The Hilali Conjecture for the Configuration Spaces

Hicham YAMOUL

Department of Mathematics
Faculty of Science Ain Chock Casablanca
Rational Homotopy Theory Moroccan Research Group

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Introduction

The Hilali Conjecture (or H Conjecture) is well known in Rational Homotopy Theory, it still an open problem for some kind of spaces. In this talk we try to give a short survey on some new results concerning this Conjecture for the Configuration Spaces as well as some open questions in relation in this subject, we investigate the Conjecture in the case of closed manifolds. We leave the discussion open for the case of open manifolds and unordered configuration spaces...

The Hilali Conjecture

The Hilali Conjecture predicts that :

Conjecture Topological version.

If X is an elliptic and simply connected topological space, then

$$\dim \pi_*(X) \otimes \mathbb{Q} \leq \dim H^*(X;\mathbb{Q}).$$

Conjecture Algebraic version.

If $(\Lambda V, d)$ is a 1-connected and elliptic model of Sullivan, then

$$\dim V \leq \dim H^*(\Lambda V, d).$$

Cases of Validation

Until now, this conjecture was resolved in many interesting cases : for pure spaces, these are spaces whose Euler-Poincaré characteristic is zero, for H..spaces, for nilmanifolds, for symplectic and cosymplectic manifolds, for coformal spaces whose rational homotopy is concentrated in odd degrees, and for formal spaces, Murillo and Al have extended the Hilali conjecture from pure spaces to hyperelliptic spaces. have checked the conjecture for elliptic spaces under some restrictive assumptions on the formal dimension.

Models for $F(M,2)$

We begin with some useful classical results :

Theorem (Lambrechts-Stanley)

If M is a connected closed and oriented manifold of dimension m such that $H^1(X;\mathbb{Q}) = H^2(X;\mathbb{Q}) = 0$. If (A,d) is a minimal model of M such that A is a connected Poincaré duality algebra. Then, there exists a minimal model of $F(M,2)$ of the form

$$\frac{A\otimes A}{(\Delta)},$$

where $\Delta := \sum_{i=1}^{n}(-1)^{|a_i|}a_i\otimes a_i^* \in (A\otimes A)^m$ is called the diagonal class, $(a_i)_{1\leq i\leq n}$ denotes the homogeneous basis of A and $(a_i^*)_{1\leq i\leq n}$ its dual.

If $q_1, q_2, ..., q_n$ are n distinct points in M, the rational homotopy type of $F(M,k)$ related to that of $M - Q_i$ where $Q_i = \{q_1, ..., q_i\}$ is given by the following :

Theorem

If the cohomology algebra $H^*(M;\mathbb{Q})$ requires at least two generators, then we have an isomorphism

$$\pi_*(F(M,k)) \otimes \mathbb{Q} \cong \pi_*(M \setminus Q_i) \otimes \mathbb{Q}.$$

We give the newest results of our work :

Theorem

The Hilali Conjecture holds for $\mathbb{CP}^n$ and for $F(\mathbb{CP}^n, 2)$

Theorem

If M is a smooth projective and closed variety, then M and $F(M, 2)$ verify the Hilali conjecture.

Lemma

If M is simply connected closed manifold of dimension at least 3, and if $X = M^{\circ}$ has a non-trivial rational homotopy group in dimension > 1, then $F(X, 2)$, and $F(M, 3)$ (or $F(M, k)$ for $k > 3$), is rationally hyperbolic.

Theorem

If M is closed manifold whose rational cohomology is generated by one element, then M and $F(M, 2)$ verify the Hilali conjecture.

Lemma

If M is a simply connected manifold of dimension at least 3, and has at least two linearly independent elements in its rational cohomology, then $F(M,3)$ (and in general $F(M,3+k)$, $k \geq 0$) is hyperbolic.

Theorem

If M is a closed and simply connected manifold verifying the Hilali conjecture such that $F(M,k)$ is elliptic, then $F(M,k)$ verify also the Hilali conjecture.

To enrich this work, we suggest many other directions of research that can be explored. For example :

Question 1 : M verifies the Hilali conjecture $\Rightarrow C(M, n)$ verifies it also ?

Here $C(M, n)$ denotes the unordered configurations of n distinct points in M defined by

$$C(M, n) := F(M, n)/\Sigma_n.$$

Question 2 : M verifies the Hilali conjecture $\Rightarrow M^\circ$ verifies it also ?

Question 3 : Try looking at the case where M is a compact, but not necessary closed, and simply connected manifold, and ask if the main Theorem and precedent questions still true.

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