tag{4a}$$

$$y(t) = Cx(t) + Du(t) \tag{4b}$$

where $d^{\alpha}$ is the Caputo derivative, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the pseudo-state state, the input and the output vector respectively of the model.
$E \in \mathbb{R}^{n\times n}, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$.
x(t) and u(t) are supposed continuously derivable The boundary conditions are given by $x(0) = x_0$ If the pencil $E\lambda - A$ is regular , then there exists non-isngular matrices $P$, $Q \in \mathbb{R}^{n\times n}$ such that

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & M \end{bmatrix}, \ PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \tag{4}$$

where $n_1$ is equal to the degree of the polynomial $\det(E\lambda - A)$.
M$\in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with the index $\mu$

## Partie 1 : Preliminaries

We denote by $\mathbb{R}^{m\times n}$ the set of real matrices with $m$ rows and $n$ columns, by $\mathbb{R}^m$ the set of real vectors and by $I_n$ the $n\times n$ identity matrix.
The following Caputo definition of the fractional derivative of real order $\alpha > 0$ of a function N time continûement differentiable given by $f : (0,\infty) \to \mathbb{R}$ will be used [9] :

$$d^{\alpha}f(t) = \frac{1}{\Gamma(N-\alpha)}\int_0^t \frac{f^{(N)}(\tau)}{(t-\tau)^{\alpha-N+1}}d\tau, N-1 < \alpha < N \ (N \in \mathbb{N}) \tag{1}$$

$$f^{(N)}(\tau) = \frac{d^N f(\tau)}{d\tau^N}$$

and

$$\Gamma(x) = \int_0^t e^{-t}t^{x-t}dt$$

is the gamma function.
The Laplace transform of the Caputo fractional derivative is[9]

$$\mathcal{L}[d^{\alpha}f(t)] = \lambda^{\alpha}F(\lambda) - \sum_{k=0}^{N-1}\lambda^{\alpha-k+1}f^{(k)}(0) \tag{2}$$

where

$$F(\lambda) = \mathcal{L}[f(t)] = \int_0^t f(t)e^{-\lambda t}dt$$

## Partie 4 :Maints Results

**Theorem 3** *Consider a system described by the equations* $(4a)-(4b)$ *with the initial condition for* $(4a)$ *is given by* $x(0) = x_0$. *If the system* $(4a)$ *is regular its solution can be described by*

$$x(t) = Q\begin{bmatrix} I_{n_1} \\ 0_{n_2\times n_1}\end{bmatrix}\left(\Phi_{10}(t)x_{10} + \int_0^t \Phi_{11}(t-\tau)B_1u(\tau)d\tau\right) + Q\begin{bmatrix} 0_{n_1\times n_2} \\ I_{n_2}\end{bmatrix}\left(-B_2u(t) - \sum_{i=1}^{\mu-1}M^i\ B_2\ d^{i\alpha}u(t)\right)$$

*and the output is given by the formula*

$$y(t) = CQ\begin{bmatrix} I_{n_1} \\ 0_{n_2\times n_1}\end{bmatrix}\left(\Phi_{10}(t)x_{10} + \int_0^t \Phi_{11}(t-\tau)B_1u(\tau)d\tau\right) + CQ\begin{bmatrix} 0_{n_1\times n_2} \\ I_{n_2}\end{bmatrix}\left(-B_2u(t) - \sum_{i=1}^{\mu-1}M^i\ B_2\ d^{i\alpha}u(t)\right)$$

We coclude the section with an example.

**Example 4** *Consider the system* $(4a-4b)$ *with*

$$E = \begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, A = \begin{bmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1&1&0&1 \end{bmatrix}, D = 0$$

*and the initial condition*

$$x_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

*It is easy to make sure that*

$$P = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}, Q = \begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{bmatrix}$$

$$x(t) = Q\begin{bmatrix} 1 \\ \frac{t}{\Gamma(\alpha+1)} \\ 0 \\ -u(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \frac{t}{\Gamma(\alpha+1)} \\ -u(t) \end{bmatrix}, y(t) = Cx(t) = 1 - u(t)$$

## Partie 2 : Linear Fractional Continuous Systems

Consider the continuous time fractional linear system defined by the representation

$$d^{\alpha}x(t) = Ax(t) + Bu(t), 0 < \alpha \leq 1 \tag{3}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$ are the pseudo-state and input vectors. $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$.

**Theorem 1** [*Kaczorek*(2008, 2009)] *the solution of equation* (3) *is given by*

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

*with*

$$\Phi_0(t) = \sum_{k=0}^{\infty}\frac{t^{k\alpha}A^k}{\Gamma(k\alpha+1)}$$

$$\Phi(t) = \sum_{k=0}^{\infty}\frac{t^{(k+1)\alpha-1}A^k}{\Gamma[(k+1)\alpha]}$$

**Remark 2** *for* $\alpha = 1$, *we obtain*

$$\Phi_0(t) = \Phi(t) = e^{tA}$$

## Références

[1] Bouagada, D and Van Dooren, P. On the Solution of Implicit Fractional Continuous Time State Space Systems. Proceedings of the MAMERN'11 conférence Saidia, Morocco, May 23-26 (2011).

[2] Dai, L. (1989). Singular Control Systems. Lecture Notres in Control and Information Sciences. Springer-Verlag. Berlin, New York.

[3] Gantmacher, F.R. (1960). The Theory of Matrices. Vol.2. Chelsea Publishing Company. New York.

[4] Kaczorek,T.(2008). Fractional Positive Continuous-Time Linear Systems and Their Reachability. International Journal of Applie Mathematics and Comuter Science 18(2) :223-228, DOI : 10.2478/v10006-OO8-0020-0.

[5] Kaczorek,T. Singular Fractional Linear Systems and Electrical Circuits. International Journal of Applie Mathematics and Comuter Science., 2011, Vol. 21, No. 2, 379-384, DOI ; 10.2478/v10006-O11-0028-8.

[6] Lin, J. Modélisation et Identification de Systèmes d'Ordre Non Entier. PHD Thesis, Université de Poitiers(2001).

[7] Miller, K.S, and Ross.B. An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1994).

[8] Oustalup, A.(1995). La Dérivation Non Entière, Théorie, Synthèse et Applications. Paris : Edition Hermès.

[9] Podlubny, I.(1999). Fractional Differential Equations. San Diego : Academic Press.