Cours d'homotopie rationnelle (Cours 4) Pr. Hilali

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<u>Rappel</u> $X \in Top_1 \longrightarrow A_{PL}(X)$

$$\begin{array}{c} A_{PL}(X) \\ \uparrow \varrho \\ \mathcal{M}(X) = (\Lambda V, d) \end{array}$$

<u>Théorème</u>. $\forall A$ adgc avec $A^0 = \mathbb{Q}$, $A^1 = 0$

A admet un modèle minimal

<u>lemme</u> extension de Hirsch

$\mathcal{M} = \bigoplus_{n \geq} \mathcal{M}(n)$, $\mathcal{M}(n) = \bigoplus_{k=0}^{n} \mathcal{M}^k$ $\mathcal{M}(1) = \mathcal{M}(0) = \mathbb{Q}$

$\mathcal{M}$ est minimal si $\mathcal{M}(n+1) = \mathcal{M}(n) \otimes \Lambda V^{n+1}$ où V^{n+1} $\mathbb{Q}$-espace vectoriel concentré en degré $n+1$.

$\forall v \in V^{n+1}$ $dv \in Z^{n+2}(\mathcal{M}(n))$

dv est décomposable

$dv = w + \omega$, $\omega \in \Lambda^{\geq 2} V^{\leq n}$ $\Rightarrow w = 0$ car $|dv| = n+2$

donc dv décomposable.

pour démontrer le théorème on utilise par récurrence l'extension de Hirsch

$n = 0$ $\mathcal{M}(0) = \mathbb{Q}$

$$\begin{array}{rcl} \rho_0 : \mathcal{M}(0) & \longrightarrow & A \\ 1 & \longmapsto & 1 \end{array}$$

on suppose qu'on a construit $\mathcal{M}(n-1) = (\Lambda V^{\leq n-1}, d)$

$\rho_{n-1} : \mathcal{M}(n-1) \longrightarrow A$ vérifiant

(i) $(\rho_{n-1})_* : H^k(\mathcal{M}(n-1)) \xrightarrow{\cong} H^k(A)$, $\forall k \leq n-1$

(ii) $(\rho_{n-1})_* : H^n(\mathcal{M}(n-1)) \hookrightarrow H^n(A)$ injective

<u>A construire</u>

. $V^n = V_1 \oplus V_2$?

. $\rho_n : \mathcal{M}(n) = \mathcal{M}(n-1) \otimes \Lambda V^n \longrightarrow A$

vérifiant

. $dV^n \subset Z^{n+1}(\mathcal{M}(n-1))$

. $(\rho_n)_* : H^k(\mathcal{M}(n)) \longrightarrow H^k(A)$

isomorphisme $k \leq n$

monomorphisme en $k = n+1$

on pose $\cdot\ V_1 = \mathrm{Coker}\,(\rho_{n-1})_*$

$dv_1 = 0 \quad , \forall v_1 \in V$

. d'autre part $(\rho_{n-1})_* : H^{n-1}(\mathcal{M}(n-1)) \longrightarrow H^{n+1}(A)$

on pose $V_2 = \ker(\rho_{n-1})_*$

$v_2 \in V_2$

$dv_2 = ?$

$v_2 \in H^{n+1}(\mathcal{M}(n-1)) \xrightarrow{\beta} Z^{n+1}(\mathcal{M}(n-1))$

$dv_2 = \beta(v_2) \quad (\neq 0)$

$\cdot\ \rho_n : \mathcal{M}(n) = \mathcal{M}(n-1) \otimes \Lambda(V_1 \oplus V_2) \longrightarrow A$

$\rho_n|_{\mathcal{M}(n-1)} = \rho_{n-1}$

$\rho_n(v_1) \in A ? \qquad v_1 \in \mathrm{Coker}\,\rho_{n-1}$

$\mathrm{Coker}\,\rho_{n-1} = H^n(A)/H^n(\mathcal{M}(n-1)) \xrightarrow{\alpha} A$

$\rho_n(v_1) = \alpha(v_1)$

$H^n(A) \cong H^n(\mathcal{M}(n-1)) \oplus H^n(V_1) \quad , (H^n(V_1) = V_1)$

donc $(\rho_n)_* : H^k(\mathcal{M}(n)) \longrightarrow H^k(A)$ isomorphisme $\forall k \leq n$.

car $H^k(\mathcal{M}(n)) = H^k(\mathcal{M}(n-1)) \ , \forall k \leq n-1$

$Z^n(\mathcal{M}(n)) = Z^n(\mathcal{M}(n-1)) \oplus V_1$

$\cdot\ \rho_n(v_2) = ?$

$$\begin{array}{ccc} v_2 \in H^{n+1}(\mathcal{M}(n-1)) & \xrightarrow{\beta} & Z^{n+1}(\mathcal{M}(n-1)) \\ & & \downarrow \rho_{n-1} \\ & & Z^{n+1}(A) \end{array}$$

$\rho_n(v_2) = \rho_{n-1}(\beta v_2)$

$(\rho_n)_*(v_2) = (\rho_{n-1})_*(\beta v_2)$

$= d(\rho_{n-1})_* v_2$

or $H^{n+1}(\mathcal{M}(n-1))/dV_2 \overset{inj}{\hookrightarrow} H^{n+1}(A)$

or $H^{n+1}(\mathcal{M}(n)) = H^{n+1}(\mathcal{M}(n-1))/dV_2 \hookrightarrow H^{n+1}(A)$

$(Z^{n+1}(\mathcal{M}(n)) = Z^{n+1}(\mathcal{M}(n-1))$

<u>Exemple</u>.

$A = A_{PL}(S^2)$

$H^k(A) = \begin{cases} \mathbb{Q} & , \quad k=0,\ k=2 \\ 0 & \quad \text{sinon} \end{cases}$

$\mathcal{M}(1) = \mathcal{M}(0) = \mathbb{Q}$

$\mathcal{M}(2) = \mathbb{Q} \otimes \Lambda V^2$, $V^2 = V_1 \oplus V_2$

$(\rho_1)_* : H^2(\mathcal{M}(1)) = 0 \hookrightarrow H^2(A) = \mathbb{Q}a$

$(\rho_1)_* : H^3(\mathcal{M}(1)) = 0 \longrightarrow H^3(A) = 0$

$V_1 = \operatorname{coker}(\rho_1)_*$, $V_2 = \ker(\rho_1)_* = 0$

$V_1 = \mathbb{Q}a$, $|a| = 2$

$\mathcal{M}(2) = \mathbb{Q} \oplus \Lambda a$

$\mathcal{M}(3) = \mathcal{M}(2) \underset{d}{\oplus} \Lambda V^3$

$(\rho_2)_* : H^3(\mathcal{M}(2)) \longrightarrow H^3(A) = 0$

$(\rho_2)_* : H^4(\mathcal{M}(2)) \longrightarrow H^4(A) = 0$

$V^3 = V_1 \oplus V_2$

$V_1 = \operatorname{coker}(\rho_2)_* = 0$

$H^4(\mathcal{M}(2)) = \mathbb{Q}a^2 \Longrightarrow V_2 = \ker(\rho_2)_* \cong \mathbb{Q}a^2$

$V_2 = \mathbb{Q}b$ avec $db = a^2$

$\Longrightarrow V_2 = \ker(\rho_2)_* \cong \mathbb{Q}a^2$

$V_2 = \mathbb{Q}b$ avec $db = a^2$

$\mathcal{M}(n) = \mathcal{M}(n-1) \otimes \Lambda V^n$, $n \geq 4$

$(\rho_{n-1})^* : \underbrace{H^{n+1}(\mathcal{M}(n-1))}_{=0} \longrightarrow H^{n+1}(A) = 0$

car $a^k = a^2 a^{k-2} = d(b a^{k-2}) \Longrightarrow V^n = 0$

$\Longrightarrow \mathcal{M}(S^2) = (\Lambda(a,b), d)$ $db = a^2$, $da = 0$

<u>Pont algèbre → Topologie</u>

$X \in Top_1$ $\xrightarrow[\text{Tour de Postnikov}]{\text{rationalisation}}$ X_0

$$\begin{tikzcd}
 & X_n \arrow[d] \\
 & X_{n-1} \arrow[d] \\
 & \vdots \arrow[d] \\
X_0 \arrow[uuur] \arrow[uur] \arrow[r] & X_2
\end{tikzcd}$$

$$\begin{tikzcd}
K(\pi_n(X)\otimes\mathbb{Q}, n) \arrow[r, equal] \arrow[d] & K(\pi_n(X)\otimes\mathbb{Q}, n) \arrow[d] \\
X_n \arrow[d] & PK \arrow[d] \\
X_{n-1} \arrow[r, "k^{n-1}"'] & K(\pi_n(X)\otimes\mathbb{Q}, n-1)
\end{tikzcd}$$

<u>lemme de Hirsch</u>

on a équivalence

(i) $p: E \longrightarrow B$ $\quad K(V,n)$ - fibration principale

(ii) $\exists$ une extension

$m(B) \underset{d}{\otimes} \Lambda(V^*)^n \xrightarrow{\ \Psi\ } A_{PL}(E)$ quasi-isomorphisme