A PRIMER ON THE GROUP OF SELF-HOMOTOPY EQUIVALENCES: A RATIONAL HOMOTOPY THEORY APPROACH

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Abstract. These notes outline the contents of the course that will be given in the 5th GeToPhyMa Summer School on “Rational Homotopy Theory and its Interaction” (July 11-21, 2016, Rabat, Morocco) celebrating Jim Stasheff and Dennis Sullivan for their respective 80th and 75th birthdays.

1. Introduction

In what follows, all spaces have the homotopy type of a CW-complex. Given a space $X$, we define the space of pointed (respectively, free) homotopy equivalences, denoted by $\text{aut}(X)$ (respectively by $\text{aut}_{\text{free}}(X)$) to be the associative topological monoid of pointed (respectively, free) self-maps of $X$ that induce homotopy equivalences. Then we define the group of self-homotopy equivalences, denoted by $\mathcal{E}(X)$, to be the group of pointed homotopy classes of elements in $\text{aut}(X)$, and $\mathcal{E}_{\text{free}}(X) = \pi_0(\text{aut}_{\text{free}}(X))$. By abuse of notation, we identify the maps between spaces and its homotopy classes.

Given a group $G$, we denote by $\text{Aut}(G)$ the automorphism group of $G$, $\text{Out}(G)$ the group of outer-automorphisms of $G$, and finally $K(G,n)$ denotes the Eilenberg-MacLane space with single non trivial homotopy group isomorphic to $G$ in dimension $n$.

We understand $p$-localisation as in [26]. Basic facts on the theory of minimal models are recalled now, we refer to [20] for more details. Let $Z$ be a graded vector space. The free commutative graded algebra on $Z$, $\Lambda Z$, is by definition the tensor product of the symmetric algebra on $Z$ even with the exterior algebra on $Z$ odd.

A commutative differential graded algebra $(A,d)$ is called a Sullivan algebra if as an algebra $A = \Lambda Z$ for some $Z$ and $Z$ admits a basis $\{x_\alpha: \alpha \in I\} \subset A$ indexed by a well-ordered set $I$ such that $d(x_\alpha) \in \Lambda(x_\beta)_{\beta < \alpha}$. The Sullivan algebra $(\Lambda Z,d)$ is called minimal if $dZ \subset \Lambda^{\geq 2}Z$. For any commutative differential graded algebra $(A,d)$ whose cohomology is connected and finite type, there is a unique (up to isomorphism) minimal algebra $(\Lambda Z,d)$ provided with a quasi-isomorphism $\phi : (\Lambda Z,d) \rightarrow (A,d)$. The differential algebra is called a minimal model of $(A,d)$. Henceforth, we will indistinctly talk about a minimal Sullivan algebra $\mathcal{M}$ or the rational space $X$ uniquely determined by $\mathcal{M}$.

We end with some notation. For a given space $X$ and $m \in \mathbb{N}$:

- the $m$-fold cartesian product of $X$ is denoted by $X \times^m$,
- the $m^{th}$-stage Postnikov piece of $X$ is denoted by $X^{(m)}$, and
• the \(m\)-connected cover of \(X\) is denoted by \(X\langle m \rangle\).

2. Pointed versus free homotopy equivalences: the case of Eilenberg-MacLane spaces

In this section we compute the space of pointed (respectively, free) homotopy equivalences of Eilenberg-MacLane spaces. Those examples allow us to illustrate how different pointed and free homotopy equivalences can be.

Recall (e.g. [36, Proposition 1.2]) that the universal Hurewicz fibration with fibre \(X\) is\n\[
X \xrightarrow{i} \text{B} \text{aut} \left( \text{free} \left( X \right) \right) \rightarrow \text{B} \text{aut} \left( \text{free} \left( X \right) \right). \tag{2.1}
\]

When considering \(X = K(G, 1)\) we can use Brown representability to reduce the long exact sequence of homotopy groups associated to (2.1), to the following exact sequence:
\[
1 \rightarrow \pi_2 \left( \text{B} \text{aut} \left( \text{free} \left( X \right) \right) \right) \rightarrow G \xrightarrow{\pi_1 i} \pi_1 \left( \text{B} \text{aut} \left( X \right) \right) \rightarrow \pi_1 \left( \text{B} \text{aut} \left( \text{free} \left( X \right) \right) \right) = \text{Out} (G) \rightarrow 1 \tag{2.2}
\]

The image of \(G\) by \(\pi_1 i : G \rightarrow \pi_1 \left( \text{B} \text{aut} \left( X \right) \right) = \text{Aut} (G)\) can be identified with the group of inner automorphisms. Therefore \(\pi_2 \left( \text{B} \text{aut} \left( \text{free} \left( X \right) \right) \right)\) must be isomorphic to \(Z(G)\), the center of \(G\). This completely describes the homotopy type of the spaces involved:

- \(\text{B} \text{aut} \left( K(G, 1) \right) = K(\text{Aut} (G), 1)\), and
- \(\text{B} \text{aut} \left( K(G, 1) \right)\) is a two-stage Postnikov piece with fundamental group isomorphic to \(\text{Out} (G)\) and second homotopy group isomorphic to \(Z(G)\).

Similar reasoning can be made for \(X = K(G, n), n > 1\). In that case \(G\) must be abelian, thus \(\text{Aut} (G) = \text{Out} (G)\) and \(Z(G) = G\), and therefore:

- \(\text{B} \text{aut} \left( K(G, n) \right) = K(\text{Aut} (G), 1)\), and
- \(\text{B} \text{aut} \left( K(G, n) \right)\) is a two-stage Postnikov piece with fundamental group isomorphic to \(\text{Aut} (G)\) and the \((n + 1)\text{th}\)-homotopy group isomorphic to \(G\).

Arguments above show how self homotopy equivalences of Eilenberg-MacLane spaces reduce to Group Theory. Thus, we can now illustrate the differences between the pointed and the free case.

For example, one of the classical problems in the study of the group of pointed self homotopy equivalences, the so-called Realizability Problem (see Section 6). Thought relevant progress has been made in the case of finite groups [11], and of algebraic linear groups [12], the general question still remains open. In contrast, if we consider the group of free self homotopy equivalences we can prove the following:

**Proposition 2.3.** Let \(G\) be a group. Then there is a space \(X\) such that \(\text{B} \text{aut} \left( \text{free} \left( X \right) \right) \simeq K(G, 1)\), and therefore \(E_{\text{free}}(X) \cong G\).

**Proof.** Given \(G\), there exists a simple group \(H\) such that \(\text{Out} (H) \cong G\) [16, Theorem 1.1]. Define \(X = K(H, 1)\). Since \(H\) is simple \(Z(H) = \{1\}\) and therefore \(\text{B} \text{aut} \left( \text{free} \left( X \right) \right) \simeq K(G, 1)\).

The technique used in the proof of the result above can be applied in order to construct interesting examples. For any space \(X\), let \(\text{cat}(X)\), \(\text{cocat}(X)\), and \(\text{TC}_k(X)\), denote the
Lusternik-Schnirelmann category, cocategory, and the \( k \)-higher topological complexity of \( X \) respectively (see [35]). Then we can prove (compare with [38, Problem 2.7]):

**Proposition 2.4.** Let \( n \in \mathbb{N} \). Then:

1. There is a space \( X_1 \) such that \( \text{cat} (B \text{aut}_\text{free}(X_1)) = n \).
2. There is a space \( X_2 \) such that \( \text{cocr}(B \text{aut}_\text{free}(X_2)) = n \).
3. Given \( k \in \mathbb{N} \), there is a space \( X_3 \) such that \( \text{TC}_k (B \text{aut}_\text{free}(X_3)) = (k - 1)n \).

**Proof.** Consider \( G_1 = \mathbb{Z}^{\oplus n} \), the direct sum of \( n \) copies of the integers, and \( G_2 = D_{2^{n+1}} \), the dihedral group of order \( 2^{n+1} \). For \( i = 1, 2 \), let \( H_i \) be a simple group such that \( \text{Out}(H_i) \cong G_i \) [16, Theorem 1.1]. If we define \( X_i = K(H_i, 1) \), then \( B \text{aut}_\text{free}(X_i) = K(G_i, 1) \).

Now, \( K(G_1, 1) = (S^1)^n \) and therefore \( \text{cat} (B \text{aut}_\text{free}(X_1)) = n \). On the other hand, \( \text{cocr} (B \text{aut}_\text{free}(X_2)) = \text{cocr} (K(D_{2^{n+1}}, 1)) = \text{nil}(D_{2^{n+1}}) = n \). Finally, define \( X_3 = X_1 \), and notice that \( B \text{aut}_\text{free}(X_3) = (S^1)^n \) is indeed an \( H \)-space. Therefore, according to [29, Theorem 1] we get that

\[
\text{TC}_k (B \text{aut}_\text{free}(X_3)) = \text{cat} (B \text{aut}_\text{free}(X_3)^{k-1}) = \text{cat} ((S^1)^{\times (k-1)n}) = (k - 1)n.
\]

\( \square \)

Observe that the groups provided by [16, Theorem 1.1] are not finitely presented, and therefore, the related Eilenberg-MacLane spaces are not of finite type. Hence, spaces given by Propositions 2.3 and 2.4 may be somehow considered as “non-natural” ones.

### 3. Spaces with finiteness conditions

The previous section indicates that we have to consider simply connected spaces \( X \), so \( \mathcal{E}(X) = \mathcal{E}_\text{free}(X) \), with some finiteness conditions on the geometrical structure. Under these hypothesis, we leave behind pure group theoretical arguments to introduce Rational Homotopy Theory arguments. To that purpose, we review the work of Wilkerson [42] and Sullivan [39].

Recall that two groups \( G \) and \( H \) are commensurable if there exists a finite string of maps

\[
G \to \ldots G_{i-1} \to G_i \leftarrow G_{i+1} \ldots \leftarrow H
\]

such that each map has finite kernel and the image has finite index. If \( G \) and \( H \) are finitely generated nilpotent groups, this is equivalent to say that the rationalizations of \( G \) and \( H \) are isomorphic, \( G(0) \cong H(0) \). Finite presentation and finite number of conjugacy classes of finite subgroups are preserved by this commensurability relation.

A matrix group \( G \subset GL(n, \mathbb{C}) \) is a linear algebraic group defined over \( \mathbb{Q} \) if it consists of all invertible matrices whose coefficients annihilate some set of polynomials with rational coefficients \( \{ P_i[X_{1,1}, \ldots, X_{n,n}] \} \) in \( n^2 \) indeterminates. Given a subring \( R \subset \mathbb{C} \), let \( G_R \) be the subgroup of elements in \( G \) that have coefficients in \( R \) and whose determinant is a unit of \( R \). Then we say that \( G_Z \) is an arithmetically defined subgroup of \( G_R \), or more briefly, an arithmetic subgroup of \( G_R \) [7]. Arithmetic subgroups of \( G_R \) are finitely presented groups, and their finite subgroups form a finite number of conjugacy classes [6, Section 5].

We can now state Sullivan-Wilkerson’s theorem [42, Theorem B], [39, Theorem (10.3)]:
Theorem 3.1 (Sullivan-Wilkerson). Let $X$ and $Y$ be simply connected finite CW-complexes. Then

1. $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ are commensurable groups if $X(0) \simeq Y(0)$.

2. $\mathcal{E}(X(0)) \cong G_\mathbb{Q}$ for a linear algebraic group $G$ defined over $\mathbb{Q}$, and $\mathcal{E}(X)$ is commensurable with an arithmetic subgroup of $\mathcal{E}(X(0))$. Hence $\mathcal{E}(X)$ is a finitely presented group and contains finitely many conjugacy classes of finite subgroups.

The techniques used by Wilkerson and Sullivan, respectively, have different flavours: while Wilkerson uses simplicial nilpotent group approximations for spaces, Sullivan’s proof relies on Postnikov systems and Sullivan minimal models. We sketch Sullivan’s approach since it also covers the case of finite type CW-complexes with finitely many homotopy groups, and the arguments will be useful in this exposition.

The first step is to describe the homotopy groups of simply connected finite CW-complexes. Recall that given $C$, a Serre class of abelian groups [37, Chapitre I], a group morphism $f: A \to B$ is called a $C$-isomorphism if $\ker f \in C$ and $\operatorname{coker} f \in C$ [37, pag. 260].

Proposition 3.2. Let $X$ be a simply connected finite CW-complex. Then for any $m \in \mathbb{N}$, $\pi_m(X)$ is a finitely generated abelian group.

Proof. Let $C$ be the Serre class of finitely generated abelian groups, and consider the constant map $f: X \to \ast$. Since both $X$ and $\ast$ are simply connected spaces and $\pi_2(f)$ is surjective, according to [37, Chapitre III, Théorème 3], $H_m(f)$ is a $C$-isomorphism for every $m \in \mathbb{N}$, if and only if $\pi_m(f)$ is so. Since $X$ is finite, $H_m(X)$ is finitely generated, and therefore $H_m(f)$ is a $C$-isomorphism for every $m \in \mathbb{N}$. Then $\pi_m(f)$ is also a $C$-isomorphism, which immediately implies $\pi_m(X) \in C$. □

We now show that computing pointed self homotopy equivalences of simply connected finite dimensional CW-complexes is equivalent to computing pointed self homotopy equivalences of finite stage Postnikov pieces. Given an space $X$ and $m \in \mathbb{N}$, the $m^{th}$-stage Postnikov piece of $X$, $X^{(m)}$, and the $m$-connected cover of $X$, $X^{(m)}$, fit in a fibration sequence

$$X^{(m)} \to X \xrightarrow{p_m} X^{(m)}.$$  \tag{3.3}

Proposition 3.4. Let $X$ be a finite dimensional CW-complex, and let $n = \dim X$. Then $\mathcal{E}(X) \cong \mathcal{E}(X^{(n)})$.

Proof. Since taking the $n^{th}$-stage Postnikov piece of a space can be done functorially, we do have a well defined group morphism

$$\psi: \mathcal{E}(X) \to \mathcal{E}(X^{(n)}).$$

Moreover, $X^{(n)}$ can be constructed from $X$ by adding $m$-cells for $m > n$, hence we may assume $X$ is the $n$-skeleton of $X^{(n)}$. Therefore, by the cellular approximation theorem, every homotopy equivalence of $X^{(n)}$ induces a homotopy equivalence of $X$ what shows that $\psi$ is surjective.

To show that $\psi$ is injective we use classical obstruction theory associated to the lifting problem for the fibration (3.3): given $f \in \mathcal{E}(X)$ the obstruction to $f$ being the unique lifting
to \( f^{(n)}p_n \) lives in \( H^m(\langle X, \pi_m(\langle X \rangle) \rangle) \) (e.g. see [25, Problem 24, pag. 420]). Since \( X \) is \( n \)-dimensional while \( X \langle n \rangle \) is \( n \)-connected, \( H^m(\langle X, \pi_m(\langle X \rangle) \rangle) = 0 \) for every \( m > 0 \). Thus \( \psi \) is surjective.

If \( X \) is a finite stage Postnikov piece, namely \( X = X^{(n)} \), whose homotopy groups are finitely generated, then \( M_n \), the minimal Sullivan model of \( X_{(0)} \), is a finitely generated DGA. We now describe \( E(X_{(0)}) \):

**Lemma 3.5.** Let \( X_{(0)} \) be a rational 1-connected space with with finitely generated minimal Sullivan model. Then \( E(X_{(0)}) = G_\mathbb{Q} \) for a linear algebraic group \( G \) defined over \( \mathbb{Q} \).

**Proof.** Let \( M_n \) be the minimal Sullivan model of \( X_{(0)} \). Then \( E(X_{(0)}) = \text{Aut}(M_n)/\text{Aut}_1(M_n) \) where \( \text{Aut}_1(M_n) \) is the subgroup of automorphisms which are homotopy equivalent to the identity. Now, \( \text{Aut}(M_n) \) is the group of rational points of an algebraic linear group defined over \( \mathbb{Q} \) since every automorphism of \( M_n \) is a \( \mathbb{Q} \)-linear map subject to the multiplicative and differential structure of \( M_n \) (which can be codified by polynomials with rational coefficients). Moreover, \( \text{Aut}_1(M_n) \) is the group of unipotent elements \( f \in \text{Aut}(M_n) \) of the form \( f = \text{Id}_{M_n} + e^{\text{iod}+\text{odi}} \) where \( i \) is a derivation of degree \(-1\). Hence \( E(X_{(0)}) \) is a quotient of rational points of algebraic groups defined over \( \mathbb{Q} \) and therefore, it is so.

We have all the ingredients for:

**Sketch of proof of Theorem 3.1.** According to Proposition 3.4, we may assume that \( X \) is a finite stage Postnikov piece, namely \( X = X^{(n)} \). We proceed by induction on \( n \).

For \( n = 2 \), since \( X \) is simply connected, \( X = K(\pi, 2) \) and \( X_{(0)} = K(\pi \otimes \mathbb{Q}, 2) \). Let \( \text{Tor}(\pi) \) denote the normal subgroup of \( \pi \) generated by torsion elements, and define the quotient \( \text{Free}(\pi) = \pi/\text{Tor}(\pi) \). Notice that since \( \pi \) is finitely generated, \( \text{Free}(\pi) \) is a finite direct sum of copies of \( \mathbb{Z} \), and \( \pi \otimes \mathbb{Q} \cong \text{Free}(\pi) \otimes \mathbb{Q} \). Therefore, \( E(X) = \text{Aut}(\pi) \) is commensurable with \( \text{Aut}(\text{Free}(\pi)) \), which is an arithmetic subgroup of \( \text{Aut}(\text{Free}(\pi) \otimes \mathbb{Q}) = E(K(\pi \otimes \mathbb{Q}, 2)) = E(X_{(0)}) \).

Assume now the result holds for \((n-1)\)th-stage Postnikov pieces. Then, following the ideas in the proof of Proposition 3.4, obstruction theory gives rise to a diagram of exact sequences:

\[
\begin{array}{ccc}
H^n(X; \pi_nX) & \longrightarrow & E(X) \longrightarrow E(X^{(n-1)}) \\
\downarrow & & \downarrow \\
H^n(X_{(0)}; \pi_nX \otimes \mathbb{Q}) & \longrightarrow & E(X_{(0)}) \longrightarrow E(X_{(0)}^{(n-1)}),
\end{array}
\]

where \( H^n(X; \pi_nX) \) is commensurable with an arithmetic subgroup of \( H^n(X_{(0)}; \pi_nX \otimes \mathbb{Q}) \), \( E(X^{(n-1)}) \) is commensurable with an arithmetic subgroup of \( E(X_{(0)}^{(n-1)}) \) by induction, and therefore \( E(X) \) is commensurable with an arithmetic subgroup of \( E(X_{(0)}) \) by [41, Proposition (3.3)].

Along the proof of Theorem 3.1 we have shown that the commensurability of \( E(X) \) is obtained via the group morphism \( E(X) \xrightarrow{\partial_0} E(X_{(0)}) \) induced by rationalization. Thus,

**Corollary 3.7.** Let \( X \) be a simply connected finite CW-complex. Then the kernel of the group morphism \( E(X) \xrightarrow{\partial_0} E(X_{(0)}) \) is finite. Therefore if \( E(X_{(0)}) \) is finite then \( E(X) \) is so.
Notice that the converse of the second statement of Corollary 3.7 does not hold in general: taking \( X = S^n \) we obtain that \( E(X) \cong \mathbb{Z}/2 \) is finite while \( E(X_{(0)}) \cong \mathbb{Q}^* \), the multiplicative group of non zero rational numbers, is not.

We finish this section by pointing out that although \( E(X) \) is finitely presented for \( X \) a virtually nilpotent finite CW-complex [14, Theorem 1.1], the thesis in Theorem 3.1.(2) cannot be extended to finite CW-complexes in general. We illustrate that fact with an easy example coming from group theory [9]:

Let \( G(n) \) be the 1-relator group with presentation
\[
G(n) = \langle a, t : t \nabla a^{-1}ta^n t^{-1}ata^{-1} \rangle,
\]
and consider \( X = K(G(n), 1) \). Since the relator \( t^{-1}a^{-1}ta^n t^{-1}ata^{-1} \) is not a proper power, then \( X \) is the presentation complex \( X \cong S^1 \cup S^1 \cup e^2 \) [17, Theorem 2.1], so \( X \) is a finite CW-complex of dimension 2. Therefore \( E(X) = \text{Aut}(G(n)) \), and \( E_{free}(X) = \text{Out}(G(n)) \). In [8] it is shown that \( \text{Out}(G(n)) = \mathbb{Z}[\frac{1}{n}] \), the additive group of rational numbers with denominator a nonnegative power of \( n \). The group \( \mathbb{Z}[\frac{1}{n}] \) is infinitely generated, since it is locally cyclic, but not cyclic. Therefore \( E_{free}(X) \) is infinitely generated and so is \( E(X) \).

4. Generating normal subgroups in \( E(X) \)

It is commonly accepted that the study of a group \( G \) should be reduced to the study of its composition series (if it exists!), that is, to the study of a finite chain of subnormal subgroups
\[
1 = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_n = G
\]
such that each factor \( G_i/G_{i-1} \) is simple. The idea is that, by an inductive process, the group theoretic properties of \( G_i \) should be related to those of \( G_{i-1} \) and \( G_i/G_{i-1} \), which are assumed to be structurally less complicated groups.

Therefore, we are interested in finding normal subgroups of \( E(X) \). The standard technique goes as follows: given a category \( C \), and a functor \( F : \mathcal{H}o\mathcal{T}op \rightarrow C \), for every space \( X \) we obtain a well defined group morphism
\[
F_1 : E(X) \rightarrow \text{Aut}_C(F(X)).
\]
The kernel of \( F_1 \), that we denote by \( E_F(X) \), is a normal subgroup of \( E(X) \). In this way, we define the following classical normal subgroups of \( E(X) \):

1. For \( C = \mathcal{G}ps \) and \( F(X) = \bigoplus_{i=1}^{m} \pi_i(X) \), then \( E^{(m)}_1(X) = E_F(X) \). When \( X \) is finite dimensional, and \( m = \text{dim}(X) \), we simply write \( E_F(X) = E^{(m)}_1(X) \).

2. For \( C = \mathcal{G}ps \) and \( F(X) = \bigoplus_{i=1}^{\infty} H_i(X; G) \) where \( G \) is an abelian group, then \( E^G_1(X) = E_F(X) \). When \( G = \mathbb{Z} \), we simply write \( E_1(X) = E^\mathbb{Z}_1(X) \).

3. For \( C = \mathcal{H}o\mathcal{T}op \), and \( F(X) = \Omega X \), then \( E_{\Omega}(X) = E_F(X) \).

4. For \( C = \mathcal{H}o\mathcal{T}op \), and \( F(X) = \Sigma X \), then \( E_{\Sigma}(X) = E_F(X) \).

We focus now on the groups \( E_{\Omega}(X) \) and \( E^\infty_1(X) \). In general, \( E_{\Omega}(X) \leq E^\infty_1(X) \), and the inclusion may be strict [22, Remarks.(a)]. Nevertheless, in our setting they behave nicely:

**Proposition 4.1.** Let \( X \) be a rational space with finitely generated Sullivan minimal model. Then \( E_{\Omega}(X) = E^\infty_1(X) \), and moreover it is a nilpotent group.
Proof. Let $\mathcal{M} = (\Lambda(x_1, \ldots, x_n), d)$ be a Sullivan minimal model for $X$, where $|x_i| \leq |x_{i+1}|$. Then a Sullivan minimal model for $\Omega X$ is $\overline{\mathcal{M}} = (\Lambda(y_1, \ldots, y_n), 0)$, where $y_i$ is the looping of the class $x_i$ (hence $|y_i| = |x_i| - 1$).

Now, define

$$\text{Aut}_\sharp(\mathcal{M}) = \{g \in \text{Aut}(\mathcal{M}): g(x_i) = x_i + \text{decomposables}, i = 1, \ldots, n\},$$

that is, $\text{Aut}_\sharp(\mathcal{M})$ is the subgroup of elements in $\text{Aut}(\mathcal{M})$ that induce the identity on the module of indecomposable elements of $\mathcal{M}$ (the homotopy groups of $X$). Therefore $\mathcal{E}_2^\infty(X)$ is the quotient of homotopy classes of elements in $\text{Aut}_\sharp(\mathcal{M})$.

Given $f \in \mathcal{E}_2^\infty(X)$, with algebraic model $g \in \text{Aut}_\sharp(\mathcal{M})$, the algebraic model for $\Omega f$, must be defined by $\overline{f}(y_i) = y_i$. Hence $f \in \mathcal{E}_\Omega(X)$.

Finally, we show that $\text{Aut}_\sharp(\mathcal{M})$ is a nilpotent group, and therefore $\mathcal{E}_2^\infty(X)$ is so. Let $m = |x_n|$, the highest degree among generators, and let $B = \{b_1, \ldots, b_s\}$ be a base of $\mathcal{M}^{\leq m}$ consisting of monomials written in standard form (i.e. $b_j = \prod_{i=1}^m x_i^{n_i(j)}$) and lexicographically ordered. Notice the elements in $B$ may not be ordered by degree, since every monomial containing $x_1$ will appear earlier in $B$ than anyone without an $x_1$. Nevertheless, every decomposable element in $B$ having the same degree as $x_i$ must show up earlier than $x_i$ since it has to contain some $x_j$ for $j < i$.

Then every element $g \in \text{Aut}_\sharp(\mathcal{M})$ is completely determined by the linear map induced by the restriction $g|_{\mathcal{M}^{\leq m}}$, and the matrix associated to $g|_{\mathcal{M}^{\leq m}}$ in base $B$ is upper triangular. Therefore we can identify $\text{Aut}_\sharp(\mathcal{M})$ with a subgroup of upper triangular matrices, and then $\text{Aut}_\sharp(\mathcal{M})$ is nilpotent. \qed

The second statement of Proposition 4.1 is a particular case of the celebrated Dror-Zabrodsky’s Theorem [15, Theorem A].

Theorem 4.2 (Dror-Zabrodsky). Let $X$ be a finite dimensional CW-complex, $n = \dim(X)$. Then $\mathcal{E}_2^{(n)}(X)$ is nilpotent.

Proof. Recall that since $X$ is a finite dimensional CW-complex, $n = \dim(X)$, then $\mathcal{E}(X) \cong \mathcal{E}(X^{(n)})$ (Proposition 3.4), and therefore we may assume that $X$ is a $n$-stage Postnikov piece.

We proceed by induction on $n$. Applying obstruction theory and following the ideas in the proof of Proposition 3.4, the fibration

$$K(\pi_n X, n) \longrightarrow X \longrightarrow X^{(n-1)}$$

gives rise to an exact sequence

$$H^n(X; \pi_n X) \xrightarrow{i} \mathcal{E}_2^{(n)}(X) \xrightarrow{p} \mathcal{E}_2^{(n-1)}(X^{(n-1)})$$

(4.3)

where $\mathcal{E}_2^{(n-1)}(X^{(n-1)})$ is nilpotent by induction. Therefore $p$ is an $\mathcal{E}_2^{(n)}(X)$-equivariant map since $\mathcal{E}_2^{(n)}(X)$ acts on $\mathcal{E}_2^{(n-1)}(X^{(n-1)})$ nilpotently by inner conjugations.

Since $\mathcal{E}_2^{(n)}(X)$ acts trivially on $\pi_n X$, an inductive argument on the Serre spectral sequence associated to the Postnikov decomposition of $X$ shows that $\mathcal{E}_2^{(n)}(X)$ acts nilpotently on $H^*(X; \pi_n X)$ (e.g. [13, pag. 360]), and the map $i$ is an $\mathcal{E}_2^{(n)}(X)$-equivariant map. So, (4.3) is
a sequence of $\mathcal{E}^{(n)}_2(X)$-groups, where kernel and cokernel are $\mathcal{E}^{(n)}_2(X)$-nilpotent. Therefore, $\mathcal{E}^{(n)}_2(X)$ is $\mathcal{E}^{(n)}_2(X)$-nilpotent (that is, nilpotent).

Indeed, Dror and Zabrodsky proved that if $X$ is a nilpotent finite dimensional CW-complex, and $G \leq \mathcal{E}(X)$ acts nilpotently on $H_j(X; \mathbb{Z})$ for $0 \leq j \leq n = \dim(X)$, then $G$ is nilpotent [15, Theorem D]. So if $X$ is a nilpotent finite dimensional CW-complex, then $\mathcal{E}^G_*(X)$ and $\mathcal{E}_*(X)$ are nilpotent groups [5, Proposition 4.9].

5. Properties of $\mathcal{E}_*(X)$

We have seen in Theorem 3.1 that the group morphism $\mathcal{E}(X) \xrightarrow{\partial_0} \mathcal{E}(X_{(0)})$ has finite kernel, and the image has finite index in an arithmetic subgroup of $\mathcal{E}(X_{(0)})$. In this section we give a better description that can be obtained when we consider the restriction of $\partial_0$ to $\mathcal{E}_*(X)$. Notice that given $P$, an arbitrary collection of prime numbers, the $P$-localization of simple spaces can be thought as the group theoretical $P$-localization at the level of homotopy groups [26, Theorem II.3B]. Moreover, the group theoretical $P$-localization, which usually refers to abelian groups, has a natural extension to the class of nilpotent groups [26, Chapter I]. Then we can prove [30, Theorem 0.1]:

**Theorem 5.1** (Maruyama). Let $X$ be a simple CW-complex and $P$ be an arbitrary collection of prime numbers. Assume that $m \geq \dim(X)$. Then the group morphism

$$\mathcal{E}^{(m)}_*(X) \xrightarrow{\partial_P} \mathcal{E}^{(m)}_*(X_{(P)})$$

induced by the $P$-localization of spaces, is the $P$-localization at the level of groups. In other words $\mathcal{E}^{(m)}_*(X)_{(P)} = \mathcal{E}^{(m)}_*(X_{(P)})$.

**Proof.** This proof follows the ideas we have already developed. Since $X$ is a finite dimensional CW-complex, say $n = \dim(X)$, then $\mathcal{E}(X) \cong \mathcal{E}(X^{(n)})$ (Proposition 3.4), and therefore we may assume that $X$ is an $n$-stage Postnikov piece.

We proceed by induction on $n$. For $n = 1$, $X = K(G, 1)$ where $G$ is abelian, and therefore $X_{(P)} = K(G \otimes \mathbb{Z}(P), 1)$. Then $\mathcal{E}^{(m)}_*(X) = \mathcal{E}^{(m)}_*(X_{(P)}) = \{1\}$ for $m \geq 1$ and the result holds.

Now, assume the result holds for $n - 1 \geq 1$. Combining diagrams (3.6) and (4.3), and considering $P$-localization of spaces instead of rationalization, we obtain

$$\begin{align*}
H^n(X; \pi_n X) &\twoheadrightarrow \mathcal{E}^{(m)}_*(X) \xrightarrow{\partial_P} \mathcal{E}^{(m)}_*(X_{(P)}) \\
H^n(X_{(P)}; \pi_n X \otimes \mathbb{Z}(P)) &\twoheadrightarrow \mathcal{E}^{(m)}_*(X_{(P)}) \xrightarrow{\partial_P} \mathcal{E}^{(m)}_*(X_{(P)}^{(n-1)})
\end{align*}$$

(5.2)

where the first and third vertical arrows are, respectively, the $P$-localization at the level of groups by the Universal Coefficient Theorem (see also [26, Theorem I.1.12 and Corollary I.1.14]) and the induction hypothesis. Finally, since $P$-localization is an exact functor [26, Theorem I.2.4], the vertical arrow in the middle is as well a $P$-localization. □

Different generalizations of Theorem 5.1 are possible. Møller, in [33], discusses the behaviour of $H_*(-; \mathbb{Z}/p)$-localization of spaces and $\text{Ext } p$-completion of nilpotent groups when
Theorem 5.3 (Félix-Murillo). Let $X$ be a simply connected finite CW-complex. Then
\[ \text{nil} \left( \mathcal{E}_t(X(0)) \right) \leq \text{cat}(X(0)) - 1 \]

**Proof.** Let $\mathcal{M} = (\Lambda V, d)$ be the minimal Sullivan model of $X$, and let $\text{Aut}_\sharp(\mathcal{M})$ the group of automorphisms of $\mathcal{M}$ which induce the identity on indecomposables (see also the proof of Proposition 4.1).

We first describe the commutator of elements in $\text{Aut}_\sharp(\mathcal{M})$. Given $g \in \text{Aut}_\sharp(\mathcal{M})$, define $l(g)$ to be the length of the shortest decomposable term obtained by applying $g$ to each generator of $\mathcal{M}$. According to [3, Corollary 3.3], if $g_i \in \text{Aut}_\sharp(\mathcal{M})$ for $i = 1, \ldots, r$, and $[g_1, g_2, \ldots, g_r]$ is the $r$-fold commutator, then
\[ l([g_1, g_2, \ldots, g_r]) \geq l(g_1) + l(g_2) + \ldots + l(g_r) - (r - 1). \]  

(5.4)

Notice that $\text{nil} \left( \mathcal{E}_t(X(0)) \right) \leq r - 1$ if and only if every $r$-fold commutator in $\text{Aut}_\sharp(\mathcal{M})$ is homotopy equivalent to the identity. Moreover, if $g \in \text{Aut}_\sharp(\mathcal{M})$ is not the identity map, then $l(g) \geq 2$.

We now describe $\text{cat}(X(0))$. Given $r \in \mathbb{N}$, let $\Phi_r: \mathcal{M} \to (\Lambda V \otimes Z(r), d)$ be the relative Sullivan model of the canonical projection $p_r: \mathcal{M} \to (\Lambda V/\Lambda^{>r}V, d)$. Then $\text{cat}(X(0)) \leq r$ if and only if $\Phi_r$ admits a section [20, Proposition 29.4]. Notice that every element in $\Lambda^{>r}V$ is a boundary in $(\Lambda V \otimes Z(r), d)$, hence if $\Phi_r$ admits a section, $\Lambda^{>r}V$ is a boundary in $\mathcal{M}$.

Finally, assume that $\text{cat}(X(0)) \leq r$. Given $g_i \in \text{Aut}_\sharp(\mathcal{M})$, for $i = 1, \ldots, r$, define $f = [g_1, g_2, \ldots, g_r]$. Then we have to show that $f \sim \text{Id}_\mathcal{M}$.

We may assume that $g_i \neq \text{Id}_\mathcal{M}$, for $i = 1, \ldots, r$, since otherwise $f = \text{Id}_\mathcal{M}$. Therefore
\[ l([g_1, g_2, \ldots, g_r]) \geq l(g_1) + l(g_2) + \ldots + l(g_r) - (r - 1) \geq 2r - r + 1 = r + 1, \]

and for each generator $x_i$ of $\mathcal{M}$ there exists $w_i \in \Lambda^{>r}V$ such that $f(x_i) = x_i + w_i$. Since $\text{cat}(X(0)) \leq r$, there exists $z_i \in \mathcal{M}$ such that $w_i = d(z_i)$, and therefore for each generator $x_i$ of $\mathcal{M}$, $f(x_i) = x_i + d(z_i)$. Then, a standard argument shows that $f \sim \text{Id}_\mathcal{M}$. \qed

The previous theorem has an integral version in [22], with techniques out of the scope of these notes. The authors there prove that for a finite $X$, $\text{nil} \left( \mathcal{E}_\Omega(X) \right) \leq \text{cat}(X) - 1$.

6. The Realisation Problem for Self-Homotopy Equivalences

In the previous sections we have described properties of the group of self-homotopy equivalences and some of its subgroups. Nevertheless, we have not shown which groups can appear as the group $\mathcal{E}(X)$ for some space $X$. This is the so-called Realizability Problem for groups of self-homotopy equivalences that we introduce here below:
Problem 6.1. Given a group $G$, find a space $X$ such that $\mathcal{E}(X) \cong G$. In that case we say that $G$ can be realized as the group of self-homotopy equivalences of a space.

Considered by many authors (e.g. see [2, 27, 28, 36]) this problem has been listed the first to solve in the collection of open problems about self-equivalences [1] (see also [19]). Apart from the case of $G = \text{Aut}(\pi)$ and $X = K(\pi, n)$, for which we have shown that $\text{Aut}(\pi) \cong \mathcal{E}(K(\pi, n))$ in Section 2, no systematic procedure was known to tackle this problem until [11]. Ad-hoc techniques are developed in literature, for example the infinite cyclic group $G = \mathbb{Z}$ is realized by a non-finite space in [27] or by a finite space in [31]. Also, finite cyclic groups (excluding a few cases of 2-torsion) are realized by finite spaces in [34]. A special mention deserves the group $G = \mathbb{Z}/2$ since it appears as the group of self-homotopy equivalences of a rational space [4, Example 5.2], pointing out the surprising appearance of a non trivial finite group in rational homotopy theory, and providing a counterexample to an old conjecture of Copeland-Shar [10, Conj. 5.8]. This fact motivates the authors in [4] to raise the following question:

Problem 6.2. Which finite groups can be realized as the group of self-homotopy equivalences of a rational space?

Of course, the realizability problem can be approached by studying distinguished subgroups, like those described in Section 4. Within this setting, Federinov-Félix have shown that every 2-solvable rational nilpotent group is realizable as $\mathcal{E}_\sharp(X)$ for some simply connected rational space $X$ [18, Theorem 1].

We devote what follows in this section to overview the solution to Problem 6.2 provided by [11]:

Theorem 6.3 (Costoya-Viruel). Every finite group $G$ can be realized as the group of self-homotopy equivalences of infinitely many (non homotopy equivalent) rational elliptic spaces.

The new insight in [11] is that although we have presented the realization problem under an homotopy-theoretic point of view, it can also be addressed into a graph-theoretic nature. In fact, in 1938 Frucht proves he following (see [23, 24]).

Theorem 6.4 (Frucht). Given a finite group $G$, there exist infinitely many non-isomorphic connected (finite) graphs $\mathcal{G}$ whose automorphism group is isomorphic to $G$.

Then, Theorem 6.3 follows from the following

Theorem 6.5. Let $\mathcal{G}$ be a finite connected graph with more than one vertex. Then there exist an elliptic space $X$ such that the group of automorphisms of $\mathcal{G}$ is realizable by the group of self-homotopy equivalences of $X$, i.e. $\text{Aut}(\mathcal{G}) \cong \mathcal{E}(X)$.

Proof. Let $\mathcal{G} = (V, E)$ be a connected graph such that $\# V > 1$, and let $X$ be a rational space whose minimal model $\mathcal{M}$ is

$$\left(\Lambda(x_1, x_2, y_1, y_2, y_3, z) \otimes \Lambda(x_v, z_v | v \in V), d\right)$$
where dimensions and differentials are

\[
\begin{align*}
|x_1| &= 8, \quad d(x_1) = 0 \\
|x_2| &= 10, \quad d(x_2) = 0 \\
|y_1| &= 33, \quad d(y_1) = x_1^3 x_2 \\
|y_2| &= 35, \quad d(y_2) = x_1^2 x_2 \\
|y_3| &= 37, \quad d(y_3) = x_1 x_2^3 \\
|x_v| &= 40, \quad d(x_v) = 0 \\
|z| &= 119, \quad d(z) = y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15} + x_2^{12} \\
|z_v| &= 119, \quad d(z_v) = x_v^3 + \sum_{(v,w) \in E} x_v x_w x_w^3.
\end{align*}
\]

Recall that every \(\sigma \in \text{Aut}(G)\) is a permutation on \(V\) such that \((v, w) \in E\) if and only if \((\sigma(v), \sigma(w)) \in E\). Therefore, \(\sigma\) induces \(f_\sigma \in \text{Aut}(M)\) given by

\[
\begin{align*}
 f_\sigma(x_1) &= x_1 \\
 f_\sigma(x_2) &= x_2 \\
 f_\sigma(y_1) &= y_1 \\
 f_\sigma(y_2) &= y_2 \\
 f_\sigma(y_3) &= y_3 \\
 f_\sigma(x_v) &= x_{\sigma(v)}, \quad v \in V \\
 f_\sigma(z) &= z \\
 f_\sigma(z_v) &= z_{\sigma(v)}, \quad v \in V.
\end{align*}
\]

Notice that given \(\sigma, \tau \in \text{Aut}(G)\), \(f_\sigma \circ f_\tau = f_{\sigma \circ \tau}\), and if moreover \(\sigma \neq \tau\), then \(f_\sigma\) and \(f_\tau\) induce different isomorphisms in the module of indecomposable elements of \(M\), hence \(f_\sigma \nmid f_\tau\). Therefore, \(\Upsilon: \text{Aut}(G) \to \mathcal{E}(X)\) given by \(\Upsilon(\sigma) = [f_\sigma]\) is a well defined group monomorphism.

A more demanding task is showing that given \(f \in \text{Aut}(M)\), there exists \(\sigma \in \text{Aut}(G)\), such that

\[
\begin{align*}
f(x_1) &= x_1 \\
f(x_2) &= x_2 \\
f(y_1) &= y_1 \\
f(y_2) &= y_2 \\
f(y_3) &= y_3 \\
f(x_v) &= x_{\sigma(v)}, \quad v \in V \\
f(z) &= z + d(m_z) \\
f(z_v) &= z_{\sigma(v)} + d(m_{z_v}), \quad v \in V
\end{align*}
\]

with \(m_z\) and \(m_{z_v}\) elements of degree 118 in \(M\). Notice that in this case, \(f \sim f_\sigma\) what shows that \(\Upsilon\) is surjective, and therefore an isomorphism.

To finish the proof, we have to show that \(M\) is elliptic. But \(M\) is elliptic if and only if \(\tilde{M}\), its associated pure Sullivan algebra, is so. We get that

\[
\tilde{M} = (\Lambda(x_1, x_2, x_v | v \in V) \otimes \Lambda(y_1, y_2, y_3, z, z_v | v \in V), \tilde{d})
\]
with
\[
\begin{align*}
\tilde{d}(x_1) &= 0 \\
\tilde{d}(x_2) &= 0 \\
\tilde{d}(x_v) &= 0 \\
\tilde{d}(y_1) &= x_1^3x_2 \\
\tilde{d}(y_2) &= x_1^2x_2^2 \\
\tilde{d}(y_3) &= x_1x_3^2 \\
\tilde{d}(z) &= x_1^{15} + x_2^{12} \\
\tilde{d}(z_v) &= x_v^3 + \sum_{(v,w) \in E} x_v x_w x_2^4.
\end{align*}
\]

Since
\[
\begin{align*}
x_1^{17} &= \tilde{d}(z x_2^3 - y_2 x_2^{10}) \\
x_2^{13} &= \tilde{d}(z x_2 - y_1 x_1^{12}) \\
[x_v^3]_4 &= \left[ - \sum_{(v,w) \in E} x_v x_w x_2^4 \right]_4 = 0
\end{align*}
\]
every element in \( H^*(\tilde{M}) \) is nilpotent, hence \( \tilde{M} \) is elliptic and therefore \( M \) is so. \qed

References

[38] S. Smith, The classifying space for fibrations and Rational Homotopy Theory, 5th GeToPhyMa Lecture Notes.