# Explicit formula for the LS-category of a family of elliptic spaces and Topological complexity of flag manifolds

**Khalid BOUTAHIR**
boutahir@gmail.com

Faculté des sciences de Meknès

01 October 2016

# Table of Contents

# Part I

# Research framework

# Research framework
## Algebraic Topology

Algebraic topology is a branch of mathematics that uses tools from abstract algebra to study topological spaces. The basic goal is to find algebraic invariants that classify topological spaces up to homeomorphism, though usually most classify up to homotopy equivalence.
Algebraic topology is concerned with the construction of algebraic invariants (usually groups) associated to topological spaces which serve to distinguish between them. Most of these invariants are “homotopy” invariants. In essence, this means that they do not change under continuous deformation of the space and homotopy is a precise way of formulating the idea of continuous deformation.

# Research framework

## Homotopy theory

Homotopy theory is the study of the invariants and properties of topological spaces $X$ and continuous maps $f$ that depend only on the homotopy type of the space and the homotopy class of the map. Classical examples include the homology groups $H_*(X;\mathbb{Z})$, the cohomology algebra $H^*(X;\mathbb{Z})$ and the homotopy groups $\pi_*(X)$. But in practice, those invariants are sometimes difficult to compute, even for simple spaces. A solution is to discard some informations until we keep something that we can manipulate. The problem is that if we discard too much informations, we don't have anything interesting left.

# Research framework

## Rational homotopy theory

A good compromise has been discovered with the rational homotopy theory: we keep the rational information by localizing with respect to $\mathbb{Q}$. The result is a theory that is both interesting and usable in practice. Invariants of rational homotopy theory include the rational homology groups $H_*(X;Q)$, the rational cohomology algebra $H^*(X;\mathbb{Q})$, the rational homotopy groups $\pi_*(X) \otimes \mathbb{Q}$. An important aspect of that theory is the concept of rational model : each space is replaced with algebraic models that mimic its properties.

# Research framework

## Rational homotopy theory

Rational homotopy theory is the study of the rational homotopy type of a space, it assigns to topological spaces invariants which are preserved by continuous maps $f$ for which $H_*(f;\mathbb{Q})$ is an isomorphism. The two standard approaches of the theory are due respectively to Quillen (1969) and Sullivan (1977). Each constructs from a class of CW complexes $X$ an algebraic model $\mathcal{M}_X$, and then constructs from $\mathcal{M}_X$ a CW complex $X_{\mathbb{Q}}$, together with a map $\varphi_X : X \longrightarrow X_{\mathbb{Q}}$. Both $H_*(X_{\mathbb{Q}};\mathbb{Z})$ and $\pi_n(X_{\mathbb{Q}})$ are rational vector spaces, and with appropriate hypotheses

$$H_*(\varphi_X) : H_*(X)\otimes\mathbb{Q} \longrightarrow H_*(X_{\mathbb{Q}};\mathbb{Z}), \text{ and}$$
$$\pi_n(\varphi_X) : \pi_n(X)\otimes\mathbb{Q} \longrightarrow \pi_n(X_{\mathbb{Q}}),\ n\geq 2,$$

are isomorphisms.

# Research framework

## Rational homotopy theory

In each case the model $\mathcal{M}_X$ belongs to an algebraic homotopy category, and a homotopy class of maps $f : X \longrightarrow Y$ induces a homotopy class of morphisms $\mathcal{M}_f : \mathcal{M}_X \longrightarrow \mathcal{M}_Y$ (in Quillen approach) and a homotopy class of morphisms $\mathcal{M}_f : \mathcal{M}_Y \longrightarrow \mathcal{M}_X$ (in Sullivan's approach). These are referred to as *representatives* of $f$.

The power of rational homotopy theory lies precisely in the bijection

$$\left\{ \begin{array}{l} \text{rational homotopy types} \\ \text{of simply connected CW} \\ \text{complexes with rational} \\ \text{homology of finite type} \end{array} \right\} \overset{\cong}{\longleftrightarrow} \left\{ \begin{array}{l} \text{isomorphism classes of minimal} \\ \text{Sullivan algebras over } \mathbb{Q} \text{ which} \\ \text{are 1-connected and of finite} \\ \text{type} \end{array} \right\}$$

It reduces all topological computations in rational homotopy theory to computations on an algebraic object, the minimal Sullivan algebra.

# Part II

# Tools

# Tools

## Rational homotopy type

Sullivan's approach associates to each path connected space $X$ a cochain algebra $\mathcal{M}_X$ of the form $(\Lambda V, d)$ in which the free commutative graded algebra $\Lambda V$ is generated by $V = V^{\geq 1}$, and $V = \oplus_m \Lambda^m V$ with $\Lambda^m V = V \wedge \cdots \wedge V$ ($m$ factors) and $d$ the differential. Additionally, each $\Lambda V^{\geq 1}$ is preserved by $d$, and $d$ also satisfies a "nilpotence"condition: $(\Lambda V, d)$ is called a *minimal Sullivan algebra*.

# Tools

## Rational homotopy type

### Definition

A simply connected space $X$ is rational if $\pi_* X$ is a $\mathbb{Q}$-vector space. A map $f : X \longrightarrow Y$ is a *rationalization* of $X$ if $Y$ is simply connected, rational and if

$$\pi_* f \otimes \mathbb{Q} : \pi_* X \otimes \mathbb{Q} \longrightarrow \pi_* Y \otimes \mathbb{Q} \cong \pi_* Y$$

is an isomorphism.

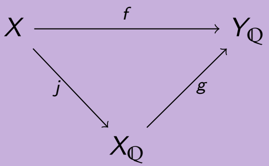
Clearly, a rational space can not have torsion in its homotopy. Being a rational space is a big constraint, it is not obvious that rational spaces/rationalization exists. Luckily, the next theorem proves their existence and unicity.

# Tools

## Rational homotopy type

**Theorem**

*Let $X$ be a simply connected space. There exists a rational space $X_{\mathbb{Q}}$ and a rationalization $j : X \longrightarrow X_{\mathbb{Q}}$. Also, if $Y_{\mathbb{Q}}$ is a simply connected rational space, then any continuous map $f : X \longrightarrow Y_{\mathbb{Q}}$ can be extended over $X_{\mathbb{Q}}$, i.e., there is a continuous map $g : X_{\mathbb{Q}} \longrightarrow Y_{\mathbb{Q}}$ which is unique up to homotopy, such that*




*commutes.*

# Algebraic tools

## Graded vector spaces

### Definition

- A *graded vector space* is a family $V = \{V^i\}_{i \geq 0}$ of vector spaces (over the field $\mathbb{K}$), indexed by the non-negative integers.
- An element $v \in V^i$ is said to have *degree* $i$ and we denote this by $|v| = i$.
- A graded vector space $V$ is concentrated in degree $i \in I$ ($I \subset \mathbb{N}$) if $V^i = 0$ for every $i \notin I$, and in this case, we would write $V = \{V^i\}_{i \in I}$.
- A graded vector space $V$ is said to be of *finite type* if each $V^i$ is finite dimensional.
- A graded vector space $V$ is *finite dimensional* if each $V^i$ is finite dimensional and $V^i = 0$ for all but finitely many $i$'s.

# Algebraic tools
## Graded vector spaces

### Definition

A linear map of degree $n$ from a graded vector space $V$ to a graded vector space $W$ is a family of linear maps $f_i : V^i \longrightarrow W^{i+n}$ (for each $i \geq 0$).

### Definition

A differential on a graded vector space $V$ is a linear map $d : V \longrightarrow V$ of degree 1 such that $d_{n+1} o d_n = 0$ for any $n \geq 0$.

# Algebraic tools
## Graded algebras

### Definitions

- A *graded algebra* $A$ is a graded vector space equipped with a linear map $A \otimes A \longrightarrow A$ of degree zero, called multiplication and denoted by $x \otimes y \longrightarrow xy$, together with an identity element $1 \in A^0$, such that for all $x, y, z \in A$,
$$(xy)z = x(yz) \text{ and } 1x = x1 = x.$$
- A graded algebra $A$ is *commutative* if
$$xy = (-1)^{|x||y|} yx \text{ for all homogeneous elements } x, y \in A.$$
- A *derivation of degree* $k$ in a graded algebra $A$ is a morphism $d : A \longrightarrow A$ of degree $k$ such that
$$d(xy) = (dx)y + (-1)^{k|x|} x(dy) \text{ for all } x, y \in A.$$
This is essentially the graded version of the Leibniz product rule from differential calculus.

# Algebraic tools
## Graded algebras

### Definition

For any free graded vector space $V$, the *tensor algebra* $TV$ is the graded vector space defined by

$$TV = \bigoplus_{q=0}^{\infty} T^q V \text{ where } T^0 V = \mathbb{K} \text{ and } T^q V = \underbrace{V \otimes \cdots \otimes V}_{q \text{ times}} \; (q \geq 1).$$

Multiplication is given by $a.b = a \otimes b$ for $a \in T^q V$ and $b \in T^p V$.
Elements $v_1 \otimes \cdots \otimes v_q \in T^q V$ have degree$= |v_1 \otimes \cdots \otimes v_q| = \sum_{j=1}^{q} |v_j|$; and word length $q$.

Suppose $1/2 \in \mathbb{K}$ and let $V$ be a free graded module. The elements $v \otimes w - (-1)^{|v||w|} w \otimes v$ $(v, w \in V)$ generate an ideal $I \subset TV$.
The quotient graded algebra $\Lambda V = TV/I$ is called the *free commutative graded algebra* on $V$.
$\Lambda V = \bigoplus_{q=0}^{\infty} \Lambda^q V$, where $\Lambda^q V$ is the linear span of the elements $v_1 \wedge \cdots \wedge v_q$, $v_i \in V$.

# Algebraic tools
## Sullivan model

### Definitions

- A *Sullivan algebra*, is a commutative cochain algebra of the form $(\Lambda V, d)$, with
    - $V = \{V^p\}_{p\geq 1}$;
    - $V = \bigcup_{k=0}^{\infty} V(k)$, where $V(0) \subset V(1) \subset \dots$ is an increasing sequence of graded subspaces such that $d = 0$ in $V(0)$ and $d : V(k) \longrightarrow \Lambda V(k-1)$ for all $k \geq 1$.
- Let $(A, d)$ be a *cca*, $(\Lambda V, d)$ be a *Sullivan cca*, and $\phi : (\Lambda V, d) \xrightarrow{\cong} (A, d)$ be a quasi-isomorphism, then we say that $\phi$ is a *Sullivan model* for $(A, d)$.
- A Sullivan algebra (or model) $(\Lambda V, d)$ is called *minimal* if $Im(d) \subset \Lambda^{\geq 2} V$.

# Algebraic tools
## Sullivan model

### Remarks

1. *If* $V^1 = 0$ *and* $d(V) \subset \Lambda^{\geq 2} V$, *then* $(\Lambda V, d)$ *is automatically a minimal Sullivan algebra.*
2. *A Sullivan algebra is minimal if and only if* $d_0 = 0$, *where* $d_0$ *is the linear part of* $d$.

### Definition

Let $(A, d)$ be a cdga. A *Sullivan model* for $(A, d)$ is a quasi-isomorphism $m : (\Lambda V, d) \xrightarrow{\simeq} (A, d)$ from a Sullivan algebra $(\Lambda V, d)$.

# Algebraic tools

## Sullivan model

**Definition**

A Sullivan algebra $(\Lambda V, d)$ is *formal* if there exists a quasi-isomorphism

$$\varphi : (\Lambda V, d) \xrightarrow{\simeq} (H(\Lambda V, d), 0)$$

where $(H(\Lambda V, d), 0)$ is the cga $H(\Lambda V, d)$ equipped with the 0 differential.

**Definition**

A simply connected space $X$ with minimal Sullivan model $(\Lambda V, d)$ is *formal* if $(\Lambda V, d)$ is formal. Equivalently, $X$ is formal if there exists a quasi-isomorphism

$$\varphi : (\Lambda V, d) \xrightarrow{\simeq} (H^*(X, \mathbb{Q}), 0)$$

# Algebraic tools

## Rationally elliptic space

### Definitions

A simply connected topological space $X$ is *rationally elliptic* if

$$\dim H^*(X;\mathbb{Q}) < \infty \text{ and } \dim \pi_*(X) \otimes \mathbb{Q} < \infty.$$

The *formal dimension* fdim$X$ of a rationally elliptic space $X$ is defined by

$$fdimX := max\{k | H^k(\Lambda V, d) \neq 0\}.$$

A minimal Sullivan algebra $(\Lambda V, d)$ is *elliptic* if its associated rational space is elliptic. The *formal dimension* of an elliptic minimal Sullivan algebra is the formal dimension of the associated space.

### Definition

A minimal Sullivan algebra $(\Lambda V, d)$ is *pure* if $\dim V < \infty$, $d|_{V^{even}} = 0$ and $d(V^{odd}) \subseteq \Lambda V^{even}$. A space is *pure* if its Sullivan model is pure.

Note that a pure space $X$ is elliptic if and only if $\dim H^*(X;\mathbb{Q}) < \infty$.

# Part III

# Outcomes

Minimal Sullivan algebras $(\Lambda V, d)$ are equipped with a homotopy theory and a range of invariants like the *sectional category*.

**Definition**

The sectional category of a map $p : E \longrightarrow B$ , denoted $secat(p)$, is the minimum number of open sets needed to cover $B$, on each of which $p$ admits a homotopy section.

It was first studied extensively by *Švarc* for fibrations (under the name *genus*) and later by *Berstein* and *Ganea* for arbitrary maps. The notion of sectional category generalizes the classical (*Lusternik-Schnirelmann*) category, since $secat(p) = cat(B)$ whenever $E$ is contractible and $p$ is surjective.

The Lusternik-Schnirelmann category is an old and well-known numerical invariant of the homotopy type of spaces which may be defined as follows:

**Definition**

A space $X$ has category $n$ if this is the least integer for which $X$ can be covered by $n$ open sets contractible in $X$.

This invariant, $cat(X)$, was introduced since 1934 by L. Lusternik and L. Schnirelmann through their study of variational problems and it was introduced by Y. S. Félix and Halperin 1982 as the first practical use of D. Sullivan models (1977).

# Explicit formula for the LS-category of a family of elliptic spaces

Félix and Halperin developed a deep approach, within rational homotopy theory, for computing the LS-category. Later on, and also concerning this hard task, Félix, Halperin and Lemaire showed that for Poincaré duality spaces (and hence for elliptic spaces) the rational LS-category coincide with the Toomer invariant which, at first sight, may look easier to compute. After that, Lechuga and Murillo found a formula for this invariant which generalizes and in some cases it complements previous results, that is, for a finite type simply connected rationally elliptic CW-complex $X$ with Sullivan minimal model $(\Lambda V, d)$ and for $k \geq 2$ the biggest integer such that $d = \sum_{i \geq k} d_i$ with $d_i(V) \subseteq \Lambda^i V$, and if $(\Lambda V, d_k)$ is moreover elliptic then $cat(\Lambda V, d) = \dim(V^{even})(k-2) + \dim(V^{odd})$. Our first main goal is to study the case outside the previous restrictive condition.

# Explicit formula for the LS-category of a family of elliptic spaces

Let $X$ be a finite type simply connected CW-complex with Sullivan minimal model $(\Lambda V, d)$ and let $k \geq 2$ the biggest integer such that $d = \sum_{i \geq k} d_i$ with $d_i(V) \subseteq \Lambda^i V$ and $\dim(V) < \infty$.

Consider on $(\Lambda V, d)$ the filtration given by

$$F^p = \Lambda^{\geq (k-1)p} V = \bigoplus_{i=(k-1)p}^{\infty} \Lambda^i V. \tag{1}$$

$F^p$ is preserved by the differential $d$ and satisfies $F^p(\Lambda V) \otimes F^q(\Lambda V) \subseteq F^{p+q}(\Lambda V)$, $\forall p, q \geq 0$, so it is a filtration of differential graded algebras. Also, since $F^0 = \Lambda V$ and $F^{p+1} \subseteq F^p$ this filtration is decreasing and bounded, so it induces a convergent spectral sequence. Its $0^{th}$-term is

$$E_0^{p,q} = \left( \frac{F^p}{F^{p+1}} \right)^{p+q} = \left( \frac{\Lambda^{\geq (k-1)p} V}{\Lambda^{\geq (k-1)(p+1)} V} \right)^{p+q}.$$

# Explicit formula for the LS-category of a family of elliptic spaces

Hence, we have the identification:

$$E_0^{p,q} = \left(\Lambda^{p(k-1)} V \oplus \Lambda^{p(k-1)+1} V \oplus \ldots \oplus \Lambda^{p(k-1)+k-2} V\right)^{p+q} \tag{2}$$

In this general situation, the $1^{st}$-term is the graded algebra $\Lambda V$ provided with a differential $\delta$, which isn't necessarily a derivation on the set $V$ of generators. That is $(\Lambda V, \delta)$ is a commutative differential graded algebra, but it is not a Sullivan algebra. The spectral sequence is therefore:

$$H^{p,q}(\Lambda V, \delta) \Rightarrow H^{p+q}(\Lambda V, d). \tag{3}$$

Hence if $\dim(V) < \infty$ and $(\Lambda V, \delta)$ has finite dimensional cohomology, then $(\Lambda V, d)$ is elliptic. This gives a new family of rationally elliptic spaces for which $d = \sum_{i \geq k} d_i$.

# Explicit formula for the LS-category of a family of elliptic spaces

Our aim is to give an almost explicit formula for $cat(\Lambda V, d)$ when $(\Lambda V, d)$ is elliptic but $(\Lambda V, d_k)$ not necessarily elliptic.
In the first step, we shall treat the case where $k = 3$ under the hypothesis assuming $H^N(\Lambda V, \delta)$ one dimensional, being $N$ the formal dimension of $(\Lambda V, d)$.

# Explicit formula for the LS-category of a family of elliptic spaces

In what follows, we give the expression for $\delta$ in the case where k=3. Our filtration is one of filtered differential graded algebras, hence in this we have :

$$E_0^{p,q} = \left(\Lambda^{2p} V \oplus \Lambda^{2p+1} V\right)^{p+q}$$

with the product given by:

$$(u,v)\otimes(u',v') = (uu', uv' + vu'), \quad \forall (u,v) \in E_0^{p,q}, \forall (u',v') \in E_0^{p',q'}.$$

# Explicit formula for the LS-category of a family of elliptic spaces

On the other hand, since $d_1 = d_2 = 0$ the differential on $E_0$ is zero , hence $E_1^{p,q} = E_0^{p,q}$ and so the identification above gives the following diagram

$$\begin{array}{ccc}
E_1^{p,q} & \xrightarrow{\cong} & \left(\Lambda^{2p}V \oplus \Lambda^{2p+1}V\right)^{p+q} \\
\downarrow \delta & & \downarrow \delta \quad (d_3,\ d_4,\ d_3) \\
E_1^{p+1,q} & \xrightarrow{\cong} & \left(\Lambda^{2(p+1)}V \oplus \Lambda^{2(p+1)+1}V\right)^{p+q+1}
\end{array}$$

with $\delta$ defined as follows,

$$\delta(u,v) = (d_3 u, d_3 v + d_4 u)$$

# Explicit formula for the LS-category of a family of elliptic spaces

## Our first result

**Theorem (1)**

*If $(\Lambda V, d)$ is elliptic and $H^N(\Lambda V, \delta) = \mathbb{Q}.\alpha$ is one dimensional, then*

$$cat_0(X) = cat(\Lambda V, d) = sup\{s \geq 0,\ \alpha = [\omega_0] \text{ with } \omega_0 \in \Lambda^{\geq s} V\}.$$

# Explicit formula for the LS-category of a family of elliptic spaces

## The first inequality

Let us resume in what follow, the algorithm that gives the first inequality: $cat(\Lambda V,d) \geq sup\{s \geq 0,\ \alpha = [\omega_0]\ with\ \omega_0 \in \Lambda^{\geq s} V\} := r.$

i) Initially we fix a representative $\omega_0 \in \Lambda^{\geq r} V$ of the fundamental class $\alpha$ with $r$ being the largest $s$ such that $\omega_0 \in \Lambda^{\geq s} V$.

ii) A straightforward calculation gives successively:

$$\omega_0 = \omega_0^0 + \omega_0^1 + \ldots + \omega_0^l \quad with \quad \omega_0^i = (\omega_0^{i,1}, \omega_0^{i,2}) \in \Lambda^{2(p+i)} V \oplus \Lambda^{2(p+i)+1} V$$

$$d\omega_0 = a_2^0 + a_3^0 + \ldots + a_{t+l}^0 \quad with \quad a_i^0 = (a_i^{0,1}, a_i^{0,2}) \in \Lambda^{2(p+i)} V \oplus \Lambda^{2(p+i)+1} V.$$

It follows that $a_2^0 = \delta(b_2)$ for some $b_2 \in \Lambda^{2(p+2)-2} V \oplus \Lambda^{2(p+2)-1} V$.

iii) We take $t$ the largest integer satisfying the inequality: $t \leq \frac{1}{4}(N - 4p - 4l - 1)$.

iv) We continue with $\omega_1 = \omega_0 - b_2$.

v) By the imposition *iii*), the algorithm leads to a representative $\omega_{t+l-1} \in \Lambda^{\geq r} V$ of the fundamental class of $(\Lambda V, d)$ and then $e_0(\Lambda V, d) \geq r$.

# Explicit formula for the LS-category of a family of elliptic spaces

## The second inequality

Denote $s = e_0(\Lambda V, d)$ and let $\omega \in \Lambda^{\geq s} V$ be a cocycle representing the generating class $\alpha$ of $H^N(\Lambda V, d)$ . Write $\omega = \omega_0 + \omega_1 + ... + \omega_t, \;\; \omega_i \in \Lambda^{s+i} V$. We deduce that:

$$\begin{aligned} d\omega &= (d_3\omega_0 + d_3\omega_1 + d_4\omega_0) + (d_3\omega_2 ... + d_3\omega_t) + (d_4\omega_1 + ... + d_4\omega_t) + ... \\ &= \delta(\omega_0, \omega_1) + ... \end{aligned}$$

Since $d\omega = 0$, by wordlength reasons, it follows that $\delta(\omega_0, \omega_1) = 0$. If $(\omega_0, \omega_1)$ were a $\delta$-boundary, i.e., $(\omega_0, \omega_1) = \delta(x)$, then

$$\begin{aligned} \omega - dx &= (\omega_0, \omega_1) - \delta(x) + (\omega_2 + ... + \omega_t) - (d - \delta)(x) \\ &= (\omega_3 + ... + \omega_t) - (d - \delta)(x) \end{aligned}$$

so $\omega - dx \in \Lambda^{\geq s+2} V$ which contradicts the fact $s = e_0(\Lambda V, d)$. Hence $(\omega_0, \omega_1)$ represents the generating class of $H^N(\Lambda V, \delta)$. But $(\omega_0, \omega_1) \in \Lambda^{\geq s} V$ implies that $s \leq r$. Consequently, $e_0(\Lambda V, d) \leq r$. We conclude that

$$e_0(\Lambda V, d) = r.$$

# Explicit formula for the LS-category of a family of elliptic spaces

## Second result

The second step in our program reads as follow:

**Theorem (2)**

*If $(\Lambda V, d)$ is elliptic and $dim H^N(\Lambda V, \delta) = m$ with basis $\{\alpha_1, \dots, \alpha_m\}$ as before. Then, there exists a unique $p_j$, such that $cat_0(X) = r_j$ with $r_j = 2p_j$ or else $r_j = 2p_j + 1$.*

# Explicit formula for the LS-category of a family of elliptic spaces

## Example

Let $(\Lambda V, d)$ be the pure model defined by $V^{even} =< x_2, x_6 >$, $V^{odd} =< y_5, y_{13}, y_{23} >$ , $dx_2 = dx_6 = 0$, $dy_5 = x_2^3$, $dy_{13} = x_2x_6^2$ and $dy_{23} = x_6^4$.
Clearly we have $dimH(\Lambda V, d_3) = \infty$ and $dimH(\Lambda V, d) < \infty$.
We note also that, since $N = 35$ is odd, then any representative of the fundamental class of $(\Lambda V, d)$ will be of the form:
$n_1x_2^kx_6^ly_5 + n_2x_2^{k'}x_6^{l'}y_{13} + n_3x_2^{k''}x_6^{l''}y_{23}$, with $n_1$, $n_2$ and $n_3 \in \mathbb{N}$.

# Explicit formula for the LS-category of a family of elliptic spaces

## Example

The matrix determining the fundamental class is:

$$A = \begin{pmatrix} x_2^2 & 0 \\ x_6^2 & 0 \\ 0 & x_6^3 \end{pmatrix}$$

So $\omega_0 = -x_2^2 x_6^3 y_{13} + x_6^5 y_5 \in \Lambda^{\geq 6} V$ is an generator of this fundamental cohomology class. Another representative of this class is $\omega_1 = -x_2^3 x_6 y_{23} + x_2^2 x_6^3 y_{13}$. It is a straightforward calculation to prove that they are the unique representatives. We conclude that $e_0(\Lambda V, d) = 6$.

# Explicit formula for the LS-category of a family of elliptic spaces

## Example

On the other hand $H^N(\Lambda V, \delta)$ has at least tow generators: $(\omega_0, 0) \in \Lambda^6 V \oplus \Lambda^7 V$ and $[(0, x_6^2 y_{23})]$, hence $dim H^N(\Lambda V, \delta) > 1$. We have also $dim H^N(\Lambda V, d_3) > 1$ with $[\omega_0]$ and $[x_6^2 y_{23}]$ being two generators of $H^N(\Lambda V, d_3)$. Here the algorithm is applied to $(\omega_0, 0)$ and the one of [5] is applied to $[\omega_0]$.
Note finally that $e_0(\Lambda V, d) = 6 \neq m + n(k-2) = 5$.

# Explicit formula for the LS-category of a family of elliptic spaces

## General case

A generalization of the Theorem 1 seems natural if $d = \sum_{i \geq k} d_i$ with $k \geq 3$.

**Theorem**

*If $(\Lambda V, d)$ is elliptic, $(\Lambda V, d_k)$ is not elliptic and $H^N(\Lambda V, \delta) = \mathbb{Q}.\alpha$ is one dimensional, then*

$$cat_0(X) = cat(\Lambda V, d) = sup\{s \geq 0,\ \alpha = [\omega_0] \text{ with } \omega_0 \in \Lambda^{\geq s} V\}.$$

# Explicit formula for the LS-category of a family of elliptic spaces

## General case

Let us resume in what follow, the algorithm that gives the first inequality: $cat(\Lambda V,d) \geq sup\{s \geq 0,\ \alpha = [\omega_0]\ with\ \omega_0 \in \Lambda^{\geq s} V\} := r.$

i) Initially we fix a representative $\omega_0 \in \Lambda^{\geq r} V$ of the fundamental class $\alpha$ with $r$ being the largest $s$ such that $\omega_0 \in \Lambda^{\geq s} V$.

ii) A straightforward calculation gives successively:

$$\omega_0 = \omega_0^0 + \omega_0^1 + ... + \omega_0^l$$

with

$$\omega_0^i = (\omega_0^{i,0}, \dots, \omega_0^{i,k-2}) \in \Lambda^{(k-1)(p+i)} V \oplus \dots \oplus \Lambda^{(k-1)(p+i)+k-2} V$$

$$d\omega_0 = a_2^0 + a_3^0 + ... + a_{t+l}^0$$

with

$$a_i^0 = (a_i^{0,0}, \dots, a_i^{0,k-2}) \in \Lambda^{(k-1)(p+i)} V \oplus \dots \oplus \Lambda^{(k-1)(p+i)+k-2} V$$

It follows that $a_2^0 = \delta(b_2)$; $b_2 \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2)-(k-1)+j} V$.

# Explicit formula for the LS-category of a family of elliptic spaces

## General case

iii) We take $t$ the largest integer satisfying the inequality:

$$t \leq \frac{1}{2(k-1)}(N-2(k-1)(p+l)-2k+5).$$

iv) We continue with $\omega_1 = \omega_0 - b_2$.

v) By the imposition *iii*), the algorithm leads to a representative $\omega_{t+l-1} \in \Lambda^{\geq r} V$ of the fundamental class of $(\Lambda V, d)$ and then $e_0(\Lambda V, d) \geq r$.

Now, $\dim(V) < \infty$, imply $\dim H^N(\Lambda V, \delta) < \infty$.

# Explicit formula for the LS-category of a family of elliptic spaces

## The spectral sequence

if $d = \sum_{i \geq k} d_i$ with $k \geq 3$,

$$E_0^{p,q} = \left(\Lambda^{p(k-1)} V \oplus \Lambda^{p(k-1)+1} V \oplus \ldots \oplus \Lambda^{p(k-1)+k-2} V\right)^{p+q} \qquad (4)$$

with the product given by:

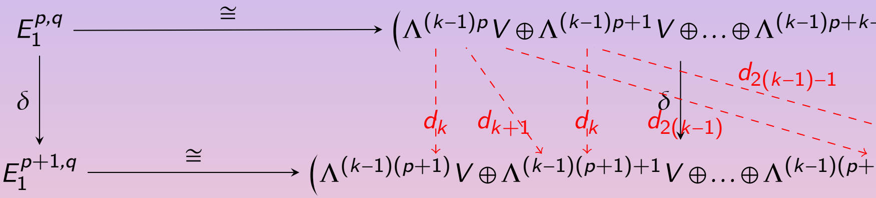
$$(u_0, u_1, \ldots, u_{k-2}) \otimes (u_0^{'}, u_1^{'}, \ldots, u_{k-2}^{'}) = (v_0, v_1, \ldots, v_{k-2})$$

for all $(u_0, u_1, \ldots, u_{k-2})$, $(u_0^{'}, u_1^{'}, \ldots, u_{k-2}^{'}) \in E_0^{p,q}$ with $v_m = \sum_{i+j=m} u_i u_j^{'}$ and $m = 0, \ldots, k-2$.

# Explicit formula for the LS-category of a family of elliptic spaces

## The spectral sequence

The differential on $E_0$ is zero, hence $E_1^{p,q} = E_0^{p,q}$ and so the identification above gives the following diagram:




with $\delta$ defined as follows,

$$\delta(u_0, u_1, \ldots, u_{k-2}) = (w_k, w_{k+1}, \ldots, w_{2k-2}) \ \ \textit{with}\ w_{k+j} = \sum_{\substack{i+i'=j \\ i'=0,\ldots,k-2}} d_{k+i} u_{i'}$$

# Explicit formula for the LS-category of a family of elliptic spaces

## The spectral sequence

Let $E_1^p = E_1^{p,*} = \bigoplus_{q \geq 0} E_1^{p,q}$ and $E_1^* = \bigoplus_{p \geq 0} E_1^{p,*}$. This gives a commutative differential graded algebra $(E_1^*, \delta)$ which is the first term of our spectral sequence:

$$E_2^{p,q} = H^{p,q}(\Lambda V, \delta) \Rightarrow H^{p+q}(\Lambda V, d).$$

# Explicit formula for the LS-category of a family of elliptic spaces

**Theorem**

*If* $(\Lambda V, d)$ *is elliptic and* $\dim H^N(\Lambda V, \delta) = m$ *with basis* $\{\alpha_1, \ldots, \alpha_m\}$ *as before. Then, there exists a unique* $p_j$*, such that*
$cat_0(X) = sup\{s \geq 0,\ \alpha_j = [\omega_{0j}]$ *with* $\omega_{0j} \in \Lambda^{\geq s} V\} := r_j$ *with* $r_j = 2p_j$ *or else* $r_j = 2p_j + 1$.
*where* $\omega_{0j} \in \Lambda^{\geq r_j} V$ *is a representative of the fundamental class* $\alpha_j$ *with* $r_j$ *being the largest* $s$ *such that* $\omega_{0j} \in \Lambda^{\geq s} V$.

**Remark**

*It suffices to apply the algorithm to each element of the base and take the sup.*

# Part IV

# On topological complexity

# Topology and Robotics
## The motion planning problem

Our second main goal is to give new results and study some properties of the invariant $TC(X)$ for some real projective spaces and some real flag manifolds.

# Topology and Robotics

## The motion planning problem

Motion planning is a central theme in robotics

### Motion planning

The motion planning problem consists of producing a continuous motion that connects a start configuration A and a goal configuration B.

### Motion planning algorithm

In terms of the configuration space $X$ the motion planning algorithm:

- Input: a point $(A, B) \in X \times X$
- Output: a path: $\alpha : [0,1] \longrightarrow X$ such that $\alpha(0) = A$ and $\alpha(1) = B$.

# Topology and Robotics

## The motion planning problem

If $X$ is the configuration space, then consider the free path fibration: $\pi : X^I \longrightarrow X \times X$ $\alpha \longmapsto (\alpha(0), \alpha(1))$ where $X^I$ denotes the space of all paths in $X$. In these terms, a motion planning algorithm is precisely a section (not necessarily continuous) of $\pi$. That is, a map

$$s : X \times X \longrightarrow X^I$$

such that is $\pi o s = id$

# Topology and Robotics

## The motion planning problem

Continuity of a motion planning algorithms is desired. It means that the suggested routes $(A; B)$ of going from $A$ to $B$ depends continuously on the states $A$ and $B$

**Theorem**

*There exists a continuous section $s : X \times X \longrightarrow X^I$ of $\pi$ if and only if the space $X$ is contractible.*

In general, motion planning algorithms have discontinuities.
We can consider local continuous sections of $\pi$. These are maps defined on an open subset $U \subset X \times X$ $s : U \longrightarrow X^I$ such that $\pi o s = inc : U \hookrightarrow X \times X$.

# Topology and Robotics

## Topological complexity

In order to study the discontinuities in these algorithms the following notion was introduced by M. Farber in 2003: 

**Definition**

The topological complexity of a topological space $X$, $TC(X)$; is the least non-negative integer $k$ such that $X \times X$ can be covered by $k$ open subsets

$$X \times X = U_1 \cup U_2 \cup \cdots \cup U_k$$

on each of which $\pi : X^I \longrightarrow X \times X$ admits a local continuous section.

# Topology and Robotics

## Topological complexity

### Examples

- $TC(S^n) = \begin{cases} 2 & \textit{if } n \textit{ is odd} \\ 3 & \textit{if } n \textit{ is even} \end{cases}$
- Let $S_g$ be the compact connected orientable surface of genus $g$.

  Then $TC(S_g) = \begin{cases} 3 & \textit{if } g \leq 1 \\ 5 & \textit{if } g \geq 2 \end{cases}$

# Topology and Robotics

## Topological complexity

In general, the computation of topological complexity is a very hard task!

### Strategy

One way of dealing with topological complexity is to consider approximations that, in some sense, are more manageable and therefore more computable.

### Definition

With any closed positive-dimensional manifold $X$ we can associate a homotopy invariant called the $\mathbb{Z}_2$-cup-length of $X$ and defined by

$$cup(X) := sup\{r : \exists\ x_1, ..., x_r \in \tilde{H}^*(X; \mathbb{Z}_2), with\ x_1 \cup ... \cup x_r \neq 0\}.$$

# Topology and Robotics
## Topological complexity

### Basic properties:

- $TC(X) = 1$ if and only if $X \simeq *$ is contractible.
- $TC(X)$ depends only on the homotopy type of $X$.
- For a path-connected topological space $X$ it holds

$$cat(X) \leq TC(X) \leq cat(X \times X) \leq 2cat(X) - 1$$

- For any path-connected paracompact locally contractible topological space $X$, we have $TC(X) \leq 2\dim(X) + 1$
- $cuplenght(X) \leq cat(X)$

# On topological complexity of some real projective spaces

An interesting result by M.Farber et al. (2003) shows that the problem of determining the topological complexity of projective spaces is equivalent to solving their immersion problem, i.e. the problem of finding the minimal number $k$ such that $\mathbb{R}P^n$ immerses in $\mathbb{R}^k$.

**Theorem**

*For $n \neq 1, 3, 7$ the topological complexity of real projective space $\mathbb{R}P^n$ is $TC(\mathbb{R}P^n) = I_n + 1$ where $I_n$ is the smallest natural number $k$ such that $\mathbb{R}P^n$ admits an immersion into $\mathbb{R}^k$. When $n = 1, 3, 7$ we have $TC(\mathbb{R}P^n) = n+1$.*

**Corollary**

*For any $n$, $TC(\mathbb{R}P^n) \leq 2n$. If $n$ is a power of 2, then it is an equality, that is $TC(\mathbb{R}P^{2^r}) = 2^{r+1}$*

# On topological complexity of some real projective spaces

Below is the table of the values $TC(\mathbb{R}P^n)$ for $n \leq 23$, given by Farber et al. (2003):

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $TC(\mathbb{R}P^n)$ | 2 | 4 | 4 | 8 | 8 | 8 | 8 | 16 | 16 | 17 | 17 | 19 |
| n | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | |
| $TC(\mathbb{R}P^n)$ | 23 | 23 | 23 | 32 | 32 | 33 | 33 | 35 | 39 | 39 | 39 | |

# On topological complexity of some real projective spaces

In the table below we have grouped known results on optimal immersions of some real projective spaces $\mathbb{R}P^n$.

| | n | $\subseteq$ | $\subsetneq$ |
|---|---|---|---|
| 1 | $2^r - 1$ with $r \equiv 1, 2 \; mod\, 4$ | $2^{r+1} - 2r - 1$ | $2^{r+1} - 2r - 2$ |
| 2 | $2^r - 1$ with $r \equiv 0 \; mod\, 4$ | $2^{r+1} - 2r - 2$ | $2^{r+1} - 2r - 3$ |
| 3 | $2^r - 1$ with $r \equiv 3 \; mod\, 4$ | $2^{r+1} - 2r - 3$ | $2^{r+1} - 2r - 4$ |
| 4 | $2^r + 1$ | $2^{r+1} - 1$ | $2^{r+1} - 2$ |
| 5 | $2^r + 2$ | $2^{r+1}$ | $2^{r+1} - 1$ |
| 6 | $2^r + 3$ | $2^{r+1}$ | $2^{r+1} - 1$ |
| 7 | $2^r + 4$ | $2^{r+1} + 2$ | $2^{r+1} + 1$ |
| 8 | $2^r + 5$ | $2^{r+1} + 6$ | $2^{r+1} + 5$ |
| 9 | $2^r + 6$ | $2^{r+1} + 6$ | $2^{r+1} + 5$ |
| 10 | $2^r + 7$ | $2^{r+1} + 6$ | $2^{r+1} + 5$ |
| 11 | $2^r + 8$ | $2^{r+1} + 7$ | ? |

# On topological complexity of some real projective spaces

| | n | $\subseteq$ | $\subsetneq$ |
|---|---|---|---|
| 12 | $2^r + 2^s, \ r > s \geq 3$ | $2^{r+1} + 2^{s+1} - 4\alpha(n) - 1$ | ? |
| 13 | $2^r + 2^s + 1, \ r > s \geq 2$ | $2^{r+1} + 2^{s+1} - 2$ | $2^{r+1} + 2^{s+1} - 3$ |
| 14 | $2^r + 2^s + 2, \ r > s \geq 2$ | $2^{r+1} + 2^{s+1} - 2$ | $2^{r+1} + 2^{s+1} - 3$ |
| 15 | $2^r + 2^s + 3, \ r > s \geq 2$ | $2^{r+1} + 2^{s+1} - 2$ | $2^{r+1} + 2^{s+1} - 3$ |

$\alpha(n)$ denote the number of 1's appearing in the dyadic expansion of n. $\subseteq$ mean: immersed in the Euclidean space of dimension, and $\subsetneq$ mean: not immersed in the Euclidean space of dimension.

# On topological complexity of some real projective spaces

From this table we can find other results for the topological complexity of the real projective space $\mathbb{R}P^n$ :

| | n | $TC(\mathbb{R}P^n)$ |
|---|---|---|
| 1 | $2^r - 1$ with $r \equiv 1, 2 \ mod\, 4$ | $2^{r+1} - 2r$ |
| 2 | $2^r - 1$ with $r \equiv 0 \ mod\, 4$ | $2^{r+1} - 2r - 1$ |
| 3 | $2^r - 1$ with $r \equiv 3 \ mod\, 4$ | $2^{r+1} - 2r - 2$ |
| 4 | $2^r + 1$ | $2^{r+1}$ |
| 5 | $2^r + 2$ | $2^{r+1} + 1$ |
| 6 | $2^r + 3$ | $2^{r+1} + 1$ |
| 7 | $2^r + 4$ | $2^{r+1} + 3$ |
| 8 | $2^r + 5$ | $2^{r+1} + 7$ |
| 9 | $2^r + 6$ | $2^{r+1} + 7$ |
| 10 | $2^r + 7$ | $2^{r+1} + 7$ |

# On topological complexity of some real projective spaces

| | n | $TC(\mathbb{R}P^n)$ |
|---|---|---|
| 11 | $2^r + 8$ | $2^{r+1} + 7$ or $2^{r+1} + 8$ |
| 12 | $2^r + 2^s,\ r > s \geq 3$ | $\leq 2^{r+1} + 2^{s+1} - 4\alpha(n)$ |
| 13 | $2^r + 2^s + 1,\ r > s \geq 2$ | $2^{r+1} + 2^{s+1} - 1$ |
| 14 | $2^r + 2^s + 2,\ r > s \geq 2$ | $2^{r+1} + 2^{s+1} - 1$ |
| 15 | $2^r + 2^s + 3,\ r > s \geq 2$ | $2^{r+1} + 2^{s+1} - 1$ |

# On topological complexity of some real projective spaces

By applying the theorem of immersion and the ascendancy property of $TC(\mathbb{R}P^n)$ in terms of $n$ we computed some other results

| n | 24 | 25 | 26 | 27 | 28 | 29 |
|---|---|---|---|---|---|---|
| $TC(\mathbb{R}\, P^n)$ | 39;40 | 47 | 47 | 47 | 48 | $48 \leq * \leq 51$ |

| n | 30 | 31 | 32 | 33 | 34 | 35 |
|---|---|---|---|---|---|---|
| $TC(\mathbb{R}\, P^n)$ | $48 \leq * \leq 52$ | 54 | 64 | 64 | 65 | 65 |

| n | 36 | 37 | 38 | 39 | 40 | 41 |
|---|---|---|---|---|---|---|
| $TC(\mathbb{R}\, P^n)$ | 67 | 71 | 71 | 71 | 71; 72 | 79 |

| n | 42 | 43 | 44 | 45 | 46 | 47 |
|---|---|---|---|---|---|---|
| $TC(\mathbb{R}\, P^n)$ | 79 | 79 | 80 | $80 \leq * \leq 82$ | $80 \leq * \leq 84$ | $80 \leq * \leq 86$ |

# On topological complexity of some real projective spaces

| n | 48 | 49 | 50 | 51 | 52 | 53 |
|---|---|---|---|---|---|---|
| $TC(\mathbb{R}P^n)$ | $86 \leq * \leq 88$ | 95 | 95 | 95 | 96 | $96 \leq * \leq 99$ |

| n | 54 | 55 | 56 | 57 | 58 | 59 |
|---|---|---|---|---|---|---|
| $TC(\mathbb{R}P^n)$ | 99;100 | 99;100 | $102 \leq * \leq 104$ | $102 \leq * \leq 107$ | 109;110 | 110 |

| n | 60 | 61 | 62 | 63 | 64 | |
|---|---|---|---|---|---|---|
| $TC(\mathbb{R}P^n)$ | $110 \leq * \leq 112$ | $110 \leq * \leq 115$ | $110 \leq * \leq 116$ | 116 | 128 | |

# On topological complexity of some real flag manifolds

## Definition

A flag manifold is the space of flags, i.e. chains of linear subspaces of $V$. Let $n_1, \ldots, n_q$ $(q \geq 2)$ be fixed positive integers, and let $F(n_1, \ldots, n_q)$ be the real flag manifold consisting of all $q$-tuples $(S_1, \ldots, S_q)$ of mutually orthogonal vector subspaces in $\mathbb{R}^n$, where $n = n_1 + \ldots + n_q$ and $\dim(S_i) = n_i$.

As a homogeneous space, we have

$$F(n_1, \ldots, n_q) \cong O(n)/O(n_1) \times \ldots \times O(n_q).$$

This makes $F(n_1, \ldots, n_q)$ into a closed manifold of dimension: $\dim(F(n_1, \ldots, n_q)) = \oplus_{1 \leq i < j \leq q} n_i n_j$.

# On topological complexity of some real flag manifolds

Over the manifold $F(n_1, n_2, ..., n_q)$, there are q canonical vector bundles $\gamma_1, ..., \gamma_q$ with $\dim(\gamma_i) = n_i$. They are characterized by the fact that the fiber of $\gamma_i$ over $(S_1, ..., S_q) \in F(n_1, ..., n_q)$ is the vector space $S_i$. The direct sum $\bigoplus_{i=1}^{q} \gamma_i$ is the trivial $n$-dimensional vector bundle.

# On topological complexity of some real flag manifolds

The manifold $F(n_1, n_2, ..., n_q)$ is nonorientable, hence has its first Stiefel-Whitney class $\omega_1(F(n_1, n_2, ..., n_q)) \in H^1(F(n_1, n_2, ..., n_q); \mathbb{Z}_2)$ non-zero, precisely when not all of the numbers $n_1, n_2, ..., n_q$ have the same parity.

Let $\omega_i(\gamma_j)$ be the Stiefel-Whitney class of the canonical vector bundle $\gamma_j$ over $F(n_1, n_2, ..., n_q)$. Then we have

$$H^*(F(n_1, n_2, ..., n_q); \mathbb{Z}_2) \cong \mathbb{Z}_2[\omega_1(\gamma_1), ..., \omega_{n_1}(\gamma_1), ..., \omega_1(\gamma_q), ..., \omega_{n_q}(\gamma_q)]/I,$$

Where the ideal $I$ is given by the identity

$$\prod_{j=1}^{q}(1 + \omega_1(\gamma_j) + ... + \omega_{n_j}(\gamma_j)) = 1$$

# On topological complexity of some real flag manifolds

**Definition**

We define $ht(\omega_1)$, the height of $\omega_1$, to be

$$ht(\omega_1) := sup\{m : \omega_1^m \neq 0 \in H^*(F(n_1, n_2, ..., n_q); \mathbb{Z}_2)\}$$

# On topological complexity of some real flag manifolds

The following result calculating $ht(\omega_1(F(n_1, n_2, ..., n_q)))$ is due to Juraj Lörinc.

## Proposition

*Let $F(n_1, n_2, ..., n_q)$ , for $q \geq 2$, be any nonorientable real flag manifold; hence not all of $n_1, n_2, ..., n_q$ have the same parity. Letting $p$ be the sum of all even numbers among $n_1, n_2, ..., n_q$, put $k = min\{p, n-p\}$. If $s$ is the uniquely determined integer such that $2^s < n \leq 2^{s+1}$, then we have*

$$ht(\omega_1(F(n_1, ..., n_q))) = \begin{cases} n-1 & \text{if} & k = 1, \\ 2^{s+1}-2 & \text{if} & k = 2 \text{ or} \\ & \text{if} & k = 3 \text{ and } n = 2^s + 1, \\ 2^{s+1}-1 & \text{otherwise} & \end{cases}$$

# On topological complexity of some real flag manifolds

We will also make use of the following results

**Theorem (J. Korbas)**

1. *For any $m \geq 1$, $k \geq 1$, one has*
$cat(F(\underbrace{1,...,1}_{k\ times},m)) = 1 + \dim(F(\underbrace{1,...,1}_{k\ times},m))$
2. *Let $m \geq 2$, $d > 0$ and $j > 0$ be integers. Taking $t$ to be the integer such that $2^t \leq m < 2^{t+1}$, suppose that $j \geq 2^{t+d} - m - 2d + 1$. Then*

$$cat(F(\underbrace{1,...,1}_{j\ times},\underbrace{2,...,2}_{d\ times},m)) = 1 + \dim(F(\underbrace{1,...,1}_{j\ times},\underbrace{2,...,2}_{d\ times},m))$$

# On topological complexity of some real flag manifolds

## Proposition (J. Korbas)

1. *For any integer* $l \geq 3$*, let* $s$ *be the only integer such that* $2^s \leq l < 2^{s+1}$*. Then* $cat(F(1,2,l)) = 3l + 3$ *if* $l = 2^{s+1} - 1$ *or if* $l = 2^{s+1} - 2$
2. $cat(F(1,2,l)) \geq 2^s + 2l + 2$ *if* $2^s \leq l \leq 2^{s+1} - 3$*,*
3. $cat(F(1,\underbrace{2,2,...,2}_{n\ times})) \geq n^2 + 1.$
4. $cat(F(\underbrace{2,2,...,2}_{n\ times})) \geq n^2 - n + 1.$

# On topological complexity of some real flag manifolds

**Theorem**

*Let $F(n_1, n_2, ..., n_q)$ , for $q \geq 2$, be any nonorientable real flag manifold; hence not all of $n_1, n_2, ..., n_q$ have the same parity. Letting $p$ be the sum of all even numbers among $n_1, n_2, ..., n_q$, put $k = min\{p, n-p\}$. If $s$ is the uniquely determined integer such that $2^s < n \leq 2^{s+1}$, then we have*

1. *If k=1, then* $TC(F(n_1, n_2, ..., n_q)) \geq n-1$
2. *If k=2, then* $TC(F(n_1, n_2, ..., n_q)) \geq n-2$
3. *If k=3 and* $n = 2^s + 1$, *then* $TC(F(n_1, n_2, ..., n_q)) \geq 2n-4$
4. *If k=3 and* $n \neq 2^s + 1$, *then* $TC(F(n_1, n_2, ..., n_q)) \geq n-1$
5. *If* $k \geq 4$, *then* $TC(F(n_1, n_2, ..., n_q)) \geq n-1$

# On topological complexity of some real flag manifolds

## Proposition

1. *For any $m \geq 1$, $k \geq 1$, one has*

$$TC(F(\underbrace{1,1,...,1}_{k\ times},m)) \geq 1 + \dim(F(\underbrace{1,1,...,1}_{k\ times},m))$$

2. *Let $m \geq 2$, $d > 0$ and $j > 0$ be integers. Taking $t$ to be the integer such that $2^t \leq m < 2^{t+1}$, suppose that $j \geq 2^{t+d} - m - 2d + 1$. Then*

$$TC(F(\underbrace{1,...,1}_{j\ times},\underbrace{2,...,2}_{d\ times},m)) \geq 1 + \dim(F(\underbrace{1,...,1}_{j\ times},\underbrace{2,...,2}_{d\ times},m))$$

# On topological complexity of some real flag manifolds

## Proposition

1. *For any integer $l \geq 3$, let $s$ be the only integer such that $2^s \leq l < 2^{s+1}$. Then $TC(F(1,2,l)) \geq 3l+3$ if $l = 2^{s+1}-1$ or if $l = 2^{s+1}-2$.*
2. $TC(F(1,2,l)) \geq 2^s + 2l + 2$ *if* $2^s \leq l \leq 2^{s+1}-3$.
3. $$TC(F(1,\underbrace{2,2,...,2}_{n\ times})) \geq n^2+1.$$
4. $$TC(F(\underbrace{2,2,...,2}_{n\ times})) \geq n^2-n+1.$$

# On topological complexity of some real flag manifolds

## Example

1. $$6 \leq TC(F(1,1,2)) \leq 11.$$
2. $$8 \leq TC(F(1,2,2)) \leq 17.$$
3. $$12 \leq TC(F(1,2,3)) \leq 23$$

# References

# References

Y. Rami and K. Boutahir
*On L.S.-category of a family of rational elliptic spaces.*
http://arxiv.org/abs/1310.6247 (under revision, Journal of Homotopy and Related Structures).

Y. Rami and K. Boutahir
*On LS-category of a family of rational elliptic spaces II.*
under revision, Extracta Mathematicae.

Y. Rami and K. Boutahir
*Note on topological complexity of some real projective spaces and real flag manifolds.*
JP Journal of Geometry and Topology, 19, Number 2 ,79-94, 2016.

O. Cornea, G. Lupton, J. Oprea, D. Tanré,(2003)
*Lusternik-Schnirelmann Category,*
Math. Surveys Monogr., vol. 103, Amer. Math. Soc., Prov., 2003.

# References


D. M. Davis.
*Table of immersions and embeddings of projective spaces*,
http://www.lehigh.edu/~ dmd1/immtable.

Alexander Dranishnikov,
*On topological complexity of twisted products*,
Topology and its Applications 179 (2015) 74âĂŞ80.

M. Farber.
*Topological complexity of motion planning*,
Discrete Comput. Geom. 29 (2003), 211-221..

M. Farber, S. Tabachnikov, and S. Yuzvinsky.
*Topological robotics: motion planning in projective spaces.*,
Int. Math. Res. Not.(2003), no. 34, 1853-1870..

# References


S. Gitler.
*Immersion and embedding of manifolds.*,
Proc.Symp.PureMath., vol.22 ,Amer.Math.Soc , Providence, 1971, pp.87-96.

J. Lörinc.
*The height of the first Stifel-Whitney class of any nonorientable real flag manifold.*,
Math. Slovaca, 53 (2003), No. 1, 91-95.

W. Lubawski, W. Marzantowicz,
*A new approach to the equivariant topological complexity*,
arXiv:1303.0171v2 [math.AT]..

Y. Félix and S. Halperin.
*Rational LS-category and its applications*. Trans. Amer. Math. Soc. 273 (1982) 1-37.

# References


- Y. Félix, S. Halperin and J.-C. Thomas.
  *Rational homotopy theory*. Graduate Texts in Mathematics 205, Springer-Verlag, 2001.
- Y. Félix, S. Halperin, and J. M. Lemaire.
  *The Rational LS-category of Products and Poincaré Duality Complexes*. Topology Vol. 37, No. 4, pp. 749-756, 1998.
- Y. Félix, S. Halperin, and J.-C. Thomas.
  *Gorenstein spaces*. Academic Press; New York and London Vol.71, No.1, September 1988
- I. M. James.
  *Lusternik-Schnirelmann category. Handbook of algebraic topology*. Journal of Elsevier 1995, 1293-1310.
- L. Lechuga and A. Murillo.
  *A formula for the rational LS-category of certain spaces*. Ann. L'inst. Fourier, 2002.

# References


A. Murillo.
*The top cohomology class of certain spaces*. Journal of Pure and Applied. Algebra, 84:209-214, 1993.

A. Murillo.
*The evaluation map of some Gorenstein algebras*. journal of Pure and Applied Algebra, 91:209-218, 1994.

L. Lechuga and A. Murillo.
*The fundamental class of a rational space, the graph coloring problem and other classical decision problems*. Bull. Belgian Math. Soc. 8 (2001),451-467.

L. Lechuga and A. Murillo.
*Complexity in rational homotopy*. Topology. 39 (2000),89-94.

D. Sullivan.
*Infinitesimal computations in topology*. Publ. Math. IHES (1978) 269-331.

Thank you