An obstruction to tangles embedding

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Overview

An *n-tangle* consists of n disjoint arcs in the 3-ball. We ask the following quastion : For a given knot K, and a tangle T, can we embed T into K ? In other words, is there a diagram of T that extends to a diagram of K ? This problem is important due to it implication in many other problems and also due to its applications in the study of DNA.

In this presentation, we present a method of using quandle 2-cocycle invariant as obstruction to embedding oriented tangles into oriented knots (or links).

All geometric objects (knots, links, tangles, or diagrams) considered in this presentation are assumed to be oriented. The convention for tangles orientation will be specified once tangles will be defined. We then leave out the "oriented" description before each occurrence of "knot", "link", "tangle", and "diagram".

Definition

A *quandle*, X, is a set with a binary operation $(a, b) \mapsto a * b$ such that :

1. For any $a \in X$, $a * a = a$.
2. For any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$.
3. For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

Note that the second condition can be replaced with the following requirement : The operation $*b : X \to X$, defined by $*b(x) = x * b$ is a bijection. The inverse map to $*b$ is denoted by $\bar{*}b$.

Example

Let n be a positive integer. For elements $i, j \in \{0, 1, ..., n-1\}$, define $i * j = 2j - i (mod\ n)$. Then $*$ defines a quandle structure on $\{0, 1, ..., n-1\}$. The quandle thus defined is called the *dihedral quandle,* R_n.

Let X be a quandle and $C_n^R(X) = \mathbb{Z}^{(X^n)}$ for every $n \in \mathbb{N}$ (note that $C_0^R(X) = \mathbb{Z}^{(*)} = \mathbb{Z}$).
For every $n \geq 2$ we put

$$\partial_n \colon C_n^R(X) \longrightarrow C_{n-1}^R(X)$$
$$(x_1, \ldots, x_n) \longmapsto \sum_{i=2}^{n} (-1)^i d_i^n(x_1, \ldots, x_n)$$

where

$$d_i^n(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) - (x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)$$

And $\partial_n = 0$ for $n \leq 1$. Then $C_*^R(X) = \{C_n^R(X), \partial_n\}_{n \in \mathbb{N}}$ is a chain complex.

Let $C_n^D(X)$ be the subset of $C_n^R(X)$ generated by n-tuples $(x_1, ..., x_n)$ with $x_i = x_{i+1}$ for some $i \in [\![1, n-1]\!]$ if $n \geq 2$; otherwise let $C_n^D(X) = 0$. One has the following

1. $d_j^n(C_n^D(X)) \subset C_{n-1}^D(X)$ for $2 \leq i+1 < j \leq n$,
2. $d_j^n(C_n^D(X)) \subset C_{n-1}^D(X)$ for $1 \leq j < i \leq n-1$,
3. $d_i^n|_{C_n^D(X)} = d_{i+1}^n|_{C_n^D(X)}$.

Thus $\partial_n(C_n^D(X)) \subset C_{n-1}^D(X)$ for every $n \in \mathbb{N}$. $C_*^D(X) = \{C_n^D(X), \partial_n\}_{n\in\mathbb{N}}$

becomes then a sub-complex of $C_*^R(X)$.

Put $C_n^Q(X) = C_n^R(X)/C_n^D(X)$ for every $n \in \mathbb{N}$. $C_*^Q(X) = \{C_n^Q(X), \partial'_n\}_{n\in\mathbb{N}}$ is again a chain complex where ∂'_n is the induced homomorphism by ∂_n.

For an abelian group A, define the chain and cochain complexes

$$C_*^W(X;A) = C_*^W(X) \otimes A, \ \partial = \partial \otimes id_A;$$

$$C_W^*(X;A) = Hom_{\mathbb{Z}}(C_*^W(X), A), \ \delta = Hom_{\mathbb{Z}}(-, A)(\partial).$$

where $W = D, R$. And

$$C_*^Q(X;A) = C_*^Q(X) \otimes A, \ \partial = \partial' \otimes id_A;$$

$$C_Q^*(X;A) = Hom_{\mathbb{Z}}(C_*^Q(X), A), \ \delta' = Hom_{\mathbb{Z}}(-, A)(\partial').$$

Definition

The *nth quandle homology group* and the *nth quandle cohomology group* of a quandle X with coefficient group A are

$$H_n^Q(X;A) = H_n(C_*^Q(X;A)) \ , \ H_Q^n(X;A) = H^n(C_Q^*(X;A))$$

The *n-cycle* and *n-boundary* groups (resp. *n-cocycle* and *n-coboundary groups*) are denoted $Z_n^W(X;A)$ and $B_n^W(X;A)$ (resp. $Z_W^n(X;A)$ and $B_W^n(X;A)$), so that

$$H_n^W(X;A) = Z_n^W(X;A)/B_n^W(X;A) \ , \ H_W^n(X;A) = Z_W^n(X;A)/B_W^n(X;A)$$

where W is one of R, D, Q.

Let $P^n(X;A) = \{f \in C_R^n(X;A)/C_n^D(X;A) \subseteq Ker(f)\}$.

If $f \in P^n(X;A)$ then f induces a unique morphism $\overline{f} \in C_Q^n(X;A)$. Then the correpondence $\psi : f \mapsto \overline{f}$ is an isomorphism between $P^n(X;A)$ and $C_Q^n(X;A)$. On the other hand, the restriction of ψ to $P^n(X;A) \cap Z_R^n(X;A)$ is an

isomorphism onto $Z_Q^n(X;A)$ and we can easily check that $\psi(\delta^n(P^{n-1}(X;A))) = \delta'^n(C_Q^{n-1}(X;A)) = B_Q^n(X;A)$.
Thus $\delta^n(P^{n-1}(X;A)) \cong B_Q^n(X;A)$. Which yield to the following lemma

Lemma

For every $n \in \mathbb{N}$, one has

$$H_Q^n(X;A) = (P^n(X;A) \cap Z_R^n(X;A))/\delta^n(P^{n-1}(X;A)).$$

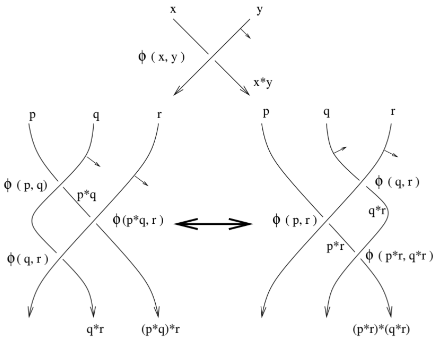

Figure: 1

The cocycle conditions are related to moves on knot diagrams as indicated in Figure1. A 2-cocycle $\phi \in H^2_Q(X; A)$ satisfies the relation :

$$\phi(p, q) + \phi(p * q; r) = \phi(p, r) + \phi(p * r, q * r) \text{ for all } p, q, r \in X$$

and $\phi(p, p) = 0$ for every $p \in X$.

Let a knot diagram be given. The *co-orientation* is a family of normal vectors to the knot diagram such that the pair (orientation , co-orientation) matches the given orientation of the plane in which lies the diagram using the right-hand rule (see Figure 2).

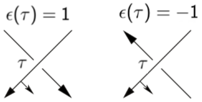

Figure: 2

At a crossing, say τ, if the pair of the co-orientation of the over-arc and that of the under-arc matches the (right hand) orientation of the plane, then we say that τ is positive and write $\epsilon(\tau) = 1$; otherwise it is negative and we write $\epsilon(\tau) = -1$.

Let K be a knot, D one of his diagrams, and X a quandle.

Definition

A coloring of D by X is a function $\mathcal{C} : \mathcal{R} \to X$, where $\mathcal{R}$ is the set of arcs in D, satisfying the condition depicted in the next figure at each crossing

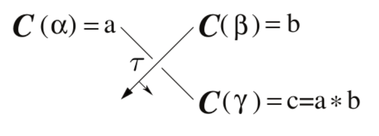

Figure: 3

The quandle element $\mathcal{C}(\alpha)$ assigned to an arc α by a coloring $\mathcal{C}$ is called a *color* of α.
In the Figure 3, the ordered colors $(\mathcal{C}(\alpha), \mathcal{C}(\beta)) := (x_\tau, y_\tau)$ are called the *source colors* in the crossing τ.

Let D be a diagram of a knot K, X a quandle, and A an abelian group. Let $\phi \in H^2_Q(X;A)$ be a quandle 2-cocycle, which can be regarded as a function $\phi : X^2 \to A$ satisfying

$$\phi(p,q) + \phi(p*q;r) = \phi(p,r) + \phi(p*r, q*r) \text{ for all } p,q,r \in X$$

and $\phi(p,p) = 0$ for every $p \in X$.

Definition

The *Boltzmann weight* associated with ϕ at a crossing τ, $\mathcal{B}_\phi(\mathcal{C},\tau)$, of D is defined as

$$\mathcal{B}_\phi(\mathcal{C},\tau) = \epsilon(\tau)\phi(x_\tau, y_\tau).$$

Let $Cr(D)$ denote the set of crossings in D and $Col_X(D)$ the set of all colorings of D by X.

Definition

The *partition function*, or the *state sum* associated with ϕ of D is the expression

$$\Phi_\phi(D) = \sum_{\mathcal{C} \in \boldsymbol{Col_X}(D)} \prod_{\tau \in \boldsymbol{Cr}(D)} \mathcal{B}_\phi(\mathcal{C}, \tau).$$

Theorem

If D' is an other diagram of K obtained from D by a finite sequence of Reidemeister movements and planar isotopies, then $\Phi_\phi(D) = \Phi_\phi(D')$. That is, Φ_ϕ defines an invariant of knots called the *2-cocycle invariant associated with* ϕ.

We note then $\Phi_\phi(D_K) = \Phi_\phi(K)$ the state sum associated with ϕ of K for any diagram D_K of K.

A *multiset* is a pair (S, m), where S is a set and m is a function that assigns to each element a in S a positive integer (called the *multiplicity*, meaning the number of occurences of a).
As an example, $\{0, 0, 1, 1, 1\}$ represent a multiset (S, m) where $S = \{0, 1\}$ and $m(0) = 2, m(1) = 3$.

Definition

The 2-cocycle invariant $\Phi_\phi(K)$ in a multiset form is defined by

$$\Phi_\phi(K) = \{ \sum_{\tau \in Cr(D_K)} \mathcal{B}(\mathcal{C}, \tau) / \mathcal{C} \in Col_X(D_K)\}.$$

Where D_K is a diagram of K.

A *tangle* is a pair (B, E), where E is a set of properly embedded oriented arcs in a 3-ball B. A tangle will have four endpoints throught this presentation.

Two tangles T_1 and T_2 are said to be *equivalent* if there exist an orientation preserving homeomorphism of pairs between T_1 and T_2 which is the identity on the boundary spheres.

A tangle T is *embedded* in a knot (or a link) K if there is an embedded ball B in 3-space such that T is equivalent to the pair $(B, B \cap K)$.

Tangles are represented by diagrams in a manner similar to knot diagrams.

The four endpoints of a given tangle diagram are located at four corners of a circle in a plane at angles $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$ when the circle is placed with the origin as its center. These endpoints are labeled NE, NW, SW, and SE, respectively, representing the directions of a compass.

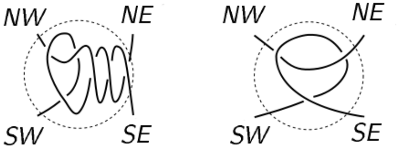

Figure: 4

Anallogically to the equivalence criterion for knots, one has the following :

Two tangles T_1 and T_2 are equivalent if and only if T_1 has a diagram D_1 and T_2 has a diagram D_2 such that D_1 and D_2 are related by a finite sequence of Reidemeister moves and planar isotopies both confined in the "tangle box" (meaning the interior of the disk whose boundary is dashed in Figure 4).

In order to set an orientation for a given tangle T, it is sufficient to choose two out of the four endpoints of a diagram of T (NW and SW for example) and indicate that we travel the diagram inward or outward from those points (NE inward and SE outward for example).

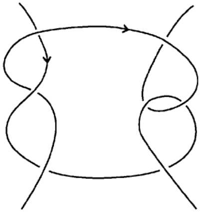

Figure: An oriented tangle diagram with the orientation (NE in, NW out)

Let T be a tangle and X be a quandle.

Definition

A *(boundary monochromatic) coloring* of T is a map $\mathcal{C} : \mathcal{A} \to X$ from the set of arcs in a diagram of T to X satisfying the same quandle coloring condition as for knot diagrams at each crossing such that the (four) boundary points of the tangle diagram receive the same element (color) of X.

Let D_T be a diagram of a tangle T and X be a quandle. Denote by $Col_{X,x}(D_T)$ and $Col_X(D_T)$, respectively, the set of boundary monochromatic colorings of D_T by X with the boundary color $x \in X$ and the set of all boundary monochromatic colorings of D_T by X.

Let A be an abelian group. We put

$$\Phi_\phi(T, x) = \sum_{\mathcal{C} \in \boldsymbol{Col_{X,x}}(D_T)} \prod_{\tau \in \boldsymbol{Cr}(D_T)} \mathcal{B}(\mathcal{C}, \tau),$$

where $\phi \in H^2_Q(X; A)$ is a 2-cocycle.

Definition

The 2-cocycle invariant associated with ϕ for T is defined by

$$\Phi_\phi(T) = \sum_{x \in X} \Phi_\phi(T, x).$$

It is an invariant of the tangle type.

It can be proven in a way similar to the case of a knot that the number of colorings $|Col_X(T)|$ does not depend on the choice of a diagram of T. If a diagram D_1 of T has a coloring $\mathcal{C}_1$, and a diagram D_2 is obtained from D_1 by a Reidemeister move, then there is a unique coloring $\mathcal{C}_2$ of D_2 induced from $\mathcal{C}_1$ such that the colors stay the same except where the move is performed.

Given two diagrams D_1 and D_2 of a tangle T, there is a one-to-one correspondence between the set of colorings of D_1 and the set of colorings of D_2, and the cocycle invariant is well defined.

Analogically to the multiset form of the 2-cocycle invariant for knots, we use the multiset form of the 2-cocycle invariant for tangles.
An adaptation of the set inclusion notion for multisets is used to give an obstruction to tangle embeddings.

Definition

Let M and N be multisets. We say that M is *multi-included* in N, and write $M \subset_m N$, if

- if $x \in M$, then $x \in N$, and
- the multiplicity of x in M is less or equal to the multiplicity of x in N.

Theorem

Let T be a tangle, X a quandle, A an abelian group, and $\phi \in H^2_Q(X; A)$ a 2-cocyle. Suppose that T embeds in a knot K. Then we have the multi-inclusion

$$\Phi_\phi(T) \subset_m \Phi_\phi(K).$$

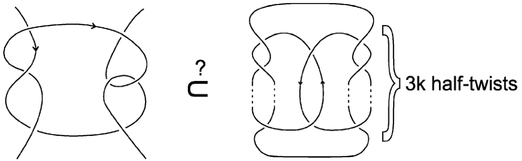

Figure: 6

Figure 6 illustrates an example of a tangle T , and a family of links that have $3k$, $k \in \mathbb{N}$, half-twists on each side. Let L denote any member of this family. We can use the dihedral quandle R_3 2-cocycle invariant to show that T does not embed into L.

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