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Arbeitsgemeinschaft: Rational Homotopy Theory in Mathematics and Physics

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Abstract. This Arbeitsgemeinschaft focused on the interplay among rational homotopy theory, differential geometry and the physics of string topology. The talks centered on one hand on how geometry and string topology make use of rational homotopy methods, elicit new questions in rational homotopy and lead to the development of new rational homotopy structures reflecting their Natures; and on the other hand on how rational homotopy theory has given concrete results in geometry.

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Introduction by the Organisers

In [9], Sullivan defined tools and models for rational homotopy inspired by already existing geometrical objects. Moreover, he gave an explicit dictionary between his minimal models and spaces, and this facility of transition between algebra and topology has created many new topological and geometrical theorems in the last 30 years.

When de Rham proved that $H^*(A_{DR}(M)) \cong H^*(M;\mathbb{R})$ for the differential algebra of differential forms $A_{DR}(M)$ on a manifold M, it immediately provided a link between the analysis on and the topology of the manifold. Sullivan suggested that even within the world of topology, there is more topological information in the de Rham algebra of M than simply the real cohomology.

In the de Rham algebra, there is information contained in two different entities: the product of forms, which tells us how two forms can be combined together to give a third one and the exterior derivative of a form. In a model, we kill the information coming from the product structure by considering free algebras $\wedge V$ (in the commutative graded sense) where V is an $\mathbb{R}$-vector space. This pushes the corresponding information into the differential and into V where it is easier to detect. More precisely, we look for a cdga (for *commutative differential graded algebra*) free as a commutative graded algebra $(\wedge V, d)$ and a morphism $\varphi\colon (\wedge V, d) \to A_{DR}(M)$ inducing an isomorphism in cohomology.

For instance, if G is a compact connected Lie group, there exists a sub-differential algebra of bi-invariant forms, $\Omega_I(G)$, inside the de Rham algebra $A_{DR}(G)$, such that the canonical inclusion $\Omega_I(G) \hookrightarrow A_{DR}(G)$ induces an isomorphism in cohomology. This is the prototype of the process for models: namely, we look for a simplification $\mathcal{M}_M$ of the de Rham algebra with an explicit differential morphism $\mathcal{M}_M \to A_{DR}(M)$ inducing an isomorphism in cohomology, exactly as bi-invariant forms do in the case of a compact connected Lie group.

The first question is, can one build such a model for any manifold? The answer is yes for connected manifolds and in fact, there are many ways to do this. So, we describe a standard way, which is called *minimal*, and which is defined by requiring that the differentials of elements of V have no linear terms. Once we have this *minimal model* (which is unique up to isomorphism), we may ask what geometrical invariants can be detected in it. In fact, there is a functor from algebra to geometry that, together with forms, creates a dictionary between the algebraic and the geometrical worlds. *But for this we have to work over the rationals and not over the reals.* As a consequence, we have to replace the de Rham algebra by other types of forms. This new construction is very similar to the de Rham algebra and allows the extension of the usual theory from manifolds to simplicial sets (or topological spaces), which is a great advantage. Denote by $A_{PL}(X)$ this analogue of the de Rham algebra for a simplicial set X. Since the minimal model construction also works perfectly well over $\mathbb{Q}$, we have the notion of a minimal model $\mathcal{M}_X \to A_{PL}(X)$ of a path connected space X.

Conversely, from a cdga (A, d) we have a topological realization $\langle (A, d) \rangle$ and if we apply this realization to a minimal model $\mathcal{M}_X$ of a space X (which is nilpotent with finite Betti numbers), then we get a continuous map $X \to \langle \mathcal{M}_X \rangle$ which induces an isomorphism in rational cohomology. The space $\langle \mathcal{M}_X \rangle$ is what, in homotopy theory, is called a rationalization of X. What must be emphasized in this process is the ability to create topological realizations of any algebraic constructions. Hence, Sullivan's theory can be seen as a rational version of classical differential geometry.

Such theories beg for applications and examples and it is possible to describe models for spheres, homogeneous spaces, biquotients, nilmanifolds, symplectic blow-ups and the free loop space. These models have geometrical applications, for instance, to complex and symplectic manifolds, the closed geodesic problem, curvature questions, actions of tori and, more recently, the Chas-Sullivan loop product. The focus of this Arbeitsgemeinschaft was the relationship between Rational

Homotopy Theory and Geometry with a natural extension to Physics via string topology. Several monographs are devoted to these theories: [1, 4, 5, 6, 10, 11].

The Arbeitsgemeinschaft consisted of 18 talks whose overriding goal was to tie together problems in geometry with rational homotopy theory. For instance, the question of the existence of infinitely many geodesics on a closed manifold was shown to be intimately tied up with the rational homotopy of the free loop space. Similarly, the rational homotopy qualities of a space were shown to depend on special properties of the geodesic flow on the manifold (considered as a Hamiltonian system). Rational homotopy was also shown to be important to understanding the difference between Kähler and symplectic manifolds as well as a key ingredient in treating certain "rational" problems about sectional curvature. (References for these items will be given in the following abstracts.) The Arbeitsgemeinschaft was attended by about 50 people, including many young mathematicians who gave excellent talks. No one was lost on the traditional hike.

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Arbeitsgemeinschaft: Rational Homotopy Theory in Mathematics and Physics

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Abstracts

Fundamentals of Geometry

Wilderich Tuschmann

After an introduction to curvature quantities in Riemannian geometry and stating some classical results about the topology of nonnegatively and nonpositively curved manifolds, I discussed the principal Riemannian finiteness theorems and known obstructions to the existence of (almost) nonnegatively curved metrics on closed manifold along with their respective interactions with (rational) homotopy theory, as well as the following open problems in the field:

(1) Conjecture (Bott, extended by Grove): Are almost nonnegatively curved manifolds rationally elliptic?
(2) Question: Do almost nonnegatively curved manifolds admit Riemannian metrics with vanishing topological entropy?
(3) Conjecture (Hopf): Do nonnegatively curved manifolds have nonnegative Euler characteristic?
(4) Question: Are positively curved manifolds formal?

Sullivan Models

Maura Macrì

This talk is an introduction to Sullivan (minimal) models focusing on their applications to geometrical examples used in later expositions: Lie groups, homogeneous spaces, principal G-bundles and biquotients. For more details we refer to [3, 4, 8].

To define Sullivan models we consider a particular algebraic object called *commutative differential graded algebra* (cdga for short), that is a graded vector space together with an associative multiplication with unit and commutative in the graded sense, endowed with a differential d ($d^2 = 0$) which is a derivation.
A basic example of cdga is the de Rham algebra of a differentiable manifold.

A cdga ΛV is *free commutative* if $\Lambda V = TV/I$ where TV is the tensor algebra over the graded vector space V and I is the bilateral ideal generated by elements $a \otimes b - (-1)^{|a||b|} b \otimes a$. A *Sullivan* cdga is a free commutative cdga ΛV such that V admits a basis $\{x_\alpha\}$ indexed by a well-ordered set, with $dx_\alpha \in \Lambda(x_\beta)_{\beta<\alpha}$. If the Sullivan cgda satisfies the additional property that the differential dx_α is a polynomial in generators x_β with no linear part, then it is called *minimal*.
Given a cdga A, a *Sullivan (minimal) model* for A is a Sullivan (minimal) cdga ΛV toghether with a quasi-isomorphism from ΛV to A.

An explicative example is the model of the de Rham algebra of the sphere S^m: the requirement of commutativity in the above definitions implies that this model has only one generator when m in odd and two generators when it is even.

Minimal models are important because they describe the rational part of homotopy groups: given a path connected space X we can always associate a rational cdga $A_{PL}(X)$ whose cohomology is isomorphic to the rational cohomology of X. If the space X is nilpotent and ΛV is the minimal model of $A_{PL}(X)$, then $V^n \simeq \mathrm{Hom}(\pi_n(X), \mathbb{Q})$, for any $n > 1$.

Even if Sullivan models are related to homotopy theory, they have several applications also in different areas. For this reason it is important to give examples of particular constructions of models.

Lie groups. Using the Hopf theorem we can easily find that the minimal model of a Lie group G is just its cohomology algebra $H^*(G) = \Lambda(x_{2p_1+1}, ..., x_{2p_r+1})$. Note that the generators have odd degree and, in fact, the number of generators is the rank of the group.

Homogeneous spaces and principal G-bundles. To construct these models we need to consider models of fibrations: given a quasi-nilpotent fibration $F \hookrightarrow E \to B$, if $(\Lambda V, d)$ is the minimal model of the base space, then $(\Lambda V \otimes \Lambda W, d)$ is a Sullivan model for the total space and the quotient

$$(\Lambda W, \bar{d}) := (\Lambda V \otimes \Lambda W, d)/(\Lambda^+ V \otimes \Lambda W)$$

is the minimal model for the fiber.
In the particular case of universal G-bundle the model becomes

$$(\Lambda V, 0) \to (\Lambda V \otimes \Lambda sV, d) \to (\Lambda sV, 0),$$

where $d(sv) = v$.
Using this model we can prove that if H is a closed connected subgroup of a compact connected Lie group G, $i : H \hookrightarrow G$ is the inclusion and $Bi : B_H \hookrightarrow B_G$ is the map induced on classifying spaces (whose cohomology algebras are $H^*(B_G, \mathbb{Q}) = \Lambda V$ and $H^*(B_H, \mathbb{Q}) = \Lambda W$ respectively), then a Sullivan model for the homogeneous space G/H is $(\Lambda W \otimes \Lambda sV, d)$ where $d(w) = 0$ and $d(sv) = H^*(Bi)(v)$.
Furthermore given a principal G-bundle $E \to B$ if $(\Lambda V, 0)$ is the minimal model of the Lie group G and $(\Lambda W, d)$ is that of the base space B, then the model of the total space E is $(\Lambda W \otimes \Lambda V, d)$.

Biquotients. A compact manifold is called a *biquotient* if it is diffeomorphic to a double quotient $K \setminus G/H$ with respect to a right action of a closed subgroups H of G and a free left action of a closed subgroup K of G on G/H [1].

To define the minimal model of a biquotient $K \setminus G/H$ we need to consider the following pullback fibration diagram

$$\begin{array}{ccc} K \setminus G/H & \longrightarrow & B_H \\ \downarrow & & \downarrow {\scriptstyle Bi_H} \\ B_K & \xrightarrow[Bi_K]{} & B_G \end{array}$$

This diagram allows us to consider a biquotient as the total space of a pullback fibration, so we can prove that if the cohomology algebras of the classifying spaces are respectively $H^*(B_G, \mathbb{Q}) = \Lambda V$, $H^*(B_H, \mathbb{Q}) = \Lambda W_H$ and $H^*(B_K, \mathbb{Q}) = \Lambda W_K$, then a Sullivan model for the biquotient is $(\Lambda W_H \otimes \Lambda W_K \otimes \Lambda sV, d)$ (cf. [2, 7]).

Nilmanifolds. A *nilmanifold* is a compact quotient $N = G/\Gamma$, where G is a real simply-connected nilpotent Lie group and Γ is a discrete cocompact subgroup of G. The cohomology of N, and hence its minimal model, depends only on the Lie algebra $\mathfrak{g}$ of the Lie group G. Indeed the Nomizu theorem states that the inclusion $\Lambda\mathfrak{g}^* \hookrightarrow A_{DR}(N)$ is a quasi-isomorphism [6].
The nilpotency of G implies that there is a basis $\{\omega_1, \ldots, \omega_n\}$ of $\mathfrak{g}^*$ such that $d(\omega_p) = \sum_{i<j<p} a^p_{ij}\omega_i \wedge \omega_j$. Since this basis satisfies the property of minimality, $(\Lambda(\omega_1, \ldots, \omega_n), d)$ is the model of $\mathfrak{g}^*$ and hence of the nilmanifold N [5].

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Group Actions

Marc Stephan

The aim of this talk is to study various interactions between G-spaces, i.e. topological spaces equipped with a continuous action by a topological group G, rational homotopy theory and geometry. Concretely, we'll discuss the toral rank of a space, the Borel construction, equivariant minimal models for finite group actions and Hamiltonian actions in symplectic geometry. The main reference is [1].

1. Toral rank

An early interaction between rational homotopy theory and geometry concerns almost free actions by a k-dimensional torus T^k. Recall that a group action on a space X is called *almost free* if for every point $x \in X$, the subgroup fixing this point is finite. For instance still open is Halperin's Toral rank conjecture:

Conjecture 1. *Let X be a nilpotent finite CW complex with an almost free T^k-action, then* $\dim H^*(X;\mathbb{Q}) \geq 2^k$.

The *rational toral rank* $rk_0(X)$ of a space X is the maximal k such that there exists a finite CW complex in the rational homotopy type of X, which admits an almost free T^k-action. For instance the rational toral rank of a compact connected Lie group G equals the dimension of its maximal torus. This calculation is a corollary of the following inequality.

Theorem 1 (Allday, Halperin (cf. [2], [1, Theorem 7.13])). *Let X be a rationally elliptic space with minimal model $(\Lambda V, d)$. Then $rk_0(X) \leq -\chi_\pi(X)$, where $\chi_\pi(X)$ denotes the* homotopy Euler characteristic $\dim V^{even} - \dim V^{odd}$ *of X.*

The proof of Theorem 1 involves the Borel construction defined next.

2. Borel construction

Let G be a topological group. The *Borel construction* of a G-space X is (the total space $X_G \coloneqq EG \times_G X$ of) the fiber bundle $EG \times_G X \to BG$ associated to the universal principal G-bundle $EG \to BG$. The equivariant cohomology of a G-space, i.e. the cohomology of its Borel construction, is related to almost free actions by Hsiang's Theorem:

Theorem 2 (Hsiang (cf. [3], [2, Proposition 1])). *Suppose a compact connected Lie group G acts almost freely on a connected finite CW complex X. Then $H^*(X_G;\mathbb{Q})$ is finite dimensional.*

In the talk, the proof of Theorem 1 will be sketched using Theorem 2.

3. Equivariant minimal models

As background material for talk 5 "Geodesics and the Free Loop Space II", equivariant minimal models for finite discrete group actions will be introduced following [1, p. 123-124].

4. Hamiltonian actions in symplectic geometry

After the "disconnected" excursion in section 3, we'll conclude "smoothly" with Hamiltonian actions by a torus T^k on a closed symplectic manifold M and the following consequence of a Hamiltonian action. For any field $\Bbbk$ of characteristic 0, the equivariant cohomology $H^*(M_{T^k};\Bbbk)$ of M is isomorphic to the tensor product of $H^*(M;\Bbbk)$ and $H^*(BT^k;\Bbbk)$ as $\Bbbk$-vector spaces (cf. [4, Theorem 3.4]).

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Geodesics and the Free Loop Space I

Stephan Wiesendorf

Geodesics are the generalization of straight lines in curved spaces. As it is well known, given two points x and y in the Euclidean space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ there is a unique path $\gamma : [0,1] \longrightarrow \mathbb{R}^n$ of minimal length joining them, namely the straight line $\gamma(t) = x + t(y - x)$, which is uniquely determined by the vanishing of its acceleration, i.e. $\frac{d}{dt}\dot{\gamma} \equiv 0$. This concept carries over to the general case of a Riemannian manifold (M, g) with Levi-Civita connection ∇, where a smooth path $\gamma : [a, b] \longrightarrow M$ is said to be a *geodesic* if its intrinsic acceleration vanishes, i.e. $\nabla_{\partial_t}\dot{\gamma} \equiv 0$. It turns out that this definition is well adapted in the sense that locally the picture is the same as in the Euclidean space. Indeed, geodesics are exactly those paths of constant speed, which locally minimize the distance between their points and given two points in M sufficiently close to each other, there is up to parameterization a unique geodesic joining them.

It is a very classical problem in differential geometry to estimate the number of geometrically distinct periodic geodesics on a complete Riemannian manifold (M, g), where two periodic geodesics $\gamma_1, \gamma_2 : \mathbb{R} \longrightarrow M$ are called geometrically distinct if $\gamma_1(\mathbb{R}) \neq \gamma_2(\mathbb{R})$. If the manifold M is compact, there are many results available and this talk deals with three of the most important ones. In chronological order the first common result in this direction is the theorem of Hadamard from 1898 [3], which is sometimes also referred to Cartan, that states that any non-trivial conjugacy class of the fundamental group $\pi_1(M)$ of a compact Riemannian manifold (M, g) contains a closed geodesic, i.e. the restriction of a periodic geodesic to a period, that is the shortest curve in this class. In the talk we will sketch the elementary proof of Hadamard before we skip to a variational approach and prove the theorem again using Morse-theoretical methods.

The most natural and successful way of thinking about closed geodesics is to consider them as critical points of the energy functional on the *free loop space* ΛM, which is the space of absolutely continuous paths $c : S^1 \cong [0,1]/\partial([0,1]) \longrightarrow M$, such that the energy functional $E : \Lambda M \longrightarrow \mathbb{R}$, given by $E(c) = \int_0^1 \|\dot{c}\|^2 dt$, is finite. Roughly speaking this is just the largest space on which the energy make sense. The space ΛM carries the structure of a Hilbert manifold, such that:

- ΛM is homotopy equivalent to the space M^{S^1} of continuous maps $S^1 \longrightarrow M$ with the compact open topology,
- $E : \Lambda M \longrightarrow \mathbb{R}$ is smooth and the critical points are exactly the closed geodesics,
- E satisfies Condition (C) of Palais and Smale [8] if M is compact.

Using the generalization of the theory of Morse [7], [6] to Hilbert manifolds [8] one can therefore apply critical point theory to detect closed geodesics as critical points of the energy E. The utility of this point is reflected for instance by the fact, that then it is clear that E assumes its minimum on any connected component, what is just the statement of the theorem of Hadamard stated above. The most important advantage of this approach is perhaps that dealing with closed geodesics becomes more intuitive. If one think about an elastic strip placed in a manifold, the strip will be tightened until it cannot be tightened any more and in that case it describes a geodesic loop. Conceptually this is exactly what the flow of the negative gradient of the energy functional does. Of course, this principle would produce only constant paths if there were no topology to pull against. But in the case of a simply connected compact manifold there are always topological constraints to pull against as one can deduce from the long exact homotopy sequence of the fibration $\Lambda M \longrightarrow M$, $c \mapsto c(0)$, with fiber homotopy equivalent to $(M,p)^{(S^1,*)}$ for a fixed point $p \in M$. Since the fibration admits a section $M \longrightarrow \Lambda M$, that maps a point p to the constant path $c_p \equiv p$, the sequence splits, i.e. $\pi_k(\Lambda M) \cong \pi_k((M,p)^{(S^1,*)}) \oplus \pi_k(M) \cong \pi_{k+1}(M) \oplus \pi_k(M)$. By Poincaré duality and the Hurewicz theorem these groups cannot all be trivial. Thus, applying the negative gradient flow to a non-trivial homotopy class in M of minimal dimension, regarded as a class in ΛM, this class will *hang up* at a non-trivial closed geodesic. This proves the second theorem of the talk, which is due to Lyusternik and Schnirelmann in 1930 [5] and says that every compact simply connected Riemannian manifolds admits at least one non-trivial closed geodesic.

Probably the most beautiful and important result in this direction is the theorem of Gromoll and Meyer from 1969 [2] due to which there are infinitely many geometrically distinct periodic geodesics on every compact simply connected Riemannian manifold M for which the sequence $\{b_k(\Lambda M)\}_k$ of Betti numbers with respect to a field of characteristic zero is not bounded. The proof of this theorem requires results of Bott [1] on the index and nullity of iterated closed geodesics and an extension of non-degenerate Morse theory to the degenerated case. We will discuss the idea of the proof and some applications. The Gromoll-Meyer theorem will be a focus of the talk because this is where rational homotopy theory comes into play. Namely, in 1976 [9] Vigué-Poirrier and Sullivan proved that the sequence of Betti numbers of the free loop space ΛM is unbounded if and only if the real cohomology ring $H^*(M;\mathbb{R})$ requires at least two generators. Since the last statement as well as the property of admitting infinitely many geometrically distinct closed geodesics is invariant with respect to the lift to finite coverings the theorem can be extended to the case of a finite fundamental group. Thus, the problem is still open in the case where $\pi_1(M)$ is infinite but has only finitely many conjugacy classes up to powers, or if $\pi_1(M)$ is finite but the real cohomology of the universal cover requires only one generator.

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Geodesics and the Free Loop Space II

Varvara Karpova

The aim of this talk was firstly to present the Sullivan-Vigué Poirrier theorem and to comment briefly on its proof. In a second part, we concentrated on analogues of the results surrounding the Closed Geodesic Problem for A-invariant geodesics. We give a quick overview of our presentation here.

Let M denote a compact simply connected Riemannian manifold. The Gromoll-Meyer theorem establishes a relation between the Betti numbers of the free loop space $\mathscr{L}M$ of M and the number of geometrically distinct geodesics on the manifold. More precisely, it tells us that if the sequence of Betti numbers of $\mathscr{L}M$ is not bounded, then the manifold M admits infinitely many geometrically distinct closed geodesics. A natural thing to wonder about are then the conditions under which the Betti numbers of the free loop space are unbounded. In [7] Micheline Vigué Poirrier and Denis Sullivan provided a solution to this problem, using algebraic minimal models for $\mathscr{L}M$. Another reasonable question to ask in this context is whether there exist analogues of Gromoll-Meyer and Sullivan-Vigué Poirrier theorems for A-invariant geodesics, where A is a (finite-order) isometry of the manifold.

Suppose we are given a simply connected space M of finite rational type. Recall that the free loop space $\mathscr{L}M$ of M is constructed as the pullback

$$\begin{array}{ccc} \mathscr{L}M & \longrightarrow & M^I \\ \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle (p_0,p_1)} \\ M & \xrightarrow{\Delta} & M \times M, \end{array}$$

where M^I denotes the path space on M, Δ is the diagonal map and $p_i(\lambda) = \lambda(i)$ for all $\lambda \in M^I$, $i = 0, 1$. In order to construct a Sullivan model for $\mathscr{L}M$, one should remember that the model of a pullback is given by the pushout of models. Roughly speaking, if $(\Lambda V, d)$ denotes a minimal model for the space M, then a model for its free loop space will be of form $(\Lambda(V \oplus sV), D)$, where sV is the suspension of the graded vector space V, and the definition of the differential D involves a certain derivation of degree -1 (see [1] for details). With an appropriate algebraic model of $\mathscr{L}M$ at hand, one is well equipped to unravel the details of the proof of the following theorem by Sullivan and Vigué Poirrier, which provided a solution to the Closed Geodesic Problem for a large class of spaces.

Theorem 3 ([7, 1])**.** *Let M be a simply connected space with minimal model $(\Lambda V, d)$, and whose rational cohomology is finite dimensional. The following conditions are equivalent.*

(1) *The sequence of rational Betti numbers of $\mathscr{L}M$ is unbounded.*
(2) *The cohomology algebra $H^*(M;\mathbb{Q})$ requires at least two generators.*
(3) *The dimension of $\pi_{\text{odd}}(M)\otimes\mathbb{Q}$ is at least two, i.e., at least two generators in $(\Lambda V, d)$ are exterior.*

Let us now turn to the analogues of the results surrounding the Closed Geodesic Problem for A-invariant geodesics.

Definition 1. *Let A be an isometry of a compact simply connected Riemannian manifold M. A geodesic $\gamma : \mathbb{R} \to M$ is called* A-invariant *if there exists some $T \in \mathbb{R}$ such that $\gamma(t+T) = A(\gamma(t))$ for all $t \in \mathbb{R}$.*

The A-invariant analogue of the free loop space, $\mathscr{L}M_A$, is given by a pullback diagram similar to the above, where the diagonal $\Delta = (\mathrm{Id}, \mathrm{Id})$ is replaced by the map (A, Id).

In [4], Karsten Grove studied existence conditions for A-invariant geodesics on connected compact Riemannian manifolds. Together with Stephen Halperin, Grove used results of [4] to obtain an existence theorem for A-invariant geodesics on a compact rationally elliptic manifold in [2].

Moreover, an A-invariant analogue of the Gromoll-Meyer theorem was provided by Minoru Tanaka in [6].

Theorem 4 ([6])**.** *Let M be a compact simply connected Riemannian manifold. If the Betti numbers of $\mathscr{L}M_A$ are unbounded, then M admits infinitely many geometrically distinct A-invariant geodesics.*

Since the isometries form a compact Lie group whose elements of finite order are dense ([5]), the isometry A of M can be assumed to be of a finite order $k \geq 1$. In this situation, the cyclic group G generated by A acts on M. The theory of algebraic models tells us how to construct a G-equivariant minimal model $(\Lambda V, d)$ for M, where the vector space V decomposes as $V = V^A \oplus J$, the direct sum of its subspace V^A generated by A-invariant vectors and of its complement. Using this decomposition, one can then produce an algebraic minimal model for the A-invariant free loop space $\mathscr{L}M_A$ (see [1] for details).

This model can be employed to give criteria under which the Betti numbers of $\mathscr{L}M_A$ are unbounded. Thereby, the following A-invariant versions of the Sullivan-Vigué Poirrier theorem were obtained by Grove, Halperin and Vigué Poirrier.

Theorem 5 ([3]). *Let M be a simply connected compact Riemannian manifold. Let A be an isometry and let $(\pi_*(M) \otimes \mathbb{Q})^A$ denote the A-invariant part of rational homotopy. If $\dim(\pi_{\mathrm{odd}}(M) \otimes \mathbb{Q})^A \geq 2$, then the Betti numbers of $\mathscr{L}M_A$ are unbounded, and M admits infinitely many nontrivial geometrically distinct A-invariant geodesics.*

Theorem 6 ([2]). *Let M be a simply connected compact Riemannian manifold, and let A be an isometry. If M is rational hyperbolic, then the Betti numbers of $\mathscr{L}M_A$ are unbounded, and M admits infinitely many geometrically distinct A-invariant geodesics.*

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Geodesic Flows

Nadine Grosse

The aim of this talk was to show that one can conclude the homotopy type of a manifold from 'special' properties of a fixed metric on this manifold. We stick to a particular example and give an overview on a result by Paternain:

Theorem 7. [3, Thm. 3.2] *Let (M, g) be a closed simply-connected Riemannian manifold with zero topological entropy. Then M is rationally elliptic.*

Topological entropy is a notion in the theory of flows in dynamical systems. In the context of the Theorem above this is meant to the entropy of the geodesic flow of M. We shortly introduce this notion:
The geodesic flow is the map $\phi_t : TM \to TM,\ \theta = (p, v) \mapsto (\gamma_v(t), \dot{\gamma}_v(t))$ where $\gamma_v(t)$ is the unique geodesic starting at $\gamma_v(0) = p$ with the velocity vector $\dot{\gamma}_v(0) = p$. Since geodesics have constant speed ϕ_t restricted on the sphere bundle $SM = \{(p, v) \in TM |\ |v| = 1\}$ is also an endomorphism. In order to define the entropy,

one fixes a distance function d on TM. From that we can define a family of new distance functions $d_T(\theta_1, \theta_2) = \max_{0\leq t\leq T} d(\phi_t(\theta_1), \phi_t(\theta_2))$ where $\theta_1, \theta_2 \in TM$.

The topological entropy is now defined (cf. [1, Chap. 3.2]) as

$$h_{top}(g) := \lim_{\epsilon\to 0} \underbrace{\limsup_{T\to\infty} \frac{1}{T} \min \#\{\text{balls of radius } \epsilon \text{ w.r.t. } d_T \text{ that cover } SM\}}_{\text{monotonically decreasing as function in } \epsilon}$$

To prove in the desired theorem that the manifold is rationally elliptic, we go for the following criterion for rational ellipticity:

Theorem 8. [5, Thm. 2.7.4] *If (M, g) is a simply-connected closed Riemannian manifold whose Betti numbers of the loop space have subexponential growth, than M is rationally elliptic.*

In the following we sketch the main ingredients of the proof:

1. Morse theory of the path space [7]
Fixing two points $p, q \in M$ we consider $\Omega_M(p, q)$ which is the space of all piecewise smooth paths $\gamma : [0, 1] \to M$ from p to q. The aim is to use Morse theory on the energy functional E defined on that space. The critical points of E are precisely the geodesics from p to q. But only for generic (i.e. not conjugate) points these critical points are nondegenerate. In that case, Morse theory can be applied. Restricting E to the subspace $\Omega^\lambda_M(p, q) \subset \Omega_M(p, q)$ of paths of length $\leq \lambda$, we obtain that $\Omega^\lambda_M(p, q)$ has the homotopy type of a finite CW-complex and, thus, we get (for generic $p, q \in M$):

$$\textbf{Morse inequality}: \quad n_\lambda(p,q) := \#\{\text{geodesics from } p \textit{ to } q \textit{ of length } \leq \lambda\}$$
$$= \#\{(\pi\circ\phi_1)^{-1}(q) \cap T_pM_{\leq\lambda}\}$$
$$\geq \sum b_i(\Omega^\lambda_M(p,q))$$

where $\pi : TM \to M$ and $T_pM_{\leq\lambda} := \{(p, v) \in TM \mid |v| \leq \lambda\}$.
2. Gromov's theorem
The right handside of the Morse inequality from above is already near to what we want to have in order to prove the rational ellipticity of M. But we actually need $b_i(\Omega(M) = (b_i\Omega_M(p, q))$ instead of $b_i(\Omega^\lambda_M(p, q))$. This is given by

Theorem 9. [6, 7.4], [2, Prop. 2.1] *Let (M, g) be a simply-connected closed Riemannian manifold. Then there is a constant $C > 0$ such that for all points $p, q \in M$ that are not conjugate and for any $i \in \mathbb{N}$ with $\lambda \geq Ci$ it holds*

$$b_i(\Omega_M(p, q)) \leq b_i(\Omega^\lambda_M(p, q)) = b_i(\Omega(M).$$

The assumption of simply-connectedness is crucial in the proof of the above theorem since it uses the existence of a homotopy equivalence from M to M which maps the 1-skeleton of M to a point.
3. Link to volume growth under the geodesic flow and Yomdin's Theorem
Connecting the Morse inequality with Gromov's Theorem we have $n_{\lambda=Cm}(p, q) \geq$

$\sum_{i=0}^{m} b_i(\Omega M)$ as long as q is not conjugate to p. But only a measure zero subset of M fails to be conjugate to a fixed p (cf. Sard's Theorem and [7]). Thus, we can integrate and get:

$$\begin{aligned}\mathrm{vol}(M)\sum_{i=0}^{m} b_i(\Omega M) &\leq \int_M n_{Cm}(p,q)\mathrm{d}q = \int_{T_pM_{\leq Cm}} |\det \mathrm{d}_\theta(\pi\circ\phi_1)|\mathrm{d}\theta \\ &\leq \int_{T_pM_{\leq Cm}} |\det \mathrm{d}_\theta\phi_1|\mathrm{d}\theta = \int_0^{Cm}\int_{S_pM} |\det \mathrm{d}_\theta\phi_t|\mathrm{d}\theta\mathrm{d}t \\ &\leq \int_0^{Cm} \mathrm{vol}_{n-1}(\phi_t(S_pM)\mathrm{d}t.\end{aligned}$$

Thus, in this last step the right side has to be estimated which is done by Yomdin's Theorem [4, Cor. 1.6]:

$$\begin{aligned}\limsup_{m\to\infty}\frac{1}{m}\log\sum_{i=0}^{m} b_i(\Omega M) &\leq \limsup_{m\to\infty}\frac{1}{m}\log\mathrm{vol}(M)^{-1}\int_0^{Cm}\mathrm{vol}(\phi_t(S_pM))\mathrm{d}t \\ &\leq \max\{0, \limsup_{m\to\infty}\frac{1}{m}\log\mathrm{vol}_{n-1}(\phi_{Cm}(S_pM))\} \\ \text{(by Yomdin's theorem)} \quad &\leq h_{top}(g) = 0.\end{aligned}$$

□

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Formality and Kähler Manifolds

Pascal Lambrechts

In this talk we explain that the rational homotopy types of compact Kähler manifolds (which are complex manifolds admitting a Riemannian metric strongly compatible with the complex structure) are very restricted. Indeed they are *formal*, which means that all the rational homotopy type is encoded in the cohomology algebra. Another fact, more classical, is that compact Kähler manifolds satisfy the Hard Lefschetz property.

A complex manifold is a manifold equipped with a holomorphic atlas modeled on $\mathbb{C}^n$. Examples of such manifolds are:

- complex tori $\mathbb{C}^n/\Gamma$ where Γ is a rank $2n$ lattice in $\mathbb{C}^n$;
- the Hopf manifold $(\mathbb{C}^n \setminus \{0\})/\simeq$ where $\simeq$ is the relation generated by $(z_1, \dots, z_n) \simeq (z_1/2, \dots, z_n/2)$. This manifold is homeomorphic to $S^1 \times S^{2n-1}$;
- smooth complex projective varieties, that is: smooth sets of solutions in $P^n(\mathbb{C})$ of systems of homogeneous polynomial equations;
- the Kodaira-Thurston manifold obtained as the nilmanifold $KT := \mathbb{R}^4/\Gamma$ where Γ is a group with 4 generators acting on $\mathbb{R}^4$ by sending (t_1, t_2, t_3, t_4) to, respectively, $(t_1 + 1, t_2, t_3, t_4)$, $(t_1, t_2 + 2, t_3, t_4)$, $(t_1, t_2, t_3 + 1, t_4)$, and $(t_1, t_1 + t_2, t_3, t_4 + 1)$. Note that KT is diffeomorphic to the product of S^1 with an S^1-bundle over $S^1 \times S^1$ with non trivial Euler class.

The tangent bundle TM of a complex manifold admits a self-map

$$J\colon TM \to TM$$

such that $J^2 = -1$, corresponding of course to the multiplication by $\sqrt{-1}$ on tangent vectors. A Riemannian metric g on M is said to be Kähler if it is strongly compatible with the complex structure J in the sense that

- $g(Jv, Jw) = g(v, w)$, and
- the Levi-Civita parallel transport associated to g preserves J.

Such a Riemannian metric g compatible with J is completely equivalent to the data of a smooth 2-form $\omega \in \Omega^2(M)$ defined by $\omega(v, w) = g(v, Jw)$ such that

- $\omega(Jv, Jw) = \omega(v, w)$, and
- $d\omega = 0$.

Such a form is called a *Kähler form* on M, and (M, ω) is called a *Kähler manifold.*

For example $\mathbb{C}^n$ and $P^n(\mathbb{C})$ are Kähler manifolds: it is not difficult to construct explicit Kähler 2-forms on those. Also it is easy to show that a holomorphic submanifold of a Kähler manifold is also Kähler, with a Kähler form obtained by restriction. Hence every smooth projective complex variety is Kähler.

Not every complex manifold admits a Kähler form; actually there are strong topological restrictions on compact Kähler manifold. In particular it must satisfy the following

Theorem 1. ***(Hard Lefschetz)***
Let (M, ω) be a compact Kähler manifold of complex dimension n. Then it satisfies the hard Lefschetz property *which means that for all $k \geq 0$, the map*

$$L_k : H^{n-k}(M) \to H^{n+k}(M), x \mapsto x.[\omega]^k$$

is an isomorphism.

The proof of this theorem is not easy. It is based on Hodge theory.

In particular $[\omega] \neq 0$ in $H^2(M)$. Thus the Hopf manifold diffeomorphic to $S^1 \times S^{2n-1}$ is not Kähler for $n > 1$.

On the other hand the Kodaira-Thurston manifold satisfies the hard Lefschetz property. Indeed KT is a nilmanifold whose associated Lie algebra is $\mathbb{R}\langle X, Y, Z, T\rangle$ with $[X, Y] = T$ and the Lie brackets of all the other pairs of generators are zero. Therefore a Sullivan model for KT is given by

$$(\wedge(x, y, z, t)\, ,\, d(t) = xy,\, d(x) = d(y) = d(z) = 0)$$

with x, y, z, t of degree 1. It is straightforward to check that the 2-form ω corresponding to $xy + zt$ induces the Hard Lefschetz isomorphisms.

Actually compact Kähler manifolds need to satisfy another more subtle topological condition: they need to be *formal.*

Definition 1. *A CDGA (commutative differential graded algebra)* (A, d) *is* formal *if it weakly equivalent as a CDGA to its cohomology algebra, in other words if there exists a zig-zag of CDGA quasi-isomorphisms*

$$(A, d) \overset{\simeq}{\leftarrow} \cdots \overset{\simeq}{\rightarrow} (H(A, d), 0).$$

A space X *is formal if the CDGA* $A_{PL}(X)$ *is formal.*

For example the Kodaira-Thurston manifold is not formal. Indeed using the above Sullivan model of KT, it is not difficult to see that it cannot be weakly equivalent to its cohomology algebra. Actually there is a non-trivial Massey product $\langle x, y, x\rangle$ in the cohomology of KT, and it is easy to check that this is an obstruction to formality. Using the same line of argument, actually one can prove

Theorem 2 ([1])**.** *A nilmanifold is formal if and only if it is abelian.*

We have the following very important

Theorem 3 ([2])**.** *Compact Kähler manifolds are formal.*

The proof of this theorem is based on the "$\partial\overline{\partial}$ lemma" that we explain now. Let M be a complex manifold. The CDGA $\Omega(M; \mathbb{C})$ of smooth forms with complex valued is bigraded where

$$\Omega^{p\cdot q}(M)$$

is the $\mathcal{C}^\infty(M, \mathbb{C})$-module generated by forms which locally are

$$dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z_{j_1}} \wedge d\overline{z_{j_2}} \wedge \cdots \wedge d\overline{z_{j_q}}$$

for some holomorphic coordinates $(z_1, \ldots, z_n)$. Then the differential d decomposes as

$$d = \partial + \overline{\partial}$$

where ∂ and $\overline{\partial}$ are of bidegree $(1, 0)$ and $(0, 1)$ respectively.

Lemma 4. ***($\partial\overline{\partial}$ lemma)***
Let M *be a compact Kähler manifold and let* $\alpha \in \Omega^{p,q}(M) \cap \ker(\partial) \cap \ker(\overline{\partial})$.
If $\alpha \in \mathrm{im}(\partial)$ *then* $\alpha \in \mathrm{im}(\partial\overline{\partial})$. *If* $\alpha \in \mathrm{im}(\overline{\partial})$ *then* $\alpha \in \mathrm{im}(\overline{\partial}\partial)$.

The proof of this lemma is based on the fact that on a Kähler manifold the laplacian associated to ∂ and $\overline{\partial}$ agree, $\Delta_\partial = \Delta_{\overline{\partial}}$. Using then Hodge theory (because M is compact) some Green operators are used to build an explicit β such that $\alpha = \partial\overline{\partial}(\beta)$.

Taking this "$\partial\overline{\partial}$ lemma" for granted, the proof of formality of compact Kähler manifold is easy. We consider a zigzag of CDGA

$$(H(\Omega^{*,*}, \partial), 0) \xleftarrow{\rho} (\Omega^{*,*} \cap \ker(\partial), \overline{\partial}) \xhookrightarrow{j} (\Omega^*, d) \simeq A_{PL}(M; \mathbb{C}).$$

The $\partial\overline{\partial}$ lemma is used to prove that the canonical projection ρ commutes with the differentials and that $H(\rho)$ and $H(j)$ are both injective and surjective. Therefore the above zigzag gives a weak equivalence between $A_{PL}(M)$ and a CDGA with differential 0, which implies formality.

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Spectral Sequences and Models

Joana Cirici

1. Introduction

From Sullivan's theory, we know that the de Rham algebra of a manifold determines all its real homotopy invariants. In addition, the Formality Theorem of [1], exhibits the use of rational homotopy in the study of complex manifolds, in that it provides homotopical obstructions for the existence of Kähler metrics.

Bearing these results in mind, and with the objective to study complex homotopy invariants, Neisendorfer and Taylor [5] define the Dolbeault homotopy groups of a complex manifold by means of a bigraded model of its Dolbeault algebra of forms. Not only interesting in themselves, these new invariants prove to be useful in the computation of classical invariants such as the real homotopy or the cohomology of the manifold.

The Frölicher spectral sequence provides a connection between Dolbeault and de Rham models, and indicates an interplay between models and spectral sequences. In [3], Halperin and Tanré analyse this issue in the abstract setting, by constructing models of filtered cdga's and establishing a relationship with the bigraded minimal models of each stage of their associated spectral sequences. This allows the study of any spectral sequence coming from a filtration of geometric nature. The Dolbeault homotopy theory of Neisendorfer and Taylor fits naturally in this wider context.

As an application, Tanré studies in [6] the Borel spectral sequence associated with an holomorphic fibration, and constructs a Dolbeault model of the total space from those of the fiber and the base. Also, the filtered model of Halperin-Stasheff that controls the formality of a cdga fits into this context by means of the trivial filtration.

Our objective is to present the homotopy theory of filtered cdga's, focusing on its applications to the study of complex manifolds.

2. Homotopy theory of filtered cdga's

A *filtered cdga* (A, d, F) is a cdga (A, d) together with a decreasing filtration

$$0 \subseteq \cdots \subseteq F^{p+1}A \subseteq F^pA \subseteq \cdots \subseteq A,$$

such that the differential and the product are compatible with the filtration.

Any filtered cdga has an associated spectral sequence, each of whose stages is a bigraded differential algebra. Furthermore, every map of filtered cdga's compatible with filtrations induces a map between their respective spectral sequences. Such a map is an *E_r-quasi-isomorphism* if the induced map at the r-stage is a quasi-isomorphism of bigraded algebras. Every E_r-quasi-isomorphism is a quasi-isomorphism but the converse is not true in general.

In order to develop an homotopy theory for filtered cdga's, we generalize Sullivan's theory and introduce E_r-minimal models which we define step by step as follows. An *E_r-minimal extension* of degree n and weight p of a filtered cdga (A, d, F) is a filtered cdga $A \otimes_d \Lambda(V)$, where V is a finite dimensional vector space of degree n and pure weight p, satisfying

$$dV \subset F^{p+r}(A^+ \cdot A^+) + F^{p+r+1}A.$$

The filtration on $A \otimes \Lambda(V)$ is defined by multiplicative extension. All cdga's are augmented, and A^+ denotes the kernel of the augmentation. An *E_r-minimal cdga* is the colimit of a sequence of E_r-minimal extensions, starting from the base field. It follows that the differentials of the associated spectral sequence of an E_r-minimal cdga satisfy $d_0 = \cdots = d_{r-1} = 0$, and d_r is decomposable.

Theorem 1 (Halperin-Tanré). *For every $r \geq 0$ and every filtered cdga (A, d, F) there exists an E_r-minimal model: that is an E_r-minimal cdga (M, D, F) together with an E_r-quasi-isomorphism $\psi : (M, D, F) \to (A, d, F)$. In particular, the induced map $E_r(\psi) : (E_r(M), d_r) \to (E_r(A), d_r)$ is a bigraded model of $(E_r(A), d_r)$.*

Observe that for the trivial filtration, an E_0-minimal model is a Sullivan model, and so the above theorem can be viewed as a generalization of the classical theory.

The homotopical approach of [2] proves to be convenient in this situation. Define *r-homotopy equivalences* by means of a filtered path object $\Lambda(t, dt)$, with t of weight 0 and dt of weight r. Every E_r-minimal cdga M is *cofibrant*: any E_r-quasi-isomorphism $w : A \to B$ induces a bijection between classes of maps modulo r-homotopy equivalence, $w^* : [M, A]_r \xrightarrow{\sim} [M, B]_r$. In addition, E_r-minimal cdga's are *minimal*, in that every E_r-quasi-isomorphism between E_r-minimal cdga's is an isomorphism. The existence of E_r-minimal models endows the category of filtered cdga's with the structure of a *Sullivan category*. As a result, for all $r \geq 0$, we obtain an equivalence of categories

$$(E_r\text{-}\mathsf{Min}/\sim_r) \longrightarrow \mathrm{Ho}_r\,(\mathbf{F}\mathsf{CDGA}) = \mathbf{F}\mathsf{CDGA}[\mathcal{E}_r^{-1}],$$

between the quotient category of E_r-minimal cdga's modulo r-homotopy equivalence, and the localized category of filtered cdga's with respect to E_r-quasi-isomorphisms.

This provides a way to derive the functor of filtered indecomposables with respect to E_r-quasi-isomorphisms, obtaining a well defined notion of *E_r-homotopy groups*, as a new family of invariants for filtered cdga's.

Also, we have the following filtered version of formality. We say that a filtered cdga (A, d, F) is *E_r-formal* if there is a chain of E_r-quasi-isomorphisms

$$(A, d) \xleftarrow{\sim} \cdots \xrightarrow{\sim} (E_{r+1}(A), d_{r+1}).$$

3. Applications

We next present some applications of the homotopy theory of filtered cdga's to the study of complex manifolds.

Dolbeault homotopy. Let X be a complex manifold. Its complex of de Rham algebra of $\mathcal{C}^\infty$ differential forms admits a bigrading by forms of type (p, q),

$$\mathcal{A}_{dR}(X) = \bigoplus \mathcal{A}^{p,q}(X).$$

The differential decomposes as $d = \partial + \overline{\partial}$. The *Frölicher spectral sequence* is the spectral sequence associated to $\mathcal{A}_{dR}(X)$, with the filtration defined by the first degree. Its 0-stage is the Dolbeault algebra $(E_0, d_0) = (\mathcal{A}^{*,*}(X), \overline{\partial})$, and its 1-stage is the Dolbeault cohomology $E_1 = H^{*,*}_{\overline{\partial}}(X)$. It converges to the de Rham cohomology $H^*_{dR}(X)$. The following result is a direct consequence of Theorem 1 applied to the Frölicher spectral sequence, and taking $r = 0$.

Theorem 2 (Neisendorfer-Taylor)**.** *There exists a de Rham model (M_X, D) of X together with a filtration such that $(E_0(M_X), d_0)$ is a Dolbeault model of X.*

In particular, given a Dolbeault model, one can build a de Rham model by defining a perturbation of its differential. If $E_1(\mathcal{A}_{dR}(X)) = E_\infty(\mathcal{A}_{dR}(X))$, then $\mathcal{A}_{dR}(X)$ is E_0-formal if and only if the manifold X is *strictly formal* in the sense of [5]. In particular, compact Kähler manifolds are E_0-formal.

Fibrations. Consider an holomorphic fibration $X_0 \to X \to Y$ of compact, connected, nilpotent complex manifolds. Assume as well that X_0 is Kähler, and that $\pi_1(Y)$ acts trivially on $H(X_0)$. In [4], Borel constructs a filtration of the Dolbeault algebra of X such that its associated spectral sequence converges to $H^{*,*}_{\overline{\partial}}(X)$, and $E_1 = (\mathcal{A}_{dR}(Y), \overline{\partial}) \otimes H_{\overline{\partial}}(X_0)$. The following result is a consequence of Theorem 1 applied to the Borel spectral sequence, with $r = 1$.

Theorem 3 (Tanré)**.** *With the previous assumptions, there exists a Dolbeault model M_X of X together with a filtration such that $(E_1(M_X), d_1) = (M_Y, \overline{\partial}) \otimes M_H$, where $(M_Y, \overline{\partial})$ is a Dolbeault model of Y and M_H is a Sullivan model of $H^*(X_0; \mathbb{C})$.*

Therefore a Dolbeault model for X can be built by taking the tensor product of a model of $H^*(X_0;\mathbb{C})$ by a Dolbeault model of Y, and defining a perturbation of the differential. An interesting application to the above result concerns compact connected Lie groups of even dimension.

Theorem 4 (Tanré). *Let $T \to G \to G/T$, where G is a compact connected Lie group of even dimension, and T a maximal torus. Then G is Dolbeault formal if and only if the Frölicher spectral sequence satisfies $E_2(G) = E_\infty(G)$.*

This result facilitates finding examples of compact complex manifolds whose spectral sequence does not collapse at the second stage.

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Formality and Symplectic Manifolds I

Victor Turchin

In this note we review some major results about the topological properties of *closed* symplectic manifolds.

A *symplectic manifold* is a pair (M, ω) where M is a smooth manifold and ω is a closed and non-degenerate at every point 2-form on M. Therefore symplectic manifolds are always of even dimension $\dim M = 2m$ and are naturally oriented since ω^m can be viewed as a volume form.

First we recall the examples of Kodaira-Thurston [10] and McDuff [7] of closed symplectic but not Kählerian manifolds. Later Rudyak and Tralle [9] and independently Babenko and Taimanov [2] showed that the Kodaira-Thurston and McDuff manifolds are not formal. This shows that the class of symplectic manifolds is larger than the class of Kählerian manifolds and moreover symplectic manifolds can be non-formal contrary to the Kählerian ones.

Given a symplectic manifold (M, ω) of dimension $2m$ one can consider a map in the de Rham cohomology

$$H^i(M) \to H^{2m-i}(M), \quad i = 0 \dots m,$$

given by multiplication by $[\omega]^{m-i}$. If for all $i = 0 \dots m$ this map is an isomorphism, the manifold (M, ω) is said to satisfy the *Hard Lefschetz property* (HL). All Kählerian manifolds satisfy HL. One can also show that HL implies that the rank of the cohomology of M in any odd degree is always even (hint: define a natural skew-symmetric bilinear form on $H^{2i-1}(M)$ and show that it follows from HL that this form is non-degenerate). Thus HL is a topological obstruction for a symplectic manifold to be Kählerian.

Non-simply connected case. A well known example of a closed symplectic non-Kählerian manifold is the Kodaira-Thurston manifold [10]. This manifold KT is a quotient of $\mathbb{R}^4$ by some subgroup Γ of affine transformations. More explicitly KT is the quotient of the group

$$G = \left\{ \left(\begin{array}{ccccc} 1 & x_1 & x_2 & 0 & 0 \\ 0 & 1 & x_3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_4 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) \middle| \; (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \right\}$$

by its subgroup Γ of matrices with integer coefficients. One can easily see that the rank of $H^1(KT)$ is 3, and therefore KT can not be HL.

It was shown, for example, in [2, 9] that KT has non-trivial triple Massey products and thus is not formal. Its non-formality is also a consequence of a more general result, since KT is a nilmanifold.

Simply connected case. McDuff [7] was the first to provide examples of simply connected closed symplectic non-Kählerian manifolds. The construction of McDuff is based on *symplectic blow up*.

Let $i\colon (M,\omega) \hookrightarrow (X,\sigma)$ be a *symplectic embedding*, which means $i^*\sigma = \omega$ and i is a smooth embedding. Gromov [4] (and in more detail McDuff [7]) defined a blow up $(\widetilde{X},\widetilde{\sigma})$ of (X,σ) along (M,ω). First notice that the normal bundle of M in X is symplectic and therefore carries a natural unique up to homotopy complex structure. The blow up $\pi\colon \widetilde{X} \to X$ has the property $\pi^{-1}(X \setminus M) \cong X \setminus M$, and $\pi^{-1}(M)$ is a fiber bundle over M with fiber $\mathbb{C}P^{k-1}$ (where $2k$ is the codimension of M in X) obtained as the projectivization of the normal bundle viewed as a complex one. McDuff shows that the blow up does not change the fundamental group $\pi_1(\widetilde{X}) = \pi_1(X)$ and as a vector space

$$H^*(\widetilde{X}) = H^*(X) \oplus H^*(M)\cdot a \oplus H^*(M)\cdot a^2 \oplus \ldots \oplus H^*(M)\cdot a^{k-1},$$

where $\deg a = 2$. In [2, 9] the authors describe the multiplicative structure on $H^*(\widetilde{X})$. On the other hand by a theorem of Tischler [11] any closed symplectic manifold with an integral symplectic form can be symplectically embedded in $\mathbb{C}P^N$ for sufficiently large N. Gromov showed [4] that one can take $N \geq 2m+1$, where $2m$ is the dimension of the manifold. From this result KT can be embedded in $\mathbb{C}P^N$, $N \geq 5$. Denote by $\widetilde{\mathbb{C}P^N}$ the blow up of $\mathbb{C}P^N$ along KT. McDuff concludes that $\pi_1(\widetilde{\mathbb{C}P^N}) = 0$ and $H^3(\widetilde{\mathbb{C}P^N})$ has rank 3. Thus $\widetilde{\mathbb{C}P^N}$, $N \geq 5$, are simply connected but are not HL and therefore cannot be Kählerian.

For the formality, Babenko-Taimanov [2] and independently Rudyak-Tralle [9] show that if the codimension of M in X is ≥ 8, and M has a non-trivial triple Massey product, then so does $\widetilde{X}$. As a corollary $\widetilde{\mathbb{C}P^N}$, $N \geq 6$, are not formal.

With a little bit more effort it is shown both in [2] and [9] that $\widetilde{\mathbb{C}P^5}$ also has a non-trivial triple Massey product.

At this point it is interesting to mention a work of Lambrechts and Stanley [5] that describes a rational model for the blow up $\widetilde{X}$ in case M has a sufficiently high codimension.

Hard Lefschetz property versus formality. One can ask what is the relation between HL and formality for the symplectic closed manifolds. But as it turns out neither property implies the other. Gompf [3] among other things provided examples of 6-dimensional (and therefore formal) simply connected closed symplectic manifolds that are not HL. On the other hand Cavalcanti [1] constructed non-formal symplectic manifolds satisfying HL. Gompf's construction is interesting to be mentioned since it provides a wide variety of symplectic manifolds. He defines a connected sum of symplectic manifolds along a codimension 2 submanifold. The construction works only in codimension 2 since only 2-disc can be symplectically inversed. Using this method he constructed symplectic 4-folds homeomorphic, but not diffeomorphic to Kählerian manifolds.

Formalizing tendancy. We mention that one still has some non-trivial "correlation" between formality and being symplectic closed. This phenomenon was studied by Lupton and Oprea in [6]. For example they show that any simply connected *coformal* symplectic closed manifold is always formal. They also show that any simply connected closed symplectic manifold that admits a pure minimal model is formal. As a consequence any homogeneous symplectic space is formal. Actually it was shown in [8] that if G/U is symplectic and G is semisimple, then G/U is Kählerian. The latter result generalizes Borel's theorem who considered U to be a maximal torus in G.

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Formality and Symplectic Manifolds II

Achim Krause

This talk was about further investigation of the difference between symplectic and Kähler manifolds. As the cutting and gluing constructions such as forming connected sums from real geometry do not preserve symplectic structures, another method for constructing new symplectic manifolds is needed. This is given by a generalization of the complex blow-up. The complex blow-up is a construction which comes originally from algebraic geometry, but has some applications in differential geometry as well. Roughly spoken, one replaces a complex submanifold of a given complex manifold with the projectivization of the (complex) normal bundle of the embedding, which yields a new complex manifold, called the blow-up of the manifold along the submanifold. One can see that one does not need to know the complex structures of the manifolds to get the real diffeomorphism type of the blow-up, but only the complex structure of the normal bundle. Thus, one can perform this construction as soon one knows a complex structure on the normal bundle of the embedding. But this is always the case in the situation of a symplectic manifold and the embedding a symplectic submanifold, since in this case one has a symplectic form on the vector bundle, and after choosing a inner product on each fiber, this is equivalent to a complex structure on the vector bundle. According to a theorem of McDuff, one can always find a symplectic structure again on this kind of blow-up (though not naturally), so practically this yields a construction for new symplectic manifolds. If one would be able to determine the rational homotopy type of this symplectic blow-up out of known data about the embedding, this would be a great opportunity to analyze how the topology of symplectic manifolds can look like. For example, the following question was answered positively by McDuff using the blow-up construction: Are there simply-connected symplectic manifolds with no Kähler structure? A generalisation of this result is due to Babenko and Taimanov: There are even simply-connected symplectic manifolds which are nonformal. To show this result, a model for the symplectic blow-up was given in the talk. This was first published by Lambrechts and Stanley in 2004 for the case that the dimension n of the embedded manifold satisfied $2n + 3 \leq m$ where m is the dimension of the ambient manifold. When $2n + 2 = m$ they constructed examples of homotopic embeddings with different rational homotopy types of blow-ups. This comes from different isotopy classes of embeddings, and apparently one cannot distinguish isotopy classes with rational homotopy invariants, so their result seems best possible. The details of the construction of their model are too complicated for a short talk, so only their explicit result was given, along with a general idea. To determine the rational homotopy type of the blow-up, one can regard it as complement of the embedding with the projectivized normal bundle of the embedding glued in along the boundary. So it suffices to construct models for the embedding of the boundary in said complement, and the gluing map of this boundary onto the projectivized normal bundle. Both can be done explicitely. Finally, the example of McDuff was analyzed using this model. This is constructed by embedding the Kodaira-Thurston manifold into some high-enough-dimensional complex projective space, and blowing up along the embedding. Such an embedding exists because of an embedding theorem for symplectic manifolds by Tischler. Using some facts about the Chern classes of the Kodaira-Thurston manifold one is now able to construct the rational homotopy type of this blow-up explicitely. The nonformality of the blow-up is now expressed in some nonvanishing Massey product, and surprisingly one sees that this Massey product is very similar to the one expressing the nonformality of KT. Of course the blow-up has to be simply-connected, as one sees from the theorem of Seifert and Van Kampen using the fact that the complex-projective spaces are simply-connected. These facts illustrate an interesting phenomenon: Although the blow-up has roughly the shape of the ambient space, such that the ambient space for example determines simply-connectedness, the formality does often only depend on the formality of the embedded submanifold.

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Curvature and Rational Homotopy I – Many manifolds with bounded curvature and diameter

Benjamin Matschke

1. Introduction

Bounding geometric invariants of Riemannian manifolds (such as sectional curvature, diameter or volume) often puts strong conditions on the diffeomorphism types of manifolds that fulfill these bounds.

The first talk of this conference gave already several instances, valid for all complete Riemannian manifolds M:

Theorem (von Mangoldt–Hadamard–Cartan). *If* $\sec(M) \leq 0$ *then the universal cover of* M *is diffeomorphic to* $\mathbb{R}^n$. *In particular,* $\pi_{*\geq 2}(M) = 0$.

Theorem (Bonnet–Myers). *If* $\sec(M) \geq \delta > 0$ *then* $\operatorname{diam}(M) \leq \pi/\sqrt{\delta}$. *In particular,* M *and its universal cover are compact and* $\pi_1(M)$ *is finite.*

Theorem (Sphere Theorem, Rauch–Berger–Klingenberg, Brendle–Schoen). *If* $1/4 < \sec(M) \leq 1$ *and* $\pi_1(M) = 0$ *then* M *is diffeomorphic to a sphere.*

Theorem (Cheeger–Peters)**.** *For all $n, C, C', D, V > 0$ there are only finitely many diffeomorphism types of closed smooth n-manifolds admitting a Riemannian metric such that $C \leq \sec(M) \leq C'$, $\operatorname{diam}(M) \leq D$, and $\operatorname{vol}(M) > V$.*

In a similar manner, Grove asked the following question (see [2]):

Question. *Does the class $M^{\leq D}_{-1 \leq \sec \leq +1}(n)$ of simply connected n-manifolds of diameter at most D and sectional curvature bounded by -1 and $+1$ contain only finitely many rational homotopy types?*

In this talk we present the solutions of Fang & Rong [1] and Totaro [3] who showed that the perhaps surprising answer to Grove's question is in general No.

Theorem 1.1 (Fang–Rong)**.** *For all $n \geq 22$ there exists a $D > 0$ such that $M^{\leq D}_{-1 \leq \sec \leq +1}(n)$ contains infinitely many rational homotopy types.*

Remark. Their proof can be easily extended: Their examples already work in dimension $n \geq 20$, and they have already pairwise non-isomorphic rational cohomology rings.

Theorem 1.2 (Totaro)**.** *There exists a $D > 0$ such that $M^{\leq D}_{-1 \leq \sec \leq +1}(7)$ contains infinitely many rational homotopy types.*

Again, Totaro's examples have already pairwise different rational cohomology rings.

2. Fang & Rong's approach

The construction of Fang & Rong's examples M_i, $i \in \mathbb{N}$, works as follows.

(1) Find a suitable principal T^3-bundle of manifolds $M \to B$, where T^3 is the 3-torus, and give M a T^3-invariant metric g.
(2) For suitable two-dimensional subtori $T_i \subseteq T^3$, let $M_i := M/T_i$.
(3) Do the construction in such a way that M_i have pairwise distinct rational homotopy.

Since the quotient maps $M \to M_i$ are Riemannian submersions and the 2-dimensional subtori T^2 of T^3 can be parametrized by the compact Stiefel manifold $V_{3,2}$, one can deduce that the sectional curvatures of all such quotients M/T^2 are uniformly bounded from above and below. Hence by scaling the metric g we can assume that $-1 \leq \sec(M_i) \leq +1$ for all i. Let $D := \operatorname{diam}(M)$. Then $\operatorname{diam}(M_i) \leq D$. In the construction we still need to take care of (3).

Fang and Rong start by constructing models $\mathcal{M}_i$ that will be part of the minimal models of M_i. First, let

$$\mathcal{M}_8 := (\Lambda(x_1, x_2, x_3, y, z), d),$$

where $|x_i| = 0$, $d(x_i) = 0$, $|y| = 5$, $d(y) = x_1^2 x^2$, $|z| = 7$, $d(z) = x_1^4 + x_2^4 + x_3^4$.

Lemma. *There is a CW-complex X whose minimal model $\mathcal{M}_E$ is $\mathcal{M}_8$.*

For this, first take $K(\mathbb{Z}^3, 2) = (\mathbb{C}P^\infty)^3$ whose minimal model is $(\Lambda(x_1, x_2, x_3), 0)$. Then take the pullback of the path-loop fibration $PK(\mathbb{Z}, 6) \to K(\mathbb{Z}, 6)$ along the map $K(\mathbb{Z}^3, 2) \to K(\mathbb{Z}, 6)$ that corresponds to the cohomology class $x_1^2 x_2 \in H^6(K(\mathbb{Z}^3, 2), \mathbb{Z})$. This adds y with $d(y) = x_1^2 x_2$ to the minimal model. Then add z in an analogous way.

Let $X^{(9)}$ be the 9-skeleton of X.

Lemma. *There exists a closed 19-manifold B and a map $X^{(9)} \to B$ that induces an isomorphism on $\pi_{*\leq 8}$.*

For this, embed $X^{(9)}$ into $\mathbb{R}^{17}$, thicken it to an open set N, and take the double $B := N \cup_{\partial N} (-N)$. Then one can show with Morse theory that B has a handle decomposition whose i-handles with $i \leq 9$ correspond to the i-cells of X.

Now $H^2(B; \mathbb{Z}) = \mathbb{Z}^3 = \langle x_1, x_2, x_3 \rangle$. Define M_i to be the principal S^1-bundle over B with Euler class $e_i = ix_1 + x_2 - x_3$. Let $\mathcal{M}_i(8)$ be the submodel of the minimal model of M_i generated by all elements in degree less then or equal to 8, i.e. the minimal model of the 8'th space in the Postnikov tower of M_i.

Lemma. *$\mathcal{M}_i(8) = (\Lambda(x_1, x_2, y, z_i), d)$, where $d(x_i) = 0$, $d(y) = x_1^2 x_2$, $d(z_i) = x_1^4 + x_2^4 + (ix_1 + x_2)^4$.*

For this, find a suitable morphism from the above to the relative minimal model of $M_i \to B$, which is $(\mathcal{M}_B \otimes \Lambda(t), d)$, $d(t) = e_i$.

Lemma. *If $i \neq j$ then $H^{*\leq 8}(M_i; \mathbb{Q}) \ncong H^{*\leq 8}(M_j; \mathbb{Q})$ as rings.*

This is straightforward computation. Now we let M be the following pullback, where f induces $\pi_2(B) \cong \pi_2(BT^3) = \mathbb{Z}^3$,

$$\begin{array}{ccc} M & \longrightarrow & ET^3 \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & BT^3. \end{array}$$

We write $T^3 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_i| = 1\}$, and let $T_i := \langle (1, z, z), (z, 1, z^i) \rangle$.

Lemma. *$M_i \cong M/T_i$.*

For this, compare the Euler classes. The lemmas complete the proof of Fang–Rong's Theorem 1.1.

3. Totaro's approach

The proof of Totaro's Theorem 1.2 relies on the deeper theorem of Barge–Sullivan, however it needs less computation and finds examples already in dimension 7. First, Totaro defines a cdga

$$H = \lambda(x_0, \ldots, x_4)/R,$$

where $|x_i| = 2$, $R = \langle x_i^2 = x_{i+1}x_{i+2}, x_i x_j = 0 \ (j \neq i, i \pm 1) \rangle$. This model satisfies Poincaré duality. Hence by the Theorem of Barge–Sullivan there exists a closed

6-manifold B whose minimal model is the minimal model of H. As above we find a principal T^5-bundle $M \to B$ that is classified by the map $B \to BT^5 = K(\mathbb{Z},2)^5$ induced by the cohomology classes $x_0,\dots,x_4$. Choose a T^5-invariant metric on M and consider quotients of M by 4-subtori $T^4 \subset T^5$. Let T_{a_0,a_1} be the subtorus such that $M_{a_0,a_1} = M/T_{a_0,a_1} \to B$ is the S^1-bundle whose Euler class is $e = a_0x_0 - a_1x_1 + (a_1^3/a_0^2)x_3$. We have $H^2(M_{a_0,a_1};\mathbb{Q}) = H^2(B;\mathbb{Q})/\langle e\rangle$.

Lemma. *The map $H^2(B)\otimes H^2(B) \to H^6(B) \cong \mathbb{Q}$ that sends $a\otimes b$ to $a\cup b\cup e$ factors through $H^2(M_{a_0,a_1})\otimes H^2(M_{a_0,a_1})$.*

This gives a quadratic form on $H^2(M_{a_0,a_1})$ whose determinant is well defined in $\mathbb{Q}/(\mathbb{Q}^*)^2$, and which can be computed to be $-a_0a_1(\mathbb{Q}^*)^2$. Hence different choices of a_0 and a_1 give infinitely many M_{a_0,a_1} with pairwise distinct rational cohomology rings.

The curvature and diameter arguments are the same as in proof of Fang–Rong, which finishes the proof of Totato's Theorem 1.2.

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Curvature II

Sebastian Goette

This talk investigates the structure of open manifolds (i.e., complete, non-compact Riemann manifolds without boundary) of nonnegative section curvature. It follows from the Soul Theorem [3] and the Splitting Theorem [4], [5] of Cheeger and Gromoll that each such manifold M is diffeomorphic to $N \times T$ after passing to a finite cover. Here, N is a simply connected open manifold with sec ≥ 0. By the Soul Theorem, there exists a totally convex and totally geodesic closed submanifold $C \subset N$, the *soul,* such that N is diffeomorphic to the normal bundle of C in N. The manifold T is a torus. The diffeomorphism $M \to N \times T$ can be chosen such that a soul of M is mapped to $C \times T$.

One may now ask the following question. Given a manifold $B = C\times T$ of sec ≥ 0, which vector bundles over B admit metrics of nonnegative sectional curvature? Here and in the following, we will use C for simply-connected compact manifolds of sec ≥ 0, and T will always denote some torus. Grove and Ziller have shown that all vector bundles over S^4 admit such metrics; however, the soul may contain many planes of sectional curvature 0. In other words, the soul S^4 may not be identic to the original base space S^4. On the other hand, Özaydin and Walschap have proved that vector bundles over a torus T carry metrics of nonnegative sectional curvature if and only if they are virtually trivial.

Now let $B = C \times T$ and consider a vector bundle $\xi\colon E(\xi) \to B$. If $M = E(\xi)$ admits a sec $\geq$ 0-metric, then there exists a soul $S \subset M$. After passing to a finite cover, we may assume that M is diffeomorphic to $N' \times T'$ with N' simply-connected and open, and with T' another torus. We also assume that a soul S of M is mapped to $C' \times T'$, with C' a compact and simply-connected soul of N'.

Forgetting about the sectional curvature now, we say that M *splits* if M is diffeomorphic to $N' \times T'$, where N' is the total space of a vector bundle over some simply-connected compact manifold C' and T' is a torus. On the other hand, we say that M *virtually comes from* C if a finite cover of M splits as $N \times T$ over a finite cover of $C \times T$, where $N \to C$ is a vector bundle. We call (C, T, k) *splitting rigid* if every vector bundle $M = E(\xi) \to B = C \times T$ of rank k that splits already virtually comes from C. If (C, T, k) is splitting rigid, then no vector bundle ξ admits a sec $\geq$ 0-metric if its rational characteristic classes do not belong to $H^\bullet(C) \otimes 1 \subset H^\bullet(C \times T)$. We apologize that we forgot to mention in the talk that such vector bundles are called *vampiric* because they do not have souls.

There are many examples of manifolds that are splitting rigid, among them spheres, projective spaces, and moreover, all known closed manifolds of positive sectional curvature. But already some homogeneous spaces are not splitting rigid for $k \geq 6$.

In the following, let $\mathrm{Char}(X, k) \subset H^\bullet(X)$ denote the natural home of the rational Euler- and Pontrijagin classes of a real vector bundle of rank k over some space X. In particular, $\mathrm{Char}(X, k)$ is the direct sum of certain even rational cohomology vector spaces of X.

Theorem (Belegradek and Kapovich [2])**.** *Let C be compact and simply-connected. If all derivations of the coholomogy algebra $(H^\bullet(C), \smile)$ of negative degree vanish on* $\mathrm{Char}(C, k) \subset H^\bullet(C)$*, then (C, T, k) is splitting rigid for all tori T.*

The proof is contained in [2], building on earlier work [1]. The same proof shows that one may replace derivations of the coholomogy algebra by derivations of a minimal model for C. With this formulation and under certain extra conditions on k, the theorem above has a partial converse that is also stated in [2].

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Poincaré Duality and Models I

Christoph Winges

The observation that the Poincaré duality of a manifold is equivalent to the non-singularity of the cup-product pairing allows one to express Poincaré duality in completely cohomological terms, and is thus also reflected in the minimal model of a given manifold. Therefore, we say that a cdga $\mathcal{A}$ over $\mathbb{Q}$ *satisfies Poincaré duality* if there is a class $[\mathcal{A}] \in H^n(\mathcal{A}) \cong \mathbb{Q}$ such that the following holds: If we define $\alpha\colon H^n(\mathcal{A}) \to \mathbb{Q}$ by requiring that $\alpha([\mathcal{A}]) = 1$, then the homomorphism $H^p(\mathcal{A}) \to \hom(H^{n-p}(\mathcal{A}), \mathbb{Q}), x \mapsto [y \mapsto \alpha(xy)]$ is an isomorphism for all p.

It is then natural to ask the following: Given a minimal Sullivan algebra which satisfies Poincaré duality, when is there a closed smooth manifold whose minimal model is this algebra? In the simply connected case, this question was completely answered by Sullivan. Additional notation is explained in the remark following the theorem.

Theorem. [7, 13.2] Let $\mathcal{A}$ be a minimal Sullivan algebra satisfying Poincaré duality with respect to some element in degree n. Suppose further that $H^1(\mathcal{A}) = 0$ and $H^*(\mathcal{A})$ is of finite type. Fix $p(\mathcal{A}) = (p_i(\mathcal{A}))_i \in \bigoplus_{i\geq 1} H^{4i}(\mathcal{A})$.

(1) If $n \not\equiv 0 \mod 4$, then there is a closed oriented n-manifold M which realizes $(\mathcal{A}, p(\mathcal{A}))$, i.e. $\mathcal{A}$ is the minimal model of M and the Pontryagin classes of M correspond to the $p_i(\mathcal{A})$ under the induced isomorphism in cohomology.

(2) If $n \equiv 0 \mod 4$, there is a closed oriented n-manifold M which realizes $(\mathcal{A}, p(\mathcal{A}))$ if and only if there is a class $[\mathcal{A}] \in H^n(\mathcal{A})$ such that:
   - Case 1: If $\alpha(L_{\frac{n}{2}}(p_1, \ldots, p_{\frac{n}{4}})) = 0$, then the obvious symmetric form $I\colon H^{2k}(\mathcal{A}) \otimes H^{2k}(\mathcal{A}) \to \mathbb{Q}$ is hyperbolic.
   - Case 2: If $\alpha(L_{\frac{n}{2}}(p_1, \ldots, p_{\frac{n}{4}})) \neq 0$, then this value agrees with the signature of I, the form I is induced by a form over $\mathbb{Z}$, and there is a closed oriented n-manifold N such that for any partition J of $\frac{n}{4}$, $p_J(N) = p_J(\mathcal{A})$.

In the above theorem, L_k denotes the k-th Hirzebruch polynomial (see [4, §19]). A partition of a natural number k is an unordered sequence of positive natural numbers whose sum equals k. Associated to such a partition $J = (j_1, \ldots, j_r)$ is a product of Pontryagin classes $p_J := p_{j_1} \cdot \ldots \cdot p_{j_r}$. If p_J lies in the top cohomology of a closed oriented manifold N, the J-th Pontryagin number $p_J(N)$ is given by evaluating p_J on the fundamental class of N. The numbers $p_J(\mathcal{A})$ can be defined in a formally identical way (cf. [4, §16]).

The proof falls into three parts, where we concentrate on dimensions ≥ 5: First, Sullivan's spatial realization functor allows one to construct a rational space $\bar{X} := \langle \mathcal{A} \rangle$ whose rational (co)homology satisfies Poincaré duality (see [7, §8], or [3, §17] for a more detailed and accessible treatment).

Second, the "Pontryagin classes" $p_i(\mathcal{A})$ enable one to construct a $\mathbb{Q}$-Poincaré complex X together with a localization map $X \to \bar{X}$ and a normal map of non-zero degree $(f, \bar{f})\colon (M, \nu) \to (X, \xi)$, i.e. a non-zero degree map $f\colon M \to X$ with an embedding i of M into some high-dimensional Euclidean space such that f is covered by a bundle map $\bar{f}$ from the normal bundle of M with respect to i to a vector bundle ξ over X (cf. [6, 3.2]).

Finally, the proof is finished by invoking a theorem due to Barge:

Theorem. [1, Théorème 4] Let $(f, \bar{f})\colon (M^n, \nu) \to (X, \xi)$ be a normal map of non-zero degree with $n \geq 5$.

If $n \not\equiv 0 \mod 4$, then $(f, \bar{f})$ is normally cobordant to a rational homotopy equivalence.

If $n \equiv 0 \mod 4$, then $(f, \bar{f})$ is normally cobordant to a rational homotopy equivalence if and only if the restriction of the intersection form of M to $\ker H_{\frac{n}{2}}(f; \mathbb{Q})$ represents the zero element in the Witt group $W(\mathbb{Q}) \cong L_0(\mathbb{Q})$, the 0-th quadratic L-group.

The proof of this theorem is via surgery theory, and works analogously to the integral case as carried out by Browder in [2], only that the arguments simplify at certain points. To give some examples:

- The odd-dimensional case is much easier since one has to deal with no torsion phenomena.
- In the case $n \equiv 2 \mod 4$, one encounters the same obstructions as in the integral case, but surprisingly surgery can still be performed on some generator of every summand since the $\mathbb{Z}_2$-valued obstruction vanishes after multiplying an arbitrary generator by 2.
- The vanishing of the surgery kernel in the Witt group implies that this form is hyperbolic by Witt's cancellation theorem (as given for example in [5, Ch. 1, Cor. 5.8]).

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Poincaré Duality and Models II

Yohan Brunebarbe

My talk was about the following question: does there exist a model for the rational homotopy of a closed simply-connected manifold M which satisfies Poincaré duality at the cochain level ?

Before getting into the precise definitions let us recall some basic results.

Let R be a commutative ring and M be a closed R-orientable (e.g. simply-connected) manifold of dimension n. The cap product endows the singular homology $\mathrm{H}_*(M,R)$ with coefficients in R with a structure of graded, left $\mathrm{H}^*(M,R)$-module. An R-orientation defines a fundamental class $[M] \in \mathrm{H}_n(M,R)$ and a morphism of graded, left $\mathrm{H}^*(M,R)$-modules $\mathrm{H}^*(M,R) \to s^{-n}\,\mathrm{H}_*(M,R)$ which associates $[M]$ to 1. Poincaré duality asserts that this is an isomorphism, or in other words that $s^{-n}\,\mathrm{H}_*(M,R)$ is a free graded, left $\mathrm{H}^*(M,R)$-module. If R is a field, the universal coefficient formula shows that $\mathrm{H}_*(M,R)$ is isomorphic to the dual of $\mathrm{H}^*(M,R)$ as a graded, left $\mathrm{H}^*(M,R)$-modules, so Poincaré duality in this context becomes an isomorphism between $\mathrm{H}^*(M,R)$ and $s^{-n}\,\mathrm{H}^*(M,R)^\vee$.

This motivates the following definition.

Definition 2. *Let k be a field. An oriented CDGA of dimension n ($n \in \mathbb{N}$) is a triple (A,d,ϵ), where (A,d) is a CDGA of finite type over k and $\epsilon \in (A^\vee)^{-n} \simeq \mathrm{Hom}(A^n,k)$ is a non-zero cocycle.*

Remark 1. *A non-zero cocycle corresponds to a morphism $A \to s^{-n}A^\vee$ of A-dg-modules.*

Definition 3. *An oriented CDGA (A,d,ϵ) satisfies Poincaré duality if the corresponding morphism of A-dg-modules $A \to s^{-n}A^\vee$ is an isomorphism.*

If (A,d,ϵ) satisfies Poincaré duality one easily checks that:

- $(\mathrm{H}^*(A),0,\tilde{\epsilon})$ satisfies Poincaré duality for the induced orientation $\tilde{\epsilon}$ on $\mathrm{H}^*(A)$,
- A is necessarily finite dimensional
- $\dim(A^k) = \dim(A^{n-k})$ for all $k \in \mathbb{N}$.

Moreover, for (A,d,ϵ) to satisfy Poincaré duality it is enough for the morphism $A \to s^{-n}A^\vee$ to be injective.

Now we can state the main theorem :

Theorem 5 (Lambrechts-Stanley [2])**.** *Let k be a field of any characteristic and let (A,d) be a CDGA of finite type over k whose cohomology is simply-connected and satisfies Poincaré duality for some orientation.*

Then there exists an oriented CDGA (A',d',ϵ') with (A',d') weakly equivalent to (A,d) which satisfies Poincaré duality.

Remark 2. *The theorem is trivial when the CDGA (A,d) is formal, for example for CDGA-models of compact Kähler manifolds, topological groups and for any simply-connected CDGA whose cohomology satisfies Poincaré duality in dimension less than or equal to* 6 [5].

Proof. Let us give an idea of the proof. Let (A, d, ϵ) be an oriented CDGA and θ be the kernel of the corresponding morphism $A \to s^{-n}A^{\vee}$. It is a two-sided homogeneous differential ideal of A. Hence the quotient $(A/\theta, \bar{d}, \bar{\epsilon})$ inherits a canonical structure of oriented CDGA and the projection is a morphism of CDGAs. Moreover $(A/\theta, \bar{d}, \bar{\epsilon})$ is easily seen to satisfy Poincaré duality. Thus we get a canonical morphism from (A, d, ϵ) to an oriented CDGA which satisfies Poincaré duality, and this morphism is a quasi-isomorphism if and only if the A-dg-module θ is acyclic (i.e. $\mathrm{H}^*(\theta) = 0$).

The following observation is crucial: if we suppose moreover that $(\mathrm{H}^*(A), 0, \tilde{\epsilon})$ satisfies Poincaré duality then an easy argument shows that θ is acyclic as soon as $H(\theta) = 0$ for $1 + \frac{n}{2} \leq i \leq n+1$.

To conclude the proof one tries to modify the CDGA (A, d, ϵ) step by step in such a way that the corresponding ideal θ becomes more and more acyclic. Several technical complications arise in the process. A key-observation is that modifying the CDGA from the middle to the top is enough. □

Let us give an application of the theorem to models of configurations spaces. If M is an n-dimensional manifold and $k \in \mathbb{N}^*$, the space of ordered configurations of k-points in M is the space

$$F(M, k) = \{(x_1, ..., x_k) \in M^k \mid x_i \neq x_j \text{ for } i \neq j\}.$$

The homeomorphism type of $F(M, k)$ depends only on the homeomorphism type of M. On the other hand it is not true that the homotopy type of $F(M, k)$ depends only on the homotopy type of M ([4]). However it holds for closed 2-*connected* manifolds w*hen* $k = 2$ ([3]). If we restrict our attention to rational homotopy type, Lambrechts and Stanley ([1]) construct a CDGA-model of $F(M, 2)$ from a Poincaré model of M for M any closed 2-connected manifold: if A is a CDGA-model of M with Poincaré duality then a CDGA-model of $F(M, 2)$ is given by $A \otimes A/(\Delta)$ where (Δ) is the differential ideal in $A \otimes A$ generated by the so-called *diagonal class.*

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Rational Homotopy Theory and String Topology

Ulrich Bunke

Let M be a closed oriented smooth manifold of dimension m and LM be its free loop space. String topology as introduced by Chas and Sullivan in [1] is about operations on the homology $H_*(LM)$ and the equivariant homology $H_*^{S^1}(LM)$. The most basic operations are

(1) the string product $H_s(LM) \otimes H_t(LM) \to H_{s+t-m}(LM)$,
(2) the BV-operator $\Delta : H_s(LM) \to H_{s+1}(LM)$, and
(3) the string bracket $\{\cdots,\cdots\} : H_s^{S^1}(LM) \otimes H_t^{S^1}(LM) \to H_{s+t-m+2}^{S^1}(LM)$.

Rational homotopy theory (see the reference book [2]) models the manifold M by a commutative differential graded algebra $\mathcal{M}_M$ such that $H^*(\mathcal{M}_M) = H^*(M;\mathbb{Q})$.

A Sullivan model $\mathcal{M}_M \cong (\Lambda V, d)$ of M determines a Sullivan model

$$\mathcal{M}_{LM} = (\Lambda(V \oplus \bar{V}), \mathcal{D})$$

in a natural way. Here $(\bar{V})^i = V^{i+1}$, and the differential $\mathcal{D}$ is given on generators by $\mathcal{D}v := dv$, $\mathcal{D}\bar{v} = -\mathcal{S}dv$ with the derivation $\mathcal{S}$ determined by $\mathcal{S}v := \bar{v}$ and $\mathcal{S}\bar{v} := 0$. Furthermore, a model for the homotopy quotient $LM/_hS^1$ is given by

$$\mathcal{M}_{LM/_hS^1} = (\Lambda(V \oplus \bar{V} \oplus \mathbb{Q}[u]), \mathcal{D}) \ ,$$

$|u| = 2$, where now $\mathcal{D}v := dv + u\bar{v}$, $\mathcal{D}\bar{v} = -\mathcal{S}dv$, and $\mathcal{D}u = 0$. We have

(1) $H^*(LM;\mathbb{Q}) \cong H^*(\mathcal{M}_{LM})$,
(2) $H^*_{S^1}(LM;\mathbb{Q}) \cong H^*(\mathcal{M}_{LM/_hS^1})$

It is now an interesting question to calculate the dual string topology operations

(1) $H^{s+t-m}(LM) \to H^s(LM) \otimes H^t(LM)$
(2) $H^{s+1}(LM) \to H^s(LM)$, and
(3) $H^{s+t-m+2}_{S^1}(LM) \to H^s_{S^1}(LM) \otimes H^t_{S^1}(LM)$

directly from the model $\mathcal{M}_M$. As shown in [3] this can be done in an essentially algorithmic way. The main purpose of my presentation was to explain the essential steps of this algorithm. It eventually produces maps

(1) $\mathcal{M}_{LM} \to \mathcal{M}_{LM} \otimes \mathcal{M}_{LM}$
(2) $\mathcal{M}_{LM} \to \mathcal{M}_{LM}$, and
(3) $\mathcal{M}_{LM/_hS^1} \to \mathcal{M}_{LM/_hS^1} \otimes \mathcal{M}_{LM/_hS^1}$

which induce the string dual operations on cohomology. The algorithm has been demonstrated in the simplest possible example $M = S^{2k+1}$.

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String Topology II

Gregory Lupton

Let M denote a 1-connected, closed, oriented smooth manifold and let LM denote its free-loop space. String Topology incorporates the following algebraic structures on $\mathbb{H}_*(M) := H_{*+d}(LM; \mathbb{Q})$:

- loop product (making $\mathbb{H}_*(M)$ a graded commutative algebra);
- Gerstenhaber algebra;
- Batalin-Vilkovisky (B-V) algebra;
- string bracket.

Following the article of Chen [2], I will indicate how (some of) these various algebraic structures of string topology may be constructed in a direct way starting from the minimal model of the manifold M. In principle, this allows for direct calculation of the various structures.

1. Frobenius Algebra

This is the fundamental structure from which other constructions flow. The paper of Abrams [1] is a good source for this topic. Suppose that A is a finite-dimensional graded algebra over $\mathbb{Q}$, which is simply connected ($\mathbb{Q}$ in degree 0 and 0 in degree 1) and which satisfies Poincaré duality. For example, we may take

(1) $A = H^*(M; \mathbb{Q})$ for M a *formal* 1-connected, closed, oriented smooth manifold;
(2) $A = \mathcal{M}_M = (\wedge V, d)$ in the case in which $V = V^{\text{odd}}$ and V is finite-dimensional.

In what follows, we may perform the various algebraic constructions starting from any cohomology algebra $H^*(M; \mathbb{Q})$. However, if the resulting structures are to correspond to those of M, we need to be able to take $H^*(M; \mathbb{Q})$ as a (rational) model for M, which entails M formal. By Poincaré duality, we mean that there is a symmetric, non-degenerate, bilinear pairing

$$\langle\ ,\ \rangle \colon A \otimes A \to \mathbb{Q}$$

which satisfies $< ab, c >=< a, bc >$.

In case (1) above (for the rational cohomology of any M, formal or not), we suppose that $\mu \in H^d(M; \mathbb{Q})$ is a fundamental class and set $x \cup y =< x, y > \mu$ for $x, y \in H^*(M; \mathbb{Q})$, where $\cup$ denotes the cup product.

In case (2) above, $\wedge V$ is oddly generated, and so is an exterior algebra (but with non-trivial differential, generally). The bilinear pairing is defined here in a similar way to that of (1).

Warning: We regrade A negatively; an element of degree n now becomes an element of degree $(-n)$. If M is of dimension d, so that $H^i(M; \mathbb{Q})$ is non-zero for $i = 0, \ldots, d$, we re-grade so that now $H^i(M; \mathbb{Q})$ is non-zero for $i = -d, \ldots, 0$. After this re-grading, in case (2), the differential d is of degree (-1).

Now define $A^* := \mathrm{Hom}(A, \mathbb{Q})$. Here, we mean graded linear maps $A \to \mathbb{Q}$, where $\mathbb{Q}$ is concentrated in degree 0 and where A has been re-graded negatively. Thus, A^* will be non-zero only in non-negative degrees.

Both A and A^* are A-bimodules: A with the obvious A-module structures from its own algebra structure, and A^* with the structures as follows:

$$(a \cdot \phi)(x) = (-1)^{|a||\phi|}\phi(ax) \qquad \text{and} \qquad (\phi \cdot a)(x) = \phi(ax),$$

for $a, x \in A$, and $\phi \in A^*$.

We have:

Proposition 6. *For A a finite-dimensional (DG) algebra, the following are equivalent:*

(a) *A is a (DG) Poincaré duality algebra;*
(b) *There exists an isomorphism $\iota\colon A \to A^*$ of (DG) A-bimodules.*

Proof. See [1]. □

An algebra that satisfies these equivalent conditions is called a *Frobenius algebra.*

We choose and fix a basis $\{e_i\}$ of A with e_0 of degree 0; take $\{e^i\}$ to be a dual basis of A^*; and take $\{\overline{e}_i\}$ a Poincaré dual basis of A. That is, we have $e^i(e_j) = \delta_{ij}$ and $< e_i, \overline{e}_j >= \delta_{ij}$. Then the isomorphism of part (b) of the above may be defined by $\iota(\overline{e}_i) = e^i$. Notice that this is a map of degree $(+d)$.

Since A^* is a (DG) coalgebra, the isomorphism $\iota\colon A \to A^*$ may be used to place a coalgebra structure on A, via $(\iota \otimes \iota)^{-1} \circ m^* \circ \iota$:

$$\begin{array}{ccc} A & \dashrightarrow & A \otimes A \\ {\scriptstyle \iota}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \iota\otimes\iota} \\ A^* & \xrightarrow[m^*]{} & A^* \otimes A^*. \end{array}$$

Here, $m^*\colon A^* \to A^* \otimes A^*$ is the dual of the multiplication $m\colon A \otimes A \to A$ of A.

In fact, we will place a slightly different structure on A. To motivate this choice, we summarize briefly some of Chen's basic formalism, and refer to the article [2] for details. Let $A(M)$ denote the DG algebra of $\mathbb{Q}$-polynomial forms on M. This is the same as the DG algebra of Sullivan PL-forms, except that cubical chains are used in [2], in place of the usual simplicial forms. We regrade $A(M)$ negatively, as above. Then $C(M) := \mathrm{Hom}(A(M), \mathbb{Q})$ is the dual space of "currents" on M. Since $A(M)$ is not of finite type, this is not a coalgebra but is, rather, a "complete" coalgebra (tensor product must be replaced with the complete tensor product). We supress this, and some other technical difficulties, in this brief discussion. Now in place of the above isomorphism, we have a *quasi-isomorphism* $\iota\colon A(M) \to C(M)$. This cannot be inverted, but it turns out that $m^* \circ \iota$ does factor through $A(M) \otimes C(M)$, as in the following diagram:

$$\begin{array}{ccc}
A(M) & & A(M) \otimes A(M) \\
\Big\downarrow \iota \, \simeq & \searrow & \simeq \Big\downarrow 1\otimes\iota \\
 & & A(M) \otimes C(M) \\
 & & \simeq \Big\downarrow \iota\otimes 1 \\
C(M) & \xrightarrow[m^*]{} & C(M) \otimes C(M).
\end{array}$$

Based on this, for our algebra A, we define a "comultiplication" $\Delta\colon A \to A \otimes A^*$, using ι as in the following diagram:

$$\begin{array}{ccc}
A & & A \otimes A \\
\Big\downarrow \iota \, \cong & \overset{\Delta}{\searrow} & \cong \Big\downarrow 1\otimes\iota \\
 & & A \otimes A^* \\
 & & \cong \Big\downarrow \iota\otimes 1 \\
A^* & \xrightarrow[m^*]{} & A^* \otimes A^*.
\end{array}$$

We note that this definition gives a degree 0 map $\Delta\colon A \to A \otimes A^*$. Furthermore, we may write, in terms of the above bases,

$$\Delta(x) = \sum_i xe_i \otimes e^i,$$

which really means $m^*\big(\iota(x)\big) = \sum_i \iota(xe_i) \otimes e^i$, and which (for future purposes) may be written

$$\Delta(x) = xe_0 \otimes e^0 + \sum_{i\geq 1} xe_i \otimes e^i \tag{†}$$

(recall that e_0 is the basis element of degree 0—the unit in A).

2. Chain Model for the loop product

Now we will describe a DG (chain) algebra $\big(A \otimes \Omega(A^*), D\big)$, whose homology gives $\mathbb{H}_*(M;\mathbb{Q})$ as an algebra.

The construction involves the *(reduced) cobar construction*, which is as follows: Suppose that (C, Δ, d) is a DG coalgebra, supplemented and with counit (think of $C = A^*$, in case (2) of the above). Let $\overline{C}$ denote the positive-degree part of C (properly, the cokernel of the supplement, or the kernel of the counit), and let $s^{-1}\overline{C}$ denote the same, but shifted down in degree by 1. Set $\Omega(C) := T(s^{-1}\overline{C})$, the tensor algebra on $s^{-1}\overline{C}$. Define a degree (-1) differential d_Ω on $\Omega(C)$ as follows:

$$d_\Omega(s^{-1}c) = -s^{-1}(dc) + \sum (-1)^{|c'|} s^{-1}c' \otimes s^{-1}c",$$

where, in C, we have $\Delta(c) = c\otimes 1 + 1\otimes c + \sum c' \otimes c"$. Then extend to elements of the form $[c_1|\cdots|c_n]$ (which is the standard way to denote the element $c_1 \otimes\cdots\otimes c_n$

in this context) as a derivation. We abuse notation here by dropping the "s^{-1}" symbol.

Remark 3. *The cobar construction is classical, and is useful for giving a chain model for the homology of the based loop space. Indeed, if C is the singular chain complex on M, structured as a coalgebra with the Alexander-Whitney diagonal, then we have $H(\Omega(C), d_\Omega) \cong H_*(\Omega M)$ as algebras.*

Now we describe Chen's construction of a chain model for the loop product, which includes a chain model for the (rational homology of the) free loop space on M.

We form $A \otimes \Omega(A^*)$, with differential of degree (-1) given as follows: For $x \in A$, and $[\gamma] \in \Omega(A^*)$, we set

$$D(x \otimes [\gamma]) = dx \otimes [\gamma] + (-1)^{|x|} x \otimes d_\Omega[\gamma] + \sum_{i \geq 1} x e_i \otimes [e^i, \gamma],$$

where the sum is over the same index as in (†) above, and the notation $[e^i, \gamma]$ denotes Lie bracket in the tensor algebra, thus $[e^i, \gamma] = [e^i|\gamma] - (-1)^{|e^i||\gamma|}[\gamma|e^i]$. We give $A \otimes \Omega(A^*)$ the obvious product, as follows. Define

$$\circ\colon \big(A \otimes \Omega(A^*)\big) \otimes \big(A \otimes \Omega(A^*)\big) \to A \otimes \Omega(A^*)$$

as $(x \otimes [\alpha]) \circ (y \otimes [\beta]) = (-1)^{|y||[\alpha]|} xy \otimes [\alpha|\beta]$. Then we have the following:

Theorem 7 (Chen, Th.4.2). *$\big(A \otimes \Omega(A^*), D\big)$ is a DG graded (chain) algebra, and the homology $H_*\big(A \otimes \Omega(A^*), D\big)$ is isomorphic* as a graded algebra *with the loop homology algebra $\mathbb{H}_*(M; \mathbb{Q})$.*

Remark 4. *In particular, the homology $H_*\big(A \otimes \Omega(A^*), D\big)$, as a graded vector space, is isomorphic with $H_{*+d}(LM; \mathbb{Q})$, i.e., with the ordinary (singular) rational homology of the free-loop space shifted down in degree by d, the dimension of the manifold M. At some level, this is a fairly plausible result, since $\Omega(A^*)$ models $H_*(\Omega M; \mathbb{Q})$, the homology of the based loop space; A models $H^*(M; \mathbb{Q})$; and $\big(A \otimes \Omega(A^*), D\big)$ is a twisting of these two models, which corresponds algebraically to a "twisting cochain" model for the free-loop fibration $\Omega M \to LM \to M$.*

Most of the remainder of this report will consist of a detailed presentation of a particular example, namely the case in which M is a sphere. Further developments show that $H_*\big(A \otimes \Omega(A^*), D\big)$ (and hence the loop homology $\mathbb{H}_*(M; \mathbb{Q})$) is a graded commutative algebra. This is a point we have assumed in the following example. We also state the fact of commutativity below, towards the end of the report.

3. The Loop Product for $M = S^d$

Take $A = H^*(S^d; \mathbb{Q})$. Recall that we regrade the cohomology algebra negatively, and so a vector space basis for A here is given by $\{e_0, e_1\}$, with $|e_0| = 0$ and $|e_1| = -d$. Recall also that we reduce A^* and then shift degrees down by 1, and so the cobar construction $\Omega(A^*)$ here is the tensor algebra on the single generator e^1 of degree $(d-1)$ (we have committed an abuse of notation by using the symbol e^1

to denote what should really be $s^{-1}e^1$, with e^1, in the first place, the basis element of A^* dual to e_1).

A typical element in $A\otimes\Omega(A^*)$ is here of the form $e_i\otimes[e^1|\cdots|e^1]$, with $i=0,1$, and we see that $A\otimes\Omega(A^*)$ is generated as an algebra by the two elements $e_1\otimes 1$ in degree $(-d)$ and $e_0\otimes[e^1]$ in degree $(d-1)$. Here we have written the unit in $T(e^1)$ as 1; it is often written as "the empty bracket" [ ].

The identities (†) here reduce to the following:

$$\Delta(e_0)=e_0\otimes e^0+e_1\otimes e^1 \qquad \text{and} \qquad \Delta(e_1)=e_1\otimes e^0;$$

which then give, from the formula for D,

$$D(e_1\otimes 1)=0$$

and

$$D(e_0\otimes[e^1])=(-1)^{-d}e_1\otimes[e^1,e^1].$$

These formulas should be extended to $A\otimes\Omega(A^*)$ as derivations.

At this point, we must separate out the odd and even cases.

3.1. $d=2n+1$**; the odd-dimensional sphere.** Here we have $|[e^1]|=2n$. In the tensor algebra, then, we have $[e^1,e^1]=[e^1|e^1]-(-1)^{2n\cdot 2n}[e^1|e^1]=0$. Hence, the differential D on $A\otimes\Omega(A^*)$ is zero, and we have $H_*\big(A\otimes\Omega(A^*),D\big)\cong A\otimes\Omega(A^*)$. Re-writing $e_1\otimes 1$ in degree $-(2n+1)$ as α, and $e_0\otimes[e^1]$ in degree $(2n)$ as β, therefore, we may write

$$H_*\big(A\otimes\Omega(A^*),D\big)\cong E(\alpha)\otimes T(\beta),$$

with $E(-)$ denoting exterior algebra and $T(-)$ denoting tensor algebra.

Remark 5. *Notice that the ordinary (singular) rational cohomology of LS^{2n+1} is given by $\wedge(a_{2n+1},b_{2n})$, where here $\wedge(-)$ denotes free graded commutative algebra (exterior on odd-degree generators and polynomial on even-degree generators). One way to see this is to observe that S^{2n+1} is rationally an H-space, and so we have a decomposition $LS^{2n+1}\simeq_{\mathbb{Q}}S^{2n+1}\times\Omega S^{2n+1}$. It is clear that the loop product and the ordinary cup product (even after shifting of degrees) are quite different structures.*

3.2. $d=2n$**; the even-dimensional sphere.** Here we have $|[e^1]|=2n-1$, and so $[e^1,e^1]=[e^1|e^1]-(-1)^{(2n-1)(2n-1)}[e^1|e^1]=2[e^1|e^1]$. Hence, the differential D on $A\otimes\Omega(A^*)$ is given on generators by $D(e_1\otimes 1)=0$ and $D(e_0\otimes[e^1])=e_1\otimes 2[e^1|e^1]$. Let us write $x=e_1\otimes 1$ in degree $(-2n)$ and $y=e_0\otimes[e^1]$ in degree $(2n-1)$ for the generators of $A\otimes\Omega(A^*)$. Then we have, for $r\geq 1$,

$$D(y^{2r})=0,$$

and for $r\geq 0$,

$$D(y^{2r+1})=y^{2r}D(y)=2xy^{2r+2}.$$

Also, for $k\geq 0$, we have

$$D(xy^k)=0.$$

It is now possible to describe the cohomology algebra $H_*\big(A \otimes \Omega(A^*), D\big)$ in a number of ways. One such is as follows. Write $\alpha = (x)$ in degree $(-2n)$, $\beta = (xy)$ in degree (-1), and $\gamma = (y^2)$ in degree $(4n-2)$. Here $(-)$ denotes "cohomology class represented by" in $H_*\big(A \otimes \Omega(A^*), D\big)$. Then we have, as algebras,

$$H_*\big(A \otimes \Omega(A^*), D\big) \cong \frac{\wedge(\alpha, \beta, \gamma)}{I},$$

with $\wedge(-)$ denoting free graded commutative algebra and I the ideal generated by $\{\alpha^2, \alpha\beta, \alpha\gamma, \beta^2\}$. Notice, in particular, that γ is *a polynomial generator* here.

Remark 6. *The contrast between the loop product algebra and the ordinary (singular) cup product algebra on the rational cohomology of the free-loop space of M is especially strong here. For M an even-dimensional sphere, cup products in the rational cohomology algebra of LM are zero. By contrast, as we see above, the loop homology algebra contains a polynomial generator.*

4. Gerstenhaber (pre-) bracket; other structures

Chen's article gives explicit formulae for the other structures listed at the start of this report, which may be used to define those structures concretely in examples. This allows for in-principle direct computation with these structures, in the rational setting. To illustrate, we mention briefly the development for the Gerstenhaber bracket, or loop bracket (not to be confused with the so-called string bracket). First, a degree $(+1)$ product, known as a pre-bracket, is defined, and then using this pre-bracket, a second product, which is the actual bracket, is defined as the commutator of the pre-bracket. This bracket (the commutator of the pre-bracket) is denoted by "$\{,\}$". It is defined on $A \otimes \Omega(A^*)$, at the chain level, is of degree $(+1)$, and passes to homology. At this stage of the development, we find the following result:

Theorem 8. *$H_*\big(A \otimes \Omega(A^*), D\big)$ is a graded commutative algebra, on which $\{,\}$ induces the structure of a degree $(+1)$ Lie algebra. The loop product and this bracket structure on $H_*\big(A \otimes \Omega(A^*), D\big)$ together satisfy $\{u, v \circ w\} = \{u, v\} \circ w \pm v \circ \{u, w\}$, for appropriate sign, and thus form a "Gerstenhaber algebra" structure.*

Skipping all details, we give the following explicit instance of a non-zero Gerstenhaber bracket. Return to the situation of the odd-dimensional sphere, $M = S^{2n+1}$. Recall from the above that we have elements $e_0 \otimes [e^1]$ in degree $2n$ and $e_1 \otimes [e^1]$ in degree (-1) in $H_*\big(A \otimes \Omega(A^*), D\big) \cong A \otimes \Omega(A^*)$. Using the formulae given in Chen's article, we find that

$$\{e_0 \otimes [e^1], e_1 \otimes [e^1]\} = -e_0 \otimes [e^1].$$

This shows, for example, that in the case in which $M = S^{2n+1}$, the Gerstenhaber bracket is not only non-zero, but is non-nilpotent.

Chen's article gives explicit formulae for the other stuctures listed at the start of this report, which in principle may thus be investigated via direct computation. The references in Chen's article include a number of papers on rational homotopy and string topology, in which the ability to model explicitly the various algebraic structures of string topology has already been established. It seems clear that explicit computations in this area rapidly become bogged down in technical details. Nonetheless, the ability to model these structures rationally holds great promise for better understanding of their general properties.

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Chen's Iterated Integrals and Higher Hochschild Chain Complex

Arturo Prat-Waldron

Given a 1-form on a smooth manifold M one obtains, via integration, smooth functions on the free loop space $\mathrm{LM} = \mathrm{Maps}(S^1, M)$ and on the path space $\mathrm{PM} = \mathrm{Maps}(I, M)$. During the 70's, Kuo Tsai Chen in a series of papers (c.f. [1, 2]) generalized this simple observation and developed a beautiful and powerful method of *iterated integration* for constructing more general differentials forms on mapping spaces like LM and PM in terms of differentials forms on the original manifold M. For instance, given an integer $k \geq 0$, consider the diagram

$$\begin{array}{ccc} \mathrm{LM} \times \Delta^k & \xrightarrow{\mathrm{ev}_k} & M^{k+1} \\ \downarrow{\scriptstyle \pi} & & \\ \mathrm{LM} & & \end{array} \tag{0.1}$$

where ev_k is the evaluation map given by

$$\mathrm{ev}_k(\gamma, (t_1, \dots, t_k)) = (\gamma(0), \gamma(t_1), \dots, \gamma(t_k))$$

for $\gamma \in \mathrm{LM}$ and $(t_1, \dots, t_k) \in \Delta^k = \{(t_1, \dots, t_k) \in \mathbb{R}^k | 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$ and π is the projection on the first factor. Given forms $\omega_0 \dots, \omega_k \in \Omega^*(M)$, their *iterated integral* is defined to be the differential form

$$\mathcal{I}(\omega_0, \dots, \omega_k) := \int \omega_0 \dots \omega_k := \pi_* \circ \mathrm{ev}_k^*(\omega_0 \otimes \cdots \otimes \omega_k) \in \Omega^*(\mathrm{LM})$$

of degree $-k + \sum_{i=0}^k \deg \omega_i$, where π_* is integration over the simplex Δ^k.

Chen studied some of the remarkable algebraic properties of this procedure and observed that it defines a map of cdga's $\mathcal{I}_* : \mathrm{CH}_*(\Omega^*(M), \Omega^*(M)) \to \Omega^*(\mathrm{LM})$, between the Hochschild chain complex of $\Omega^*(M)$ with coefficients on itself, endowed with the usual shuffle product, and the de Rham complex of the free loop space with wedge product of forms. Moreover, he showed that if M is simply-connected this map is a quasi-isomorphism.

In order to talk about objects like differential forms and the de Rham complex on mapping spaces like LM and PM, Chen developed a theory of *differentiable* *spaces*, nowadays also known as *diffeological spaces*. These are sets X endowed with a collection of *plots*, i.e. maps from open subsets of Euclidean space $U \to X$ which are considered to be smooth, satisfying certain compatibility and gluing conditions. Differential forms on X and all their operations are then defined plotwise. He also developed a method of *formal power series connections* which allowed him to prove a de Rham type-theorem for the cohomology of loop spaces and to use iterated integrals to study loop space homology. The method of formal power series connections provides as well a strong link between Chen's theory and rational homotopy theory. In particular, Chen proved the following

Theorem 9. *Let M be a simply-connected closed oriented smooth manifold of dimension n. The* iterated integral *map and its dual induce isomorphism*

$$H^*(\mathrm{LM}) \cong \mathrm{HH}_*(\Omega^*(M), \Omega^*(M)) \text{ and } H_{*+n}(\mathrm{LM}) \cong \mathrm{HH}^*(\Omega^*(M), \Omega^*(M))$$

of graded vector spaces.

Using simplicial methods, Jones [5] showed that, in the simply-connected case, the isomorphisms of Theorem 9 hold over $\mathbb{Z}$ after replacing $\Omega^*(M)$ by $C^*(M)$, the complex of integral singular cochains on M.

In previous lectures we saw that the shifted homology of the free loop space, $\mathbb{H}_*(\mathrm{LM}) := H_{*+n}(\mathrm{LM})$, endowed with the Chas-Sullivan product and bracket carries the structure of a BV algebra. On the other hand, the Hochschild cohomology $\mathrm{HH}^*(A, A)$ of any cdga A, carries the structure of a Gerstenhaber algebra and Tradler and Zeinalian [8] constructed a compatible BV-operator on $\mathrm{HH}^*(C^*(M), C^*(M))$ when M is closed, oriented. Félix and Thomas [3], using rational homotopy methods, proved that there exists an isomorphism of BV-algebras between $\mathbb{H}_*(\mathrm{LM})$ and $\mathrm{HH}^*(C^*(M), C^*(M))$ in the case of characteristic 0 coefficients and Merkulov [6] showed that a dual of the Chen's iterated integral map induces a morphism of Gerstenhaber algebras over $\mathbb{R}$, giving a nice geometric interpretation to the product and bracket.

The purpose of this talk is to present a generalization of some of these methods and results to more general mapping spaces, as appears in Section 2 of [4]. Here the authors consider an arbitrary pointed simplicial set $Y_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Set}_*$ and following Pirashvili [7] they define the *higher Hochschild chain complex* $\mathrm{CH}_*^{Y_\bullet}(A, N)$ of a cdga A with coefficients on a symmetric A-bimodule N with respect to $Y_\bullet$ to be the chain complex associated to the simplicial chain complex $\mathcal{L}(A, N) \circ Y_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Chain}$ where $\mathcal{L}(A, N) : \mathsf{Set}_* \to \mathsf{Chain}$ is the functor considered by Pirashvili from the category of pointed sets to chain complexes given as follows: For a finite pointed set $S = (s_0, \dots, s_k)$ with basepoint s_0, $\mathcal{L}(A, N)(S) := N \otimes A^{\otimes k}$ and for a map $f : S \to R = (r_0, \dots, r_l)$ between finite pointed sets

$$\mathcal{L}(A, N)(f)(n \otimes a_1 \otimes \cdots \otimes a_k) = (-1)^\epsilon \tilde{n} \otimes b_1 \otimes \cdots \otimes b_l$$

where $b_j = \prod_{i \in f^{-1}(j)} a_i$ for $j = 1, \dots, l$, $\tilde{n} = n \otimes \prod_{i \in f^{-1}(\{r_0\}), i \neq s_0} a_i$, and the sign ϵ is determined by the usual Koszul rule, which makes everything well defined and independent of choices of ordering on the sets. The functor is then extended to arbitrary sets by (co)limits. By taking $Y_\bullet = S^1_\bullet$ to be the standard simplicial representation for the circle, with only one non-degenerate 0 and 1-simplices one recovers the usual Hochschild chain complex. Moreover, one can define a *shuffle product* $sh^{Y_\bullet}$ on $\mathrm{CH}_*^{Y_\bullet}(A, A)$, which generalizes the usual one when $Y_\bullet = S^1_\bullet$ and turns $\mathrm{CH}_*^{Y_\bullet}(A, A)$ into a cdga.

On the other hand the authors consider the mapping space $\mathrm{Maps}(Y_\bullet, M)$ of maps $|Y_\bullet| \to M$ from the geometric realization of $Y_\bullet$ to M which are continuous and smooth in the interior of each non-degenerate simplex. They endow this space with a natural structure of diffeological space and generalize equation (0.1) to define a *(higher) Chen's iterated integral map* $\mathcal{I}_*^{Y_\bullet} : \mathrm{CH}_*^{Y_\bullet}(\Omega^*(M), \Omega^*(M)) \to \Omega^*(\mathrm{Maps}(Y_\bullet, M))$. They prove the following generalization of Chen's result:

Theorem 10. *The iterated integral map*

$$\mathcal{I}_*^{Y_\bullet} : (\mathrm{CH}_*^{Y_\bullet}(\Omega^*(M), \Omega^*(M)), sh^{Y_\bullet}) \to (\Omega^*(\mathrm{Maps}(Y_\bullet, M)), \wedge)$$

is a morphism of cdga's.

Moreover, if $\dim(Y_\bullet) = k$ *(i.e., k is the largest dimension of any non-degenerate simplex) and M is k-connected, then the map* $\mathcal{I}_*^{Y_\bullet}$ *is a quasi-isomorphism.*

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Differential Modules and Applications

Yves Félix

In this talk we give the basic definitions and properties of differential modules with some applications.

Let (A, d) be a differential graded algebra defined over a field $\mathbb{k}$. A differential graded module (M, d) over (A, d) is called semifree (or cofibrant) if M is free as an A-module, $M = A \otimes V$, and if V is equipped with a filtration $V = \cup_{n \geq 0} V(n)$, with $V(0) \subset V(1) \subset \cdots \subset V(n)$ such that $d(V(n)) \subset A \otimes V(n-1)$.

Semifree modules are important objects. First, for every differential graded module (M, d) there exists a semifree module (S, d) equipped with a quasi-isomorphism $\varphi : (S, d) \to (M, d)$. The module (S, d) is then called a semifree model of

M. For instance if $d = 0$ on A and M, then a free resolution of M is a semifree resolution of (M, d). Now let S be a semifree model of M, then for any module N we define $\mathrm{Ext}^n_A(M, N) = H^n(\mathrm{Hom}_A(S, N))$ and $\mathrm{Tor}^n_A(M, N) = H^n(S \otimes_A N)$. (In this talk I will not take attention to left and right modules. The good choice is given by the formulas).

If $F \to E \to B$ is a fibration with B simply connected with finite Betti numbers, then $C^*(E)$ is a $C^*(B)$-module and $H^*(F) \cong \mathrm{Tor}_{C^*(B)}(C^*(E), \Bbbk)$; On the other hand by the lifting of homotopies, $C_*(F)$ is a $C_*(\Omega B)$-module, and $H_*(E) \cong \mathrm{Tor}_{C_*(\Omega B)}(C_*(F), \Bbbk)$.

When B is simply connected with finite Betti numbers, $C^*(E)$ admits a semifree model of the form $(C^*(B) \otimes H^*(F), D)$. Here $C^*(B)$ can be replace by any quasi-isomorphic cdga. In particular over $\mathbb{Q}$, we can replace $C^*(B)$ by the minimal model of B.

Consider for instance the situation of a torus T^r acting almost freely on a finite CW complex X. Then the cohomology of the space $ET^r \times_{T^r} X$ is finite dimensional. The minimal model of BT^r is $(\wedge(x_1, \cdots, x_r), 0)$ with $|x_i| = 2$ and $ET^r \times_{T^r} X$ admits a semifree model of the form $(\wedge(x_1, \cdots, x_r) \otimes H^*(F), D)$. Recall here the TRC conjecture: When the action of T^r is almost free, then $\dim H^*(X) \geq 2^r$. This conjecture is in fact equivalent to a conjecture on semifree modules : Suppose $(M, d) = (\wedge(x_1, \cdots, x_r) \otimes V, D)$ is a semifree module and $\dim H^*(M, D) < \infty$, then $\dim V \geq 2^r$.

Semifree modules appear as an important tool in duality theory: A connected finite type dga (A, d) has a cohomology satisfying Poincaé duality of dimension m if and only if $H^*(A)$ is finite dimensional and $\mathrm{Ext}_A(\Bbbk, A) \cong \mathrm{Ext}^m_A(\Bbbk, A)$ has dimension 1. It follows that taking cohomology induces for every A-module N an isomorphism

$$\mathrm{Ext}^r_A(M, A) \cong \mathrm{Hom}(M^{m-r}, A^m)\,.$$

In particular for every submanifold $f : N \hookrightarrow M$ of dimension n, $C^*(N)$ is a $C^*(M)$-module, and the linear map $H^n(N) \to H^m(M)$ that maps fundamental class to fundamental class corresponds to a well defined element

$$f^! \in \mathrm{Ext}^{m-n}_{C^*(M)}(C^*(N), C^*(M))\,.$$

This can be taken as definition for the shriek map as the cochain level. More generally, suppose $E \to M$ is a fibration and $E' \to B$ the induced fibration along f. Let take a semifree model $S = (C^*(M) \otimes V, D)$ for $C^*(N)$, then $C^*(E) \otimes_{C^*(M)} S = C*(E) \otimes_{C^*(M)} (C^*(M) \otimes V)$ is a semifree model for $C^*(E')$ as $C^*(E)$-module and the isomorphism $\mathrm{Ext}_{C^*(M)}(S, C^*(M)) \simeq \mathrm{Ext}_{C*(E)}(C^*(E) \otimes_{C^*(M)} S, C^*(E))$ gives a uniquely defined shriek map in $\mathrm{Ext}^{m-n}(C^*(E'), C^*(E))$.

Examples of semifree models are given by Bar constructions : Let M be an A-module, then the double Bar construction $B(A; A; M) = A \otimes T(s\bar{A}) \otimes M$ is a semifree model for M.

The free loop space $LM = \mathrm{Map}(S^1, M)$ is at the center of relations between topology and geometry. If A is a rational model for M (1-connected with finite type Betti numbers), then the dual ΩA of the Bar construction on A is an algebra model for $C_*(\Omega M)$. A semifree model for $C^*(LM)$ as $C^*(M)$-module is given by the Hochschild complex $A\otimes B(A)$, that is, since A is commutative, a semifree A-module. On this model you can easily read the dual of the string product: Suppose A is a Poincaré duality model, i.e., a model that satisfies Poincaré duality at the cochain level and let $\Delta\in A\otimes A$ be the diagonal class. The multiplication by Δ, $q: A\to A\otimes A$ is a morphism of $A\otimes A$-module and a representant of the shriek map of degree m corresponding to the injection of the diagonal into $M\times M$. Denote now by $\nabla: BA\to BA\otimes BA$ the usual coproduct on the Bar construction. Then $1\otimes\nabla: A\otimes B(A)\to A\otimes B(A)\otimes B(A)$ is a morphism of A-modules, and the composition

$$(q\otimes 1)\circ(1\otimes\nabla): A\otimes BA\to (A\otimes BA)^{\otimes 2}$$

a model for the loop product.

Using Tor and Ext, we also have $H_*(LM)=\mathrm{Tor}_{\Omega A}(\Omega A,\mathbb{Q})$ where ΩA acts on itself by conjugation, and $H^*(LM)=\mathrm{Tor}_{A\otimes A}(A,A)$.

There exists an important relation between semifree resolutions of a module M over a dga A, and resolutions of $H(M)$ as module over $H(A)$. In fact if $(A\otimes V, D)$ is a semifree resolution of $\Bbbk$ with $\dim V<\infty$, then for some p we have $\mathrm{Ext}^p_{H(A)}(\Bbbk, H(A))\neq 0$. Applying this to the path space fibration $PX\to X$, with X a finite simply connected CW complex, we deduce that for some p, $\mathrm{Ext}^p_{H_*(\Omega X)}(\Bbbk, H_*(\Omega X))\neq 0$. This has been a great ingredient for the study of the structure of the algebra $H_*(\Omega X;\Bbbk)$.

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Problems on Curvature and RHT

John Oprea, Victor Turchin

The book [3] gives an idea of how rational homotopy theory is a useful tool when studying curvature constraints on Riemannian manifolds. But there are other results that beg the question whether other rational connections exist.

First, recall the famous result of Bochner (see [7]) that a closed manifold M with non-negative Ricci curvature obeys

$$b_1(M)\leq \dim(M),$$

where $b_1(M)$ denotes the first Betti number of M (with equality implying M is a flat torus). This result was refined in [5] by replacing $\dim(M)$ with $\mathrm{cat}(M)$, the Lusternik-Schnirelmann category of M ([2]). The important point in making the refinement was the use of elementary properties of Lusternik-Schnirelmann category in conjunction with the Cheeger-Gromoll Splitting Theorem: A closed manifold with non-negative Ricci curvature (and infinite fundamental group) has a finite cover $\widetilde{M} \to M$ with $\widetilde{M}$ diffeomorphic to a product $T^k \times N$ with N simply connected. In fact, this type of splitting also holds for almost non-negative Ricci curvature ([1]), so the LS category bound holds for these manifolds as well. A crucial point is that $b_1(M) \leq k \leq \mathrm{cup}(\widetilde{M})$, where $\mathrm{cup}(M)$, denotes the *cuplength* of M, the length of the longest non-trival product in rational cohomology. This elicits:

Problem. Are there other curvature conditions on closed manifolds that give product splittings up to a finite cover with the cuplength condition stated above?

The second question arises from a fundamental problem with using rational homotopy theory in geometry. Namely, the algebra of rational homotopy theory works best for nilpotent spaces (i.e. spaces with nilpotent fundamental groups that act nilpotently on higher homotopy groups), but nilpotency does not seem to be a prevailing condition in the non-simply-connected geometric world. Recently, however, it has been shown in [4] that the condition of almost non-negative *sectional* curvature on M implies that, up to a finite cover, M is nilpotent. Therefore, the methods of rational homotopy theory may be useful in studying these spaces. Furthermore, as a generalization of the Bochner result mentioned above, it was shown in [6] that almost non-negative curvature implies that a finite cover of M is the total space of a fibration over a torus of dimension $b_1(M)$. In [4] it was shown that a finite cover of M (which is nilpotent) is the total space of a *fibre bundle* over a nilmanifold with fibre simply connected. This all leads to the following.

Problem. Use rational homotopy theory to study almost non-negatively sectionally curved manifolds.

Problem. There are two topological obstructions for a closed symplectic manifold not to be Kählerian: Hard Lefschetz property and formality. Moreover it is known that neither property implies the other. Are there other topological obstructions? Or may be Hard Lefschetz together with formality implies that a closed symplectic manifold is weakly (or rationally) equivalent or homeomorphic to a Kählerian one?

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