HARMONIC MAPS ON GENERALIZED WARPED PRODUCT MANIFOLDS

NOUR EL HOUDA DJAA (Doctorant dans l'équipe Géométrie et Analyse non linéaire)
Institut Sciences et Technologie, Centre Universitaire de Relizane.

Abstract

In this paper, we present some new properties for harmonic maps between generalized warped product manifolds.

Introduction

Consider a smooth map $\phi : (M^m, g) \longrightarrow (N^n, h)$ between two Riemannian manifolds. The energy functional of ϕ is defined by

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g . E(\phi) = \int_M e(\phi) dv_g. \quad (0.1)$$

(or over any compact subset $K \subset M$), where $e(\phi) = \frac{1}{2}|d\phi|^2$, denote the energy density of ϕ.
A map is called harmonic if it is a critical point of the energy functional $E(\phi)$ (or $E(K)$ for all compact subsets $K \subset M$), the Euler-Lagrange equation associated to (1.1) is

$$\tau(\phi) = Tr_g \nabla d\phi \quad (0.2)$$

$\tau(\phi)$ is called the tension field of ϕ. For any smooth variation $\phi_{tt \in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d\phi_t}{dt} \mid_{t=0}$,we have

$$\frac{d}{dt} E(\phi_t) \mid_{t=0} = -\int_M h(\tau(\phi), V) dv_g \quad (0.3)$$

Then,we have

$$\tau(\phi) = Tr_g \nabla d\phi = 0 \quad (0.4)$$

One can refer to [PB], [2], [4], [5] and [9] for background on harmonic maps.

Part 1 : Some results on generalized warped product manifolds

In this section, we give the definition and some geometric properties of generalized warped product manifolds.

Definition 1 *Let (M^m, g) and (N^n, h) be two riemannian manifolds, and $f : M \times N \to \mathbb{R}$ be a smooth positive function. The generalized warped metric on $M \times_f N$ is defined by*

$$G_f = \pi^* g + (f)^2 \eta^* h \quad (0.5)$$

where $\pi : (x, y) \in M \times N \to x \in M$ and $\eta : (x, y) \in M \times N \to y \in N$ are the canonical projections. For all $X, Y \in T(M \times N)$, we have

$$G_f(X, Y) = g(d\pi(X), d\pi(Y)) + (f)^2 h(d\eta(X), d\eta(Y))$$

and we denote by :

$$\Big(X \wedge_{G_{f^2}} Y\Big) Z = G_{f^2}\Big(Z, Y\Big) X - G_{f^2}\Big(Z, X\Big) Y \quad (0.6)$$

Proposition 2 *Let (M^m, g) and (N^n, h) be two Riemannian manifolds. If $\overline{\nabla}$ denote the Levi-Civita connection on $(M \times_f N, G_f)$, then for all $X_1, Y_1 \in \mathcal{H}(M)$ and $X_2, Y_2 \in \mathcal{H}(N)$ whe have :*

$$\begin{aligned} \overline{\nabla}_X Y = \; & \nabla_X Y + X(\ln f)(0, Y_2) + Y(\ln f)(0, X_2) \quad (0.7) \\ & -\frac{1}{2} h(X_2, Y_2)(grad_M f^2, \frac{1}{f^2} grad_N f^2) \end{aligned}$$

where $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ and $\nabla_X Y = (\nabla^M_{X_1} Y^1, \nabla^N_{X_2} Y^2)$

In the general case, the geometry of product manifolds is considered in [7].

Proposition 3 *Let (M^m, g) and (N^n, h) be two Riemannian manifolds and $f : M \times N \to \mathbb{R}$ be smooth positive function. The Ricci curvature from generalized warped product manifolds $(M \times_f N, G_f)$ is given by the following formulas :*

$$\begin{aligned} Ric((X_1, 0), (Y_1, 0)) &= Ric^M(X_1, Y_1) - ng(\nabla^M_{X_1} grad_M \ln f \\ &+ X_1(\ln f) grad_M \ln f, Y_1) \\ Ric((X_1, 0), (0, Y_2)) &= -nX_1(Y_2(\ln f)) \\ Ric((0, X_2), (Y_1, 0)) &= h(X_2, grad_N(Y_1(\ln f))) - nX_2(Y_1(\ln f)) \end{aligned}$$

Part 2 : Harmonic maps on generalized warped product manifolds

$$\begin{aligned} Ric((0, X_2), (0, Y_2)) &= Ric^N(X_2, Y_2) + (2 - n)h(\nabla^N_{X_2} grad_N \ln f, Y_2) \\ &+ (2 - n)\big[h(X_2, Y_2) \mid grad_N \ln f \mid^2 \\ &- X_2(\ln f) h(grad_N \ln f, Y_2)\big] \\ &+ h(X_2, Y_2)\big[n f^2 \mid grad_M \ln f \mid^2 \\ &- \Delta_N(\ln f) - f^2 \Delta_M(\ln f)\big] \end{aligned}$$

for all $X_1, Y_1 \in \mathcal{H}(M)$ and $X_2, Y_2 \in \mathcal{H}(N)$.
Let $(M^m, g), (N^n, h)$ and (P^p, ℓ) be Riemannian manifolds of dimensions m, n and p respectively, $f : M \times N \to \mathbb{R}$ be smooth positive function, and $(M \times_f N, G_f)$ be the generalized warped product manifold.

Proposition 4 *If $\varphi : P \to M$ and $\psi : P \to N$ are regular maps, then the tension field of*

$$\begin{aligned} \phi : (P^p, \ell) &\longrightarrow (M \times_f N, G_f) \\ x &\longmapsto (\varphi(x), \psi(x)) \end{aligned}$$

is given by the following relation :

$$\begin{aligned} \tau(\phi) = \; & \Big(\tau(\varphi), \tau(\psi)\Big) + 2\Big(0, d\psi(grad_P(\ln f \circ \phi))\Big) \quad (0.8) \\ & -e(\psi)\Big(grad_M f^2, \frac{1}{f^2} grad_N f^2\Big) \end{aligned}$$

Remarks :

– If f is a constant function, then the tension field of ϕ is given by

$$\tau(\phi) = \Big(\tau(\varphi), \tau(\psi)\Big)$$

and ϕ is harmonic map if and only if φ et ψ are harmonic maps.

– If $P = M$ and $\psi = y_0$ is constant, then the tension field of $\phi : x \in M \longmapsto (\varphi(x), y_0) \in M \times N$ is given by

$$\tau(\phi) = (\tau(\varphi), 0)$$

– If $P = N$ and $\varphi = x_0$ is constant then the tension field of $\phi : y \in N \longmapsto (x_0, \psi(y)) \in M \times N$ is given by

$$\begin{aligned} \tau(\phi) = \; & (0, \tau(\psi)) + 2\Big(0, d\psi(grad_M(\ln f \circ \phi))\Big) \\ & - e(\psi)\Big(grad_M f^2, \frac{1}{f^2} grad_N f^2\Big) \end{aligned}$$

– If $P = N$ and $\psi = Id_N$, then $e(\psi) = \frac{n}{2}$ and then the tension field of $\phi : y \in N \longmapsto (\varphi(y), y) \in M \times N$ is given by

$$\tau(\phi) = \Big(\tau(\varphi) - \frac{n}{2} grad_M f^2, (2 - n) grad_N f^2\Big)$$

From definition of conformal map and Proposion 4, we deduce

Proposition 5 *Let $\varphi : M \to M$ be conformal map with dilatation λ, then the tension field of*

$$\begin{aligned} \phi : (M, g) &\longrightarrow (M \times_f M, G_f) \\ x &\longmapsto (\varphi(x), \varphi(x)) \end{aligned}$$

is given by

$$\begin{aligned} \tau(\phi) = \; & (2 - m)\Big(d\varphi(grad \ln \lambda), d\varphi(grad \ln \lambda)\Big) + 2\Big(0, d\varphi(grad(\ln f \circ \varphi))\Big) \\ & -\frac{m}{2}\lambda^2\Big(grad f^2, \frac{1}{f^2} grad f^2\Big) \circ \varphi \end{aligned}$$

For more details on conformal maps, we can refer to [1], [8].
Harmonicity conditions Let $\phi : (x, y) \in (M \times_f N, G_f) \longrightarrow \phi(x, y) \in (P, k)$ be smooth map. If we denote by

$$\begin{aligned} \phi_N = \phi^x_N : (N, h) &\longrightarrow (P, k) \\ y &\longmapsto \phi^x_N(y) = \phi(x, y) \end{aligned}$$

and

$$\begin{aligned} \phi_M = \phi^y_M : (M, g) &\longrightarrow (P, k) \\ x &\longmapsto \phi^y_M(x) = \phi(x, y) \end{aligned}$$

then for all $X \in \mathcal{H}(M)$, $Y \in \mathcal{H}(N)$ and $(x, y) \in M \times N$, we have :

Proposition 6 *The tension field of $\phi : (M \times_f N, G_f) \longrightarrow (P, k)$ is given by :*

$$\begin{aligned} \tau(\phi) = \; & \tau(\phi_M) + nd\phi_M(grad_M \ln f) \\ & + \frac{1}{f^2}\Big\{\tau(\phi_N) + (n - 2) d\phi_N(grad_N \ln f)\Big\}. \quad (0.9) \end{aligned}$$

Particular cases :

– If $f \in C^\infty(N)$ (i.e : $f(x, y) = f(y)$), then

$$\tau(\phi) = \tau(\phi_M) + nd\phi_M(grad_M \ln f) + \frac{1}{f^2}\tau(\phi_N).$$

– If $f \in C^\infty(M)$ (i.e : $f(x, y) = f(x)$), then

$$\tau(\phi) = \tau(\phi_M) + \frac{1}{f^2}\Big(\tau(\phi_N) + (n - 2) d\phi_N(grad_N \ln f)\Big).$$

– Let $\phi = \pi : (x, y) \in M \times_f N \to x \in M$, then $\tau(\pi) = n.grad_M \ln f$ and π is harmonic map if and only if f is constant on M,(i.e : $f(x, y) = f(y)$) .

– Let $\phi = \eta : (x, y) \in M \times_f N \to y \in N$, then $\tau(\eta) = \frac{n-2}{f^2} grad_N \ln f$ and η is harmonic map if and only if f is constant on N, (i.e : $f(x, y) = f(x)$), or $dimN = 2$.

– Let $\varphi : (M, g) \longrightarrow (P, k)$ be a smooth map and $\phi(x, y) = \varphi(x)$, then

$$\tau(\phi) = \tau(\varphi) + n.d\varphi(grad_M \ln f)$$

therefore, if φ is a conformal map with dilatation λ, then

$$\tau(\phi) = (2 - m) d\varphi(grad_M \ln \lambda) + n.d\varphi(grad_M \ln f)$$

and ϕ is a harmonic map if and only if $f = C(y).\lambda^{\frac{m-2}{n}}$.

– Let $\psi : (N, h) \longrightarrow (P, k)$ be a smooth map and $\phi(x, y) = \psi(y)$, then

$$\tau(\phi) = \frac{1}{f^2}\Big(\tau(\psi) + (n - 2).d\psi(grad_N \ln f)\Big)$$

therefore, if ψ is a conformal map with dilatation λ, then

–
$$\tau(\phi) = \frac{(n - 2)}{f^2}\Big(d\psi(grad_N \ln f) - d\psi(grad_N \ln \lambda)\Big)$$

and ϕ is a harmonic map if and only if $f = C(x).\lambda$ or $dimN = 2$.

– Let $\varphi : (M, g) \longrightarrow \mathbb{R}$ and $\psi : (N, h) \longrightarrow \mathbb{R}$ are a smooth functions, if $\phi(x, y) = \varphi(x)\psi(y)$, then

$$\begin{aligned} \tau(\phi) = \; & \psi\Big\{\tau(\varphi) + nd\varphi(grad_M \ln f)\Big\} \\ & + \frac{\varphi}{f^2}\Big\{\tau(\psi) + (n - 2) d\psi(grad_N \ln f)\Big\} \end{aligned}$$

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