# SVEP and Generalized Weyl's theorem


**Babbah Adel, join work with M. Zohry**

adel.groupe@gmail.com, zohry@uae.ma

**Faculty of Sciences, Univ. Tetouan, Maroc**


## Abstract


It is shown that if a bounded linear operator $T$ or its adjoint $T^*$ has the single-valued extension property, then generalized Weyl's theorem and generalized Browder's theorem hold for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$. We establish the spectral theorem for the B-Weyl spectrum and we give necessary and sufficient conditions for such operator $T$ to obey generalized Weyl's theorem.


## Introduction and denotations

Let $X$ denote an infinite-dimensional complex Banach space and $\mathcal{L}(X)$ the unital (with unit the identity operator, $I$, on $X$) Banach algebra of bounded linear operators acting on $X$. For an operator $T \in \mathcal{L}(X)$ write $T^*$ for its adjoint, $N(T)$ for its null space, $R(T)$ for its range, $\sigma(T)$ for its spectrum, $\sigma_{su}(T)$ for its surjective spectrum, $\sigma_a(T)$ for its approximate point spectrum, $\alpha(T)$ for its nullity and $\beta(T)$ for its defect. $T$ is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator if the range $R(T)$ of $T$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$). A semi-Fredholm operator is an upper or a lower semi-Fredholm operator. If both $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is called a Fredholm operator and the index of $T$ is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. For a $T$-invariant closed linear subspace $Y$ of $X$, let $T \mid Y$ denote the operator given by the restriction of $T$ to $Y$.
For a bounded linear operator $T$ and for each integer $n$, define $T_n$ to be the restriction of $T$ to $R(T^n)$ viewed as a map from $R(T^n)$ into itself. If for some integer $n$ the range $R(T^n)$ is closed and $T_n = T \mid R(T^n)$ is a Fredholm (resp. semi-Fredholm) operator, then $T$ is called a B-Fredholm (resp. semi-B-Fredholm) operator. In this case, from [?, Proposition 2.1] $T_m$ is a Fredholm operator and $\text{ind}(T_m) = \text{ind}(T_n)$ for each $m \geq n$. This permits to define the index of a B-Fredholm operator $T$ as the index of the Fredholm operator $T_n$ where, $n$ is any integer such that $R(T^n)$ is closed and $T_n$ is a Fredholm operator. It is shown (see [?, Theorem 3.2]) that if $S$ and $T$ are two commuting B-Fredholm operators then the product $ST$ is a B-Fredholm operator and $\text{ind}(ST) = \text{ind}(S) + \text{ind}(T)$. Let $BF(X)$ be the class of all B-Fredholm operators and $\rho_{BF}(X) = \{\lambda \in \mathbb{C} : T - \lambda I \in BF(X)\}$ be the B-Fredholm resolvent of $T$ and let $\sigma_{BF}(T) = \mathbb{C} \setminus \rho_{BF}(T)$ be the B-Fredholm spectrum of $T$. The class $BF(X)$ has been studied by M. Berkani (see [?, Theorem 2.7]) where it was shown that an operator $T \in \mathcal{L}(X)$ is a B-Fredholm operator if and only if $T = S_0 \oplus S_1$ where $S_0$ is a Fredholm operator and $S_1$ is a nilpotent one. He also proved that $\sigma_{BF}(T)$ is a closed subset of $\mathbb{C}$ contained in the spectrum $\sigma(T)$ and showed that the spectral mapping theorem holds for $\sigma_{BF}(T)$, that is, $f(\sigma_{BF}(T)) = \sigma_{BF}(f(T))$ for any complex-valued analytic function on a neighborhood of $\sigma(T)$ (see [?, Theorem 3.4]). From [?] we recall that for $T \in \mathcal{L}(X)$, the ascent $a(T)$ and the descent $d(T)$ are given by

$$a(T) = \inf\{n \geq 0 : \ N(T^{n+1}) = N(T^n)\}$$

and

$$d(T) = \inf\{n \geq 0 : \ R(T^n) = R(T^{n+1})\}$$

respectively, where the infinum over the emptyset is taken to be $\infty$. If $a(T)$ and $d(T)$ are both finite then $a(T) = d(T) = p$, $X = N(T^p) \oplus R(T^p)$ and $R(T^p)$ is closed.
An operator $T \in \mathcal{L}(X)$ is called semi-regular if $R(T)$ is closed and $N(T) \subseteq R(T^n)$ for every $n \in \mathbb{N}$. The semi-regular resolvent is the subset of the complex field defined by $\text{s-reg}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is semi-regular}\}$, we note that $\text{s-reg}(T) = \text{s-reg}(T^*)$ is an open subset of $\mathbb{C}$. The semi-B-Fredholm resolvent of $T$ is the subset of the complex field given by $\rho_{SBF} = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is semi-B-Fredholm}\}$.
As a consequence of [?, Théorème 2.7], we obtain the following result.

**Proposition 1.1.**
*Let $T \in \mathcal{L}(X)$.*
*(i) If $T$ has the SVEP then $\text{s-reg}(T) = \rho_a(T)$.*
*(ii) If $T^*$ has the SVEP then $\text{s-reg}(T) = \rho_{su}(T)$.*

We recall that an operator $T \in \mathcal{L}(X)$ has the single-valued extension property, abbreviated SVEP, if, for every open set $U \subseteq \mathbb{C}$, the only analytic solution $f : \ U \longrightarrow X$ of the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function on $U$. We will denote by $\mathcal{H}(\sigma(T))$ the set of all complex-valued functions which are analytic on an open set containing $\sigma(T)$.

For our investigations we need the following result.

**Proposition 1.2.**
*Let $T \in \mathcal{L}(X)$.*
*(i) If $T$ has the SVEP then $\text{ind}(T - \lambda I) \leq 0$ for every $\lambda \in \rho_{SBF}(T)$.*
*(ii) If $T^*$ has the SVEP then $\text{ind}(T - \lambda I) \geq 0$ for every $\lambda \in \rho_{SBF}(T)$.*

**Proposition :**
(i) Let $\lambda \in \rho_{SBF}(T)$, then there exists an integer $p$ such that

$$(T \mid R(T - \lambda I)^p) - \lambda I = (T - \lambda I) \mid R(T - \lambda I)^p$$

is semi-Fredholm. From the Kato decomposition, there exists $\delta > 0$ such that

$$\{\mu \in \mathbb{C} : \ 0 < |\mu - \lambda| < \delta\} \subseteq \text{ s-reg}(T - \lambda I \mid R(T - \lambda I)^p).$$

Since $T$ has the SVEP, Proposition 1.1 implies that $\text{s-reg}(T - \lambda I \mid R(T - \lambda I)^p) = \rho_{ap}(T - \lambda I \mid R(T - \lambda I)^p)$. Therefore, $N((T \mid R(T - \lambda I)^p) - \mu I) = 0$ and so $\text{ind}(T - \mu I) = \text{ind}((T \mid R(T - \lambda I)^p) - \mu I) \leq 0$, holding for $0 < |\mu - \lambda| < \delta$. Thus, by the continuity of the index, $\text{ind}(T - \lambda) \leq 0$.
(ii) This is included in part (i) since $\text{ind}(T^*) = -\text{ind}(T)$.

An operator $T \in \mathcal{L}(X)$ is said to be Weyl if it is Fredholm of index zero and Browder if it is Fredholm of finite ascent and descent. The essentiel spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Fredholm}\};$$

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Weyl}\};$$

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Browder}\}.$$

It is well known that $\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma(T)$.
An operator $T \in \mathcal{L}(X)$ is called B-Weyl if it is B-Fredholm of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ of $T$ is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not B-Weyl}\}.$$

For a subset $K$ of $\mathbb{C}$, we shall write $\text{iso}(K)$ for its isolated points. A complex number $\lambda$ is said to be Riesz point of $T$ in $\mathcal{L}(X)$ if $\lambda_0 \in \text{iso}(\sigma(T))$ and the spectral projection corresponding to the set $\{\lambda_0\}$ has finite-dimensional range. The set of all Riesz points of $T$ will be denoted by $\Pi_0(T)$. It is known that if $T \in \mathcal{L}(X)$ and $\lambda \in \sigma(T)$ then $\lambda \in \Pi_0(T)$ if and only if $T - \lambda I$ is Fredholm of finite ascent and descent (see [?]). Consequently $\sigma_b(T) = \sigma(T) \setminus \Pi_0(T)$. Let $\Pi(T)$ denote the set of all poles of the resolvent of $T$ and $E_0(T) = \{\lambda \in \mathbb{C} : \ \lambda \in \text{iso}(\sigma(T)), \ 0 < \alpha(T - \lambda I) < \infty\}$. For a normal operator $T$ acting on a Hilbert space $H$, Berkani [?, Theorem 4.5] showed that $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$ where $E(T)$ is the set of all eigenvalues of $T$ which are isolated in $\sigma(T)$. This result gives a generalization of the classical Weyl's theorem $\sigma_w(T) = \sigma(T) \setminus E_0(T)$.

## SVEP and Generalized Weyl's theorem

The concept of Drazin invertibility plays an important role for the class of B-Fredholm operators. From [?] we recall that, for an algebra $\mathcal{A}$ with unit $\mathbf{1}$ we say that an element $a \in \mathcal{A}$ is Drazin invertible of degre $k$ if there is an element $b$ of $\mathcal{A}$ such that $a^k ba = a^k$, $bab = b$ and $ab = ba$. The drazin spectrum of $a \in \mathcal{A}$ is defined by $\sigma_D(a) = \{\lambda \in \mathbb{C} : a - \lambda \mathbf{1} \text{ is not Drazin invertible}\}$. In the case of $\mathcal{A} = \mathcal{L}(X)$, it is well known that $T$ is Drazin invertible if and only if it has a finite ascent and descent which is also equivalent to the fact that $T = T_0 \oplus T_1$ where $T_0$ is an invertible operator and $T_1$ is a nilpotent one, see for instance [?, Proposition 6] and [?, Corollary 2.2].

Recall that $\sigma_w(T) = \bigcap\{\sigma(T + K) : \ K \in \mathcal{K}(X)\}$ where $\mathcal{K}(X)$ is the class of all compact operators acting on $X$. It was proved in [?, Theorem 4.3] that for $T \in \mathcal{L}(X)$, $\sigma_{BW}(T) = \bigcap\{\sigma_D(T + F) : \ F \in \mathcal{F}(X)\}$.

Let $T \in \mathcal{L}(X)$, we will say that :
(i) $T$ satisfies Weyl's theorem if $\sigma_w(T) = \sigma(T) \setminus E_0(T)$.
(ii) $T$ satisfies generalized Weyl's theorem if $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$.
(iii) $T$ satisfies Browder's theorem if $\sigma_w(T) = \sigma(T) \setminus \Pi_0(T)$.
(iv) $T$ satisfies generalized Browder's theorem if $\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T)$.

Recall from [?] that if $T \in \mathcal{L}(X)$ satisfies generalized Weyl's theorem then it also satisfies Weyl's theorem and if $T$ satisfies generalized Browder's theorem then it satisfies Browder's theorem.

We now turn to an another extension of the characterization of operators obeying Weyl's theorem ([?, Theorem 4]).

**Theorem 2.1.** [?, Theorem 2.5]
*If $T \in \mathcal{L}(X)$ then we have*
*(i) $\sigma_{BW}(T) \subset \sigma(T) \setminus E(T)$ if and only if $E(T) = \Pi(T)$.*
*(ii) $\sigma_{BW}(T) \supset \sigma(T) \setminus E(T)$ if and only if $\sigma_{BW}(T) = \sigma_D(T)$.*

From this theorem we obtain immediately the following corollary.

**Corollary 2.2.**
*Let $T \in \mathcal{L}(X)$, then $T$ satisfies generalized Weyl's theorem if and only if $\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T)$ and $E(T) = \Pi(T)$.*

Using a standard argument and the Riesz functional calculus, we obtain the following result.

**Proposition 2.3.**
*Let $T \in \mathcal{L}(X)$, then $\sigma_{BW}(f(T)) \subseteq f(\sigma_{BW}(T))$ for every $f \in \mathcal{H}(\sigma(T))$.*

**Proposition :** Let $\lambda \in \sigma_{BW}(f(T))$, then $f(T) - \lambda I$ is not a B-Weyl's operator. As $\sigma_{BW}(f(T)) \subseteq \sigma(f(T)) = f(\sigma(T))$, there exists $\mu \in \sigma(T)$ such that $\lambda = f(\mu)$. We have $f(z) - f(\mu) = (z - \mu)^m (z - \mu_1)^{m_1} \cdots (z - \mu_n)^{m_n} g(z)$ where $g$ is a non vanishing analytic function on $\sigma(T)$. So $f(T) - f(\mu)I = (T - \mu I)^m (T - \mu_1 I)^{m_1} \cdots (T - \mu_n I)^{m_n} g(T) = f(T) - \lambda I$. Since $f(T) - \lambda I$ is not a B-Weyl operator, and

$$\text{ind}(f(T) - f(\mu)I) = m\,\text{ind}(T - \mu I) + m_1 \text{ind}(T - \mu_1 I) + \cdots + m_n \text{ind}(T - \mu_n I),$$

there exists $\beta \in \{\mu, \mu_1, \cdots, \mu_n\}$ such that $T - \beta I$ is not a B-Weyl operator and since $f(\beta) = \lambda$ we get $\beta \in \sigma_{BW}(T)$.

The opposite inclusion does not hold in general. Furthermore if $f$ is injective on $\sigma_{BW}(T)$, the last inclusion becomes an equality.

The proof of the next result is similar to that one involving $\sigma_w(T)$ (see [?, Theorem 3].

**Theorem 2.4.**
*Let $T \in \mathcal{L}(X)$, if $f \in \mathcal{H}(\sigma(T))$ is injective on $\sigma_{BW}(T)$ then $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$.*

Let $BW(X)$ be the class of $T \in \mathcal{L}(X)$ such that $\text{ind}(T - \lambda I) \leq 0$ for all $\lambda \in \rho_{BF}(T)$ or $\text{ind}(T - \lambda I) \geq 0$ for all $\lambda \in \rho_{BF}(T)$. We recall that hyponormals operators on a Hilbert space $H$ lie in $BW(H)$.

The following result shows that, for operators lying in the class $BW(X)$, the spectral mapping theorem for complex polynomials implies the spectral mapping one for complex-valued analytic functions. For its proof, we can repeat the argument used in [?, Theorem 2] word for word; we have only to replace Weyl operator by B-Weyl operator, Fredholm operator by B-Fredholm operator and Weyl spectrum by B-Weyl spectrum.

**Theorem 2.5.**
*For $T \in \mathcal{L}(X)$ the following assertions are equivalent :*
*(i) $T \in BW(X)$.*
*(ii) $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ for all $f \in \mathcal{H}(\sigma(T))$.*
*(iii) $\sigma_{BW}(p(T)) = p(\sigma_{BW}(T))$ for all complex polynomial $p$.*

We are now in a position to show that SVEP implies the spectral theorem for the B-Weyl spectrum.

**Proposition 2.6.**
*If $T$ or $T^*$ has the SVEP, then $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ for any $f \in \mathcal{H}(\sigma(T))$.*

**Proposition :** Let $f \in \mathcal{H}(\sigma(T))$. If $T$ or $T^*$ has the SVEP, by Proposition 1.2, $T$ lies in $BW(X)$ and Theorem 2.5 concludes the proof.

Our next goal is to show that generalized Browder's theorem is satisfied for $f(T)$ whenever $T$ or $T^*$ has the single-valued extension property and $f \in \mathcal{H}(\sigma(T))$. To settle this, we use a characterization of the pole of the resolvent in terms of of ascent and descent given in [?].

**Theorem 2.7.**
*If $T \in \mathcal{L}(X)$ or its adjoint has the SVEP, then generalized Browder's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*