FAC. SCIENCES, MEKNES

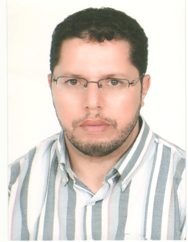
RHT-Seminar

Spectral Sequences I

7 JANUARY 2012

My Ismail Mamouni-CPGE-CPR Rabat

Professeur Agrégé-Docteur en Math
Master 1 en Sc de l'éducation, Univ. Rouen
mamouni.new.fr
mamouni.myismail@gmail.com



بِسمِ اللَّهِ الرَّحمَنِ الرَّحِيمِ

وَ قُلِ إِعمَلُوا فَسَيَرَى اللَّهُ عَمَلَكُم وَ رَسُولُهُ

وَ المُؤ مِنُون

صَدَقَ اللَّهُ العَظِيم

Introduction

- Area of application : homological algebra and algebraic topology,
- Mean Goal : Computing homology groups by taking successive approximations
- Spectral sequences are a generalization of exact sequences
- First introduction : Jean Leray in 1946

Motivation

- Jean Leray, faced with the problem of computing sheaf cohomology, introduced a computational technique now known as the Leray spectral sequence.
- The relation involved an infinite process.
- Leray found that the cohomology groups of the pushforward formed a natural chain complex, so that he could take the cohomology of the cohomology.
- This was still not the cohomology of the original sheaf, but it was one step closer in a sense. The cohomology of the cohomology again formed a chain complex, and its cohomology formed a chain complex, and so on.
- The limit of this infinite process was essentially the same as the cohomology groups of the original sheaf.

Usefulness

- Spectral sequences were found in diverse situations, and they gave intricate relationships among homology and cohomology groups coming from geometric situations such as fibrations and from algebraic situations involving derived functors.
- While their theoretical importance has decreased since the introduction of derived categories, they are still the most effective computational tool available.
- This is true even when many of the terms of the spectral sequence are incalculable.

Defects

- Because of the large amount of information carried in spectral sequences, they are difficult to grasp.
- This information is usually contained in a rank three lattice of abelian groups or modules.
- The easiest cases to deal with are those in which the spectral sequence eventually collapses, meaning that going out further in the sequence produces no new information.
- Even when this does not happen, it is often possible to get useful information from a spectral sequence by various tricks.

Main hurdle

- ⇝ The subject of spectral sequences has a reputation for being difficult for the beginner.
- ⇝ **G. W. Whitehead** : “The machinery of spectral sequences, stemming from the algebraic work of Lyndon and Koszul, seemed complicated and obscure to many topologists.”
- ⇝ **David Eisenbud** : “The subject of spectral sequences is elementary, but the notion of the spectral sequence of a double complex involves so many objects and indices that it seems at first repulsive.”

How to overcome

Timothy Y. Chow

- Spectral sequences have to be taught in a way that explains how one might have come up with the definition in the first place.
- Without an understanding of where spectral sequences come from, one naturally finds them mysterious.
- If one does see where they come from, the notation should not be a stumbling block.

Our main goal

To make

the ideas accessible to more than the lucky few who are able to have the right conversation with the right expert at the right time.

Formal definition

Definition :

Fix an abelian category, such as a category of modules over a ring. A spectral sequence is a choice of a collection of :

- Objects E_r, called "sheets", or sometimes "pages "or "terms",
- Endomorphisms $d_r : E_r \longrightarrow E_r$ satisfying $d_r^2 = 0$, called "boundary maps" or "differentials",
- Isomorphisms :

$$E_{r+1} = H_*(E_r, d_r)$$

General Assumptions

Throughout, we work over a field. All chain groups are finite-dimensional, and all filtrations (explained below) have only finitely many levels. In the “real world”, these assumptions may fail, but the essential ideas are easier to grasp in this simpler context.

Graded Complexes

Simple example

That is a chain complex

$$\cdots \xrightarrow{\partial} C_{d+1} \xrightarrow{\partial} C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots$$

where each C_d splits into a finite direct sum

$$C_d = \bigoplus_{p=1}^{n} C_{d,p}$$

and where moreover the boundary map ∂ respects the grading in the sense that

$$\partial C_{d,p} \subset C_{d-1,p}$$

This grading situation allows us to break up the computation of the homology into smaller pieces : simply compute the homology in each grade independently and then sum them all up to obtain the homology of the original complex.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
\cdots & \xrightarrow{\partial} & C_{d+1,p-1} & \xrightarrow{\partial} & C_{d,p-1} & \xrightarrow{\partial} & C_{d-1,p-1} & \xrightarrow{\partial} \cdots & H_{p-1}(C,\partial) \\
 & & C_{d+1,p} & & C_{d,p} & & C_{d-1,p} & & H_p(C,\partial) \\
\cdots & \xrightarrow{\partial} & C_{d+1,p+1} & \xrightarrow{\partial} & C_{d,p+1} & \xrightarrow{\partial} & C_{d-1,p+1} & \xrightarrow{\partial} \cdots & H_{p+1}(C,\partial) \\
 & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

This grading situation allows us to break up the computation of the homology into smaller pieces : simply compute the homology in each grade independently and then sum them all up to obtain the homology of the original complex.

$$\cdots \xrightarrow{\partial} \begin{array}{|c|}\hline \vdots \\ \hline C_{d+1,p-1} \\ \hline C_{d+1,p} \\ \hline C_{d+1,p+1} \\ \hline \vdots \\ \hline \end{array} \xrightarrow{\partial} \begin{array}{|c|}\hline \vdots \\ \hline C_{d,p-1} \\ \hline C_{d,p} \\ \hline C_{d,p+1} \\ \hline \vdots \\ \hline \end{array} \xrightarrow{\partial} \begin{array}{|c|}\hline \vdots \\ \hline C_{d-1,p-1} \\ \hline C_{d-1,p} \\ \hline C_{d-1,p+1} \\ \hline \vdots \\ \hline \end{array} \xrightarrow{\partial} \cdots \qquad \begin{array}{|c|}\hline \vdots \\ \hline H_{p-1}(C,\partial) \\ \hline H_p(C,\partial) \\ \hline H_{p+1}(C,\partial) \\ \hline \vdots \\ \hline \end{array}$$

Filtered Complex

Frequent situation

in practice it is not always so lucky as to have a grading on our complex, but we frequently have instead is a filtered complex. That is, when each C_d is equipped with a nested sequence of submodules

$$0 = \mathcal{F}_{d,0} \subset \mathcal{F}_{d,1} \subset \mathcal{F}_{d,2} \subset \cdots \subset \mathcal{F}_{d,n} = C_d$$

and where the boundary map respects the filtration in the sense that

$$\partial \mathcal{F}_{d,p} \subset \mathcal{F}_{d-1,p}$$

The index p is called the filtration degree.

Filtered Complex

Frequent situation

$\vdots$		$\vdots$		$\vdots$		$\vdots$
$\downarrow\partial$		$\downarrow\partial$		$\downarrow\partial$		$\downarrow\partial$
$0=\mathcal{F}_{d+1,0}$	$\subset$	$\mathcal{F}_{d+1,1}$	$\subset$	$\mathcal{F}_{d+1,2}$	$\subset\cdots\subset$	$\mathcal{F}_{d+1,n}=C_{d+1}$
$\downarrow\partial$		$\downarrow\partial$		$\downarrow\partial$		$\downarrow\partial$
$0=\mathcal{F}_{d,0}$	$\subset$	$\mathcal{F}_{d,1}$	$\subset$	$\mathcal{F}_{d,2}$	$\subset\cdots\subset$	$\mathcal{F}_{d,n}=C_{d}$
$\downarrow\partial$		$\downarrow\partial$		$\downarrow\partial$		$\downarrow\partial$
$0=\mathcal{F}_{d-1,0}$	$\subset$	$\mathcal{F}_{d-1,1}$	$\subset$	$\mathcal{F}_{d-1,2}$	$\subset\cdots\subset$	$\mathcal{F}_{d-1,n}=C_{d-1}$
$\downarrow\partial$		$\downarrow\partial$		$\downarrow\partial$		$\downarrow\partial$
$\vdots$		$\vdots$		$\vdots$		$\vdots$

A naturel question

- ⇝ Is there any possible connection between the two concepts : garded complex and filtred complex ?
- ⇝ is there a natural way to break up the homology groups of a filtered complex into a direct sum ?

A naturel question

The answer : **Yes**

A naturel question

The situation : **surprisingly complicated**

A naturel question

The anlysis leads to : **Spectral Sequences**

The machinery

Write

$$\begin{array}{rcl} V & = & U \oplus (V/U) \quad \text{where} \quad V = \mathcal{F}_{d,n} = C_d \quad U = \mathcal{F}_{d,n-1} \\ U & = & W \oplus (U/W) \quad \text{where} \quad U = \mathcal{F}_{d,n-1} \quad W = \mathcal{F}_{d,n-2} \end{array}$$

Put

$$E^0_{d,p} := \mathcal{F}_{d,p}/\mathcal{F}_{d,p-1}$$

Then

$$C_d = \bigoplus_{p=1}^{n} E^0_{d,p}$$

The machinery

Write

$$\begin{array}{lllll} V &=& U \oplus (V/U) & \text{where} & V = \mathcal{F}_{d,n} = C_d \quad U = \mathcal{F}_{d,n-1} \\ U &=& W \oplus (U/W) & \text{where} & U = \mathcal{F}_{d,n-1} \quad W = \mathcal{F}_{d,n-2} \end{array}$$

Put

$$E^0_{d,p} := \mathcal{F}_{d,p}/\mathcal{F}_{d,p-1}$$

Then

$$C_d = \bigoplus_{p=1}^{n} E^0_{d,p}$$

The machinery

Write

$$\begin{array}{rcl} V & = & U \oplus (V/U) \quad \text{where} \quad V = \mathcal{F}_{d,n} = C_d \quad U = \mathcal{F}_{d,n-1} \\ U & = & W \oplus (U/W) \quad \text{where} \quad U = \mathcal{F}_{d,n-1} \quad W = \mathcal{F}_{d,n-2} \end{array}$$

Put

$$E^0_{d,p} := \mathcal{F}_{d,p}/\mathcal{F}_{d,p-1}$$

Then

$$C_d = \bigoplus_{p=1}^{n} E^0_{d,p}$$

The first conversion is done

The nice thing about this direct sum decomposition is that the boundary map ∂ naturally induces a map

$$d_0 : C_d = \bigoplus_{p=1}^{n} E^0_{d,p} \longrightarrow C_{d-1} = \bigoplus_{p=1}^{n} E^0_{d-1,p}$$

such that

$$d_0(E^0_{d,p}) \subset E^0_{d-1,p}$$

The first conversion is done

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \xrightarrow{d_0} & E^0_{d+1,p-1} & \xrightarrow{d_0} & E^0_{d,p-1} & \xrightarrow{d_0} & E^0_{d-1,p-1} & \xrightarrow{d_0} & \cdots \\
 & & E^0_{d+1,p} & & E^0_{d,p} & & E^0_{d-1,p} & & \\
\cdots & \xrightarrow{d_0} & E^0_{d+1,p+1} & \xrightarrow{d_0} & E^0_{d,p+1} & \xrightarrow{d_0} & E^0_{d-1,p+1} & \xrightarrow{d_0} & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

The first term

Let

$$E^1_{d,p} := H_d(E^0_{d,p}, d_0) = \frac{\ker d_0 : E^0_{d,p} \longrightarrow E^0_{d-1,p}}{\operatorname{Im} d_0 : E^0_{d+1,p} \longrightarrow E^0_{d,p}}$$

The first complication

- We might hope that $\bigoplus_{p=1}^{n} H_d(E^0_{d,p}, d_0)$ is the homology of our original complex.
- Unfortunately, this is too simple to be true.
- Although $\bigoplus_{p=1}^{n} H_d(E^0_{d,p}, d_0)$ does indeed give the homology of the associated graded complex, it may not give the homology of the original complex.

Analyzing a simple situation

- The associated graded complex is so closely related to the original complex that even if its homology isn't exactly what we want, it ought to be a reasonably good approximation.
- To keep things as simple as possible, in order to see more clear, let us begin by considering the case $n = 2$

Analyzing a simple situation

Our array diagram has only two levels, which we shall call the "upstairs" ($p = 2$) and "downstairs" ($p = 1$) levels.

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_0} & E^0_{d+1,2} & \xrightarrow{d_0} & E^0_{d,2} & \xrightarrow{d_0} & E^0_{d-1,2} & \xrightarrow{d_0} & \cdots \\ \cdots & \xrightarrow{d_0} & E^0_{d+1,1} & \xrightarrow{d_0} & E^0_{d,1} & \xrightarrow{d_0} & E^0_{d-1,1} & \xrightarrow{d_0} & \cdots \end{array}$$

Analyzing a simple situation

- Recall that our main goal is to find a natural way of decomposing $H_d = H(C_d, \partial)$ into a direct sum.
- Write $H_d = Z_d/B_d$ where Z_d is the space of cycles in C_d and B_d that of boundaries
- Since C_d is filtered, so it is also and naturaly for Z_d and for B_d :

$$0 = Z_{d,0} \subset Z_{d,1} \subset Z_{d,2} = Z_d$$

and

$$0 = B_{d,0} \subset B_{d,1} \subset B_{d,2} = B_d$$

Analyzing a simple situation

Main idea

We use the same trick of modding out by the "downstairs part" and then direct summing with the "downstairs part" itself :

$$H_d = \frac{Z_d}{B_d} = \frac{Z_d + \mathcal{F}_{d,1}}{B_d + \mathcal{F}_{d,1}} \oplus \frac{Z_d \cap \mathcal{F}_{d,1}}{B_d \cap \mathcal{F}_{d,1}} = \frac{Z_{d,2} + \mathcal{F}_{d,1}}{B_{d,2} + \mathcal{F}_{d,1}} \oplus \frac{Z_{d,1}}{B_{d,1}}$$

Analyzing a simple situation

A naive hope

Because that the numerators and denominators in the expressions $\dfrac{Z_{d,2}+\mathcal{F}_{d,1}}{B_{d,2}+\mathcal{F}_{d,1}}$ and $\dfrac{Z_{d,1}}{B_{d,1}}$ are precisely the cycles and boundaries in the definition of $E^1_{d,p}$, can we claim that

$$E^1_{d,2} \overset{?}{\simeq} \frac{Z_{d,2}+\mathcal{F}_{d,1}}{B_{d,2}+\mathcal{F}_{d,1}} \quad , \quad E^1_{d,1} \overset{?}{\simeq} \frac{Z_{d,1}}{B_{d,1}}$$

Analyzing a simple situation

The answer :

Unfortunately, in general, the answer is negative

Analyzing a simple situation

The conclusion :

Some corrections are needed.

Let us look "downstairs"

Recall that

$$E^1_{d,p} := H_d(E^0_{d,p}, d_0) = \frac{\ker d_0 : E^0_{d,p} \longrightarrow E^0_{d-1,p}}{\operatorname{Im} d_0 : E^0_{d+1,p} \longrightarrow E^0_{d,p}}$$

- ⇝ The space of cycles in $E^1_{d,1}$ is that of cycles in $E^0_{d,1}$, which is $Z_{d,1}$
- ⇝ The space of "boundaries" in $E^1_{d,1}$ is the image I of the map $d_0 : E^0_{d+1,1} \longrightarrow E^0_{d,1}$.
- ⇝ However, this image I is not $B_{d,1}$, because that $B_{d,1}$, which is the part of B_d that lies in $\mathcal{F}_{d,1}$, contains I and maybe many other things.
- ⇝ The map ∂ may carry some elements of C_{d+1} down from "upstairs" to "downstairs," whereas I only captures boundaries of elements that were already downstairs

Conclusion $Z_{d,1}/B_{d,1}$ is a quotient of $E^1_{d,1}$.

Let us look "downstairs"

Recall that

$$E^1_{d,p} := H_d(E^0_{d,p}, d_0) = \frac{\ker d_0 : E^0_{d,p} \longrightarrow E^0_{d-1,p}}{\operatorname{Im} d_0 : E^0_{d+1,p} \longrightarrow E^0_{d,p}}$$

- $\rightsquigarrow$ The space of cycles in $E^1_{d,1}$ is that of cycles in $E^0_{d,1}$, which is $Z_{d,1}$
- $\rightsquigarrow$ The space of "boundaries" in $E^1_{d,1}$ is the image I of the map $d_0 : E^0_{d+1,1} \longrightarrow E^0_{d,1}$.
- $\rightsquigarrow$ However, this image I is not $B_{d,1}$, because that $B_{d,1}$, which is the part of B_d that lies in $\mathcal{F}_{d,1}$, contains I and maybe many other things.
- $\rightsquigarrow$ The map ∂ may carry some elements of C_{d+1} down from "upstairs" to "downstairs," whereas I only captures boundaries of elements that were already downstairs

Conclusion $Z_{d,1}/B_{d,1}$ is a quotient of $E^1_{d,1}$.

Let us look “downstairs”

Recall that

$$E^1_{d,p} := H_d(E^0_{d,p}, d_0) = \frac{\ker d_0 : E^0_{d,p} \longrightarrow E^0_{d-1,p}}{\mathrm{Im} d_0 : E^0_{d+1,p} \longrightarrow E^0_{d,p}}$$

- ⇝ The space of cycles in $E^1_{d,1}$ is that of cycles in $E^0_{d,1}$, which is $Z_{d,1}$
- ⇝ The space of “boundaries” in $E^1_{d,1}$ is the image I of the map $d_0 : E^0_{d+1,1} \longrightarrow E^0_{d,1}$.
- ⇝ However, this image I is not $B_{d,1}$, because that $B_{d,1}$, which is the part of B_d that lies in $\mathcal{F}_{d,1}$, contains I and maybe many other things.
- ⇝ The map ∂ may carry some elements of C_{d+1} down from “upstairs” to “downstairs,” whereas I only captures boundaries of elements that were already downstairs

Conclusion $Z_{d,1}/B_{d,1}$ is a quotient of $E^1_{d,1}$.

Let us look "downstairs"

Recall that

$$E^1_{d,p} := H_d(E^0_{d,p}, d_0) = \frac{\operatorname{ker} d_0 : E^0_{d,p} \longrightarrow E^0_{d-1,p}}{\operatorname{Im} d_0 : E^0_{d+1,p} \longrightarrow E^0_{d,p}}$$

- ⇝ The space of cycles in $E^1_{d,1}$ is that of cycles in $E^0_{d,1}$, which is $Z_{d,1}$
- ⇝ The space of "boundaries" in $E^1_{d,1}$ is the image I of the map $d_0 : E^0_{d+1,1} \longrightarrow E^0_{d,1}$.
- ⇝ However, this image I is not $B_{d,1}$, because that $B_{d,1}$, which is the part of B_d that lies in $\mathcal{F}_{d,1}$, contains I and maybe many other things.
- ⇝ The map ∂ may carry some elements of C_{d+1} down from "upstairs" to "downstairs," whereas I only captures boundaries of elements that were already downstairs

Conclusion $Z_{d,1}/B_{d,1}$ is a quotient of $E^1_{d,1}$.

Let us look “downstairs”

Recall that

$$E^1_{d,p} := H_d(E^0_{d,p}, d_0) = \frac{\ker d_0 : E^0_{d,p} \longrightarrow E^0_{d-1,p}}{\operatorname{Im} d_0 : E^0_{d+1,p} \longrightarrow E^0_{d,p}}$$

- ⇝ The space of cycles in $E^1_{d,1}$ is that of cycles in $E^0_{d,1}$, which is $Z_{d,1}$
- ⇝ The space of “boundaries” in $E^1_{d,1}$ is the image I of the map $d_0 : E^0_{d+1,1} \longrightarrow E^0_{d,1}$.
- ⇝ However, this image I is not $B_{d,1}$, because that $B_{d,1}$, which is the part of B_d that lies in $\mathcal{F}_{d,1}$, contains I and maybe many other things.
- ⇝ The map ∂ may carry some elements of C_{d+1} down from “upstairs” to “downstairs,” whereas I only captures boundaries of elements that were already downstairs

Conclusion $Z_{d,1}/B_{d,1}$ is a quotient of $E^1_{d,1}$.

Let us look "downstairs"

Recall that

$$E^1_{d,p} := H_d(E^0_{d,p}, d_0) = \frac{\ker d_0 : E^0_{d,p} \longrightarrow E^0_{d-1,p}}{\operatorname{Im} d_0 : E^0_{d+1,p} \longrightarrow E^0_{d,p}}$$

- ⇝ The space of cycles in $E^1_{d,1}$ is that of cycles in $E^0_{d,1}$, which is $Z_{d,1}$
- ⇝ The space of "boundaries" in $E^1_{d,1}$ is the image I of the map $d_0 : E^0_{d+1,1} \longrightarrow E^0_{d,1}$.
- ⇝ However, this image I is not $B_{d,1}$, because that $B_{d,1}$, which is the part of B_d that lies in $\mathcal{F}_{d,1}$, contains I and maybe many other things.
- ⇝ The map ∂ may carry some elements of C_{d+1} down from "upstairs" to "downstairs," whereas I only captures boundaries of elements that were already downstairs

Conclusion $Z_{d,1}/B_{d,1}$ is a quotient of $E^1_{d,1}$.

Let us look upstairs

- Recall our last question : $E^1_{d,2} \overset{?}{\simeq} \dfrac{Z_{d,2}+\mathcal{F}_{d,1}}{B_{d,2}+\mathcal{F}_{d,1}}$ and our definition

$$E^1_{d,p} := H_d(E^0_{d,p}, d_0) = \frac{\ker d_0 : E^0_{d,p} \longrightarrow E^0_{d-1,p}}{\mathrm{Im} d_0 : E^0_{d+1,p} \longrightarrow E^0_{d,p}} ;$$

- The space of boundaries of $E^1_{d,2}$ is $B_{d,2}+\mathcal{F}_{d,1}$;
- The space of cycles is the kernel K of the map

$$d_0 : E^0_{d,2} = \frac{\mathcal{F}_{d,2}}{\mathcal{F}_{d,1}} \longrightarrow E^0_{d-1,2} = \frac{\mathcal{F}_{d-1,2}}{\mathcal{F}_{d-1,1}}$$

- Thus, K contains chains that ∂ sends to zero but also chains that ∂ sends downstairs to $\mathcal{F}_{d-1,1}$.

Let us look upstairs

- Recall our last question : $E^1_{d,2} \overset{?}{\simeq} \dfrac{Z_{d,2} + \mathcal{F}_{d,1}}{B_{d,2} + \mathcal{F}_{d,1}}$ and our definition
$$E^1_{d,p} := H_d(E^0_{d,p}, d_0) = \frac{\ker d_0 : E^0_{d,p} \longrightarrow E^0_{d-1,p}}{\mathrm{Im} d_0 : E^0_{d+1,p} \longrightarrow E^0_{d,p}} ;$$
- The space of boundaries of $E^1_{d,2}$ is $B_{d,2} + \mathcal{F}_{d,1}$;
- The space of cycles is the kernel K of the map
$$d_0 : E^0_{d,2} = \frac{\mathcal{F}_{d,2}}{\mathcal{F}_{d,1}} \longrightarrow E^0_{d-1,2} = \frac{\mathcal{F}_{d-1,2}}{\mathcal{F}_{d-1,1}}$$
- Thus, K contains chains that ∂ sends to zero but also chains that ∂ sends downstairs to $\mathcal{F}_{d-1,1}$.

Let us look upstairs

- Recall our last question : $E^1_{d,2} \overset{?}{\simeq} \dfrac{Z_{d,2} + \mathcal{F}_{d,1}}{B_{d,2} + \mathcal{F}_{d,1}}$ and our definition
$$E^1_{d,p} := H_d(E^0_{d,p}, d_0) = \frac{\ker d_0 : E^0_{d,p} \longrightarrow E^0_{d-1,p}}{\operatorname{Im} d_0 : E^0_{d+1,p} \longrightarrow E^0_{d,p}};$$
- The space of boundaries of $E^1_{d,2}$ is $B_{d,2} + \mathcal{F}_{d,1}$;
- The space of cycles is the kernel K of the map
$$d_0 : E^0_{d,2} = \frac{\mathcal{F}_{d,2}}{\mathcal{F}_{d,1}} \longrightarrow E^0_{d-1,2} = \frac{\mathcal{F}_{d-1,2}}{\mathcal{F}_{d-1,1}}$$
- Thus, K contains chains that ∂ sends to zero but also chains that ∂ sends downstairs to $\mathcal{F}_{d-1,1}$.

Let us look upstairs

- Recall our last question : $E^1_{d,2} \overset{?}{\simeq} \dfrac{Z_{d,2} + \mathcal{F}_{d,1}}{B_{d,2} + \mathcal{F}_{d,1}}$ and our definition
$$E^1_{d,p} := H_d(E^0_{d,p}, d_0) = \frac{\ker d_0 : E^0_{d,p} \longrightarrow E^0_{d-1,p}}{\operatorname{Im} d_0 : E^0_{d+1,p} \longrightarrow E^0_{d,p}};$$
- The space of boundaries of $E^1_{d,2}$ is $B_{d,2} + \mathcal{F}_{d,1}$;
- The space of cycles is the kernel K of the map

$$d_0 : E^0_{d,2} = \frac{\mathcal{F}_{d,2}}{\mathcal{F}_{d,1}} \longrightarrow E^0_{d-1,2} = \frac{\mathcal{F}_{d-1,2}}{\mathcal{F}_{d-1,1}}$$

- Thus, K contains chains that ∂ sends to zero but also chains that ∂ sends downstairs to $\mathcal{F}_{d-1,1}$.

Still look upstairs

- In contrast, the elements of $Z_{d,2} + \mathcal{F}_{d,1}$ are more special : their boundaries are boundaries of chains that come from $\mathcal{F}_{d,1}$.
- Hence $\dfrac{Z_{d,2} + \mathcal{F}_{d,1}}{B_{d,2} + \mathcal{F}_{d,1}}$ is a subspace of $E^1_{d,2}$

Still look upstairs

- In contrast, the elements of $Z_{d,2} + \mathcal{F}_{d,1}$ are more special : their boundaries are boundaries of chains that come from $\mathcal{F}_{d,1}$.
- Hence $\dfrac{Z_{d,2} + \mathcal{F}_{d,1}}{B_{d,2} + \mathcal{F}_{d,1}}$ is a subspace of $E^1_{d,2}$

An intuitive Conclusion

The problem

- In the associated graded complex, we only see activity that is confined to a single horizontal level ; everything above and below that level is chopped off.
- In the original complex, the boundary map ∂ may carry things down one or more levels (it cannot carry things up one or more levels because that ∂ respects the filtration), and one must therefore correct for this inter-level activity.

The Emergence of Spectral Sequences

Claim 1

∂ induces a natural map, $d_1 : E^1_{d+1,2} \longrightarrow E^1_{d,1}$, beacuse that the boundary of any element in $E^1_{d+1,2}$ is a cycle that lies in $\mathcal{F}_{d,1}$, and thus it defines an element of $E^1_{d,1}$.

♣

The Emergence of Spectral Sequences

Claim II

If we mod $E^1_{d,1}$ by the image of d_1, then we obtain $\dfrac{Z_{d,1}}{B_{d,1}}$, because that the image of d_1 gives all the boundaries that lie in $\mathcal{F}_{d,1}$.

The Emergence of Spectral Sequences

Claim III

The kernel of d_1 is a subspace of $E^1_{d,2}$ isomorphic to $\dfrac{Z_{d+1,2} + \mathcal{F}_{d+1,1}}{B_{d+1,2} + \mathcal{F}_{d+1,1}}$

$$
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & \cdots \\
& \searrow^{d_1} & & \searrow^{d_1} & & \searrow^{d_1} & & \searrow^{d_1} & \\
\cdots & & E^1_{d+1,2} & & E^1_{d,2} & & E^1_{d-1,2} & & \cdots \\
& \searrow^{d_1} & & \searrow^{d_1} & & \searrow^{d_1} & & \searrow^{d_1} & \\
\cdots & & E^1_{d+1,1} & & E^1_{d,1} & & E^1_{d-1,1} & & \cdots \\
& \searrow^{d_1} & & \searrow^{d_1} & & \searrow^{d_1} & & \searrow^{d_1} & \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

Finaly

Put

$$E^2_{d,p} := H_d(E^1_{d,p}) = \frac{\ker d_1 : E^1_{d,p} \longrightarrow E^1_{d-1,p-1}}{\operatorname{Im} d_1 : E^1_{d+1,p+1} \longrightarrow E^1_{d,p}}$$

Due to claims II and III we conclude that $E^2_{d,1} \oplus E^2_{d,2}$ is (finally !) the correct homology of our original filtered complex.

Finaly

Put

$$E^2_{d,p} := H_d(E^1_{d,p}) = \frac{\ker d_1 : E^1_{d,p} \longrightarrow E^1_{d-1,p-1}}{\mathrm{Im} d_1 : E^1_{d+1,p+1} \longrightarrow E^1_{d,p}}$$

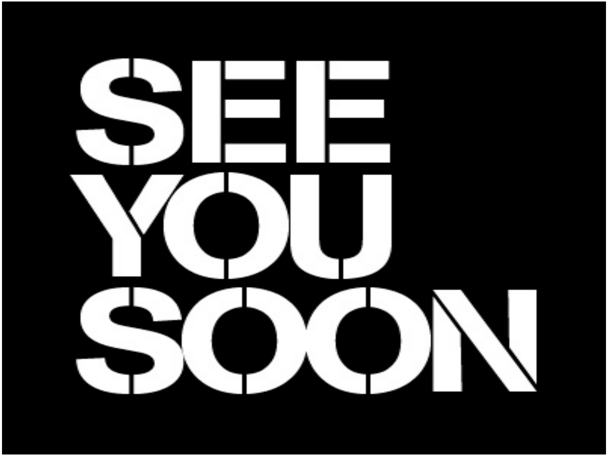
Due to claims II and III we conclude that $E^2_{d,1} \oplus E^2_{d,2}$ is (finally !) the correct homology of our original filtered complex.

Succes stroy

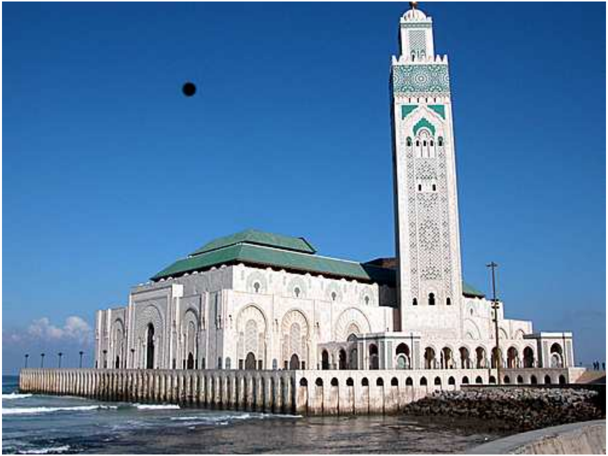
n = 2

- The sequence of terms E_0, E_1, E_2 is the spectral sequence of our filtered complex when $n = 2$.
- E^1 is a first-order approximation of the desired homology,
- E^2 is theoretically a second-order approximation, when $n = 2$, it is not just an approximation but the true answer.

What if n > 2 ?



What if n > 2 ?



Casablanca, 12 Février 2012

References

- T. Y. Chow, You Could Have Invented Spectral Sequences, Notices of the AMS, Jan 2006, 15-19.
- A. Hatcher, Spectral Sequences in Algebraic Topology,http ://www.math.cornell.edu/ hatcher/SSAT/SSATpage.html
- J. McCleary, A User's Guide to Spectral Sequences, Cambridge Studies in Advanced Mathematics, 58 (2001, 2nd ed.), Cambridge University Press, doi :10.2277/0521567599, ISBN 978-0-521-56759-6, MR1793722
- J. McCleary, A history of spectral sequences : Origins to 1953, in History of Topology, edited by Ioan M. James, North Holland, 1999, 631–663.
- B. Mitchell, Spectral sequences for the layman, Amer. Math. Monthly 76 (1969), 599–605.


köszönöm
!תודה
děkuji
mahalo
고맙습니다
thank you
merci
谢谢
danke
Ευχαριστώ
شكرا
どうもありがとう
gracias