TOPOLOGICAL COMPLEXITY AND RELATED INVARIANTS

YULI B. RUDYAK

Abstract. In beginning of 2000th Michael Farber introduced and developed the notion of topological complexity and applied this to robotics (in greater detail, to robot motion planning). This is a numerical invariant of Lusternik–Schnirelmann type. In the report we survey a recent progress in the area.

1. Motion Planning Problem

Let $X$ be a topological space that we can regard as the configuration of a mechanical system. Points of $X$ are the states the system, and a continuous motion can be regarded as a continuous map (path) $\alpha : I \to X$ where $I = [0, 1]$. Here $\alpha(0)$ is the initial point and $\alpha(1)$ is the final point.

We denote by $X^I$ the space of continuous paths $I \to X$ equipped with the compact-open topology.

We assume that $X$ is path-connected, and so we can move any given point of $X$ to another given point.

A motion planning algorithm is a rule that assigns to each pair $(x, y) \in X$ a path $\alpha : I \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$.

More on motion planning see [La1991, LV2006].

To say it more formally, consider the fibration

$$\zeta_X = \{\pi : X^I \to X \times X, \quad \pi(\alpha) = (\alpha(0), \alpha(1))\}.$$ 

Now the motion planning algorithm turns out to be a map (not necessarily continuous) $s : X \times X \to X^I$ such that $\pi(s(x, y)) = (x, y)$ for all $(x, y) \in X \times X$. In other words,

$$\pi \circ s = 1_{X \times X},$$

or we can say that $s : X \times X \to X^I$ is a section of $\zeta_X$. Now we (can) interpret a motion planning algorithm as a section of the fibration $\zeta_X$.

It is easy to see that a continuous motion planning algorithm exists if and only if $X$ is contractible, see [Fa2008, Lemma 4.2]

However, usually people do not like discontinuity or, at least, want to control this. Now we describe a mathematical apparatus that helps us to manage this situation.

Date: June 19, 2016.

2010 Mathematics Subject Classification. Primary 55M30; Secondary 55R80, 55R05, 57Q40, 68T40.

Key words and phrases. Lusternik–Schnirelmann category; Schwarz genus; topological complexity; motion planning.

This paper was written for the 5th GeToPhyMa Summer School on “Rational Homotopy Theory and its Interaction” (July 11–21, 2016, Rabat, Morocco) celebrating Jim Stasheff and Dennis Sullivan for their respective 80th and 75th birthdays.
Below “fibration” denotes “Hurewicz fibration” and the base $B$ of any fibration is assumed to be path-connected CW space of finite type. All maps are assumed to be continuous unless something other is said explicitly. All functional spaces of the form $Y^X$ are assumed to be equipped with compact-open topology.

2. **Sectional Category: Definition**

2.1. **Definition** (A. Schwarz [S1966]). Given a fibration $\xi = \{p : E \to B\}$, a sectional category or Schwarz genus of $\xi$ is the least number $k$ such that there exists an open covering $A_0, A_1, \ldots A_k$ of $B$ and, for each $A_i$, a map $s_i : A_i \to E$ having $p \circ s_i = 1_{A_i}$. In other words, each $s_i$ is a (continuous) local section of $p$.

We use the notation $\text{secat}\xi$ or $\text{secat}p$ for the sectional category of $\xi$. Clearly, $\text{secat}\xi = 0$ iff $p$ has a section.

In next three sections we give three main examples of sectional category.

3. **Lusternik–Schnirelmann Category**

Let $X$ be a path-connected CW space of finite type, let $x_0 \in X$, and let $P(X) = P(X, x_0)$ be the space of paths that start at $x_0$. So, $P(X) = \{\alpha \in X^I \mid \alpha(0) = x_0\}$. Following Serre, define the fibration $\eta_X = \{p_X : P(X) \to X\}$ by setting $p_X(\alpha) = \alpha(1)$.

3.1. **Definition.** The Lusternik–Schnirelmann category of a space $X$ (denoted by $\text{cat}\ X$) is the sectional category of $\eta_X$. So, $\text{cat}\ X := \text{secat}\eta_X$.

It is clear that the space $P(X)$ is contractible. Hence, a local section $s : A \to P(X)$ of $p$ exists if and only if the subspace $A$ of $X$ is contractible in $X$. So, Lusternik–Schnirelmann category is least number $k$ such that there exists an open covering $A_0, A_1, \ldots A_k$ of $X$ where each $A_i$ is contractible in $X$.

It is worth noting one of the main applications of the Lusternik–Schnirelmann theory: Given a smooth function $f : M \to \mathbb{R}$ on a closed smooth manifold $M$, the number of critical points of is at least $1 + \text{cat}\ M$. This result turned out to be the starting point of LS theory, [LS1929, LS1934]. Currently, the LS theory is a wide area of intensive topology research.

More on Lusternik–Schnirelmann theory can be found in [CLOT2003].

4. **Topological Complexity**

Most results of this subsection is due by Farber and his collaborators, [Fa2003, Fa2004, Fa2006, Fa2008, FG2008].

4.1. **Definition** (Farber [Fa2008]). Let $X$ be a path-connected CW space of finite type. A topological complexity of a space $X$ (denoted by $\text{TC}(X)$) is the sectional category of $\zeta_X$. So, $\text{TC}(X) := \text{secat}\zeta$.

How is it related to motion planning problem? We already noticed that a continuous motion planning algorithm exists for contractible $X$ only. So, as a first step, it makes sense to consider subsets $\{A_i\}$ with $\bigcup A_i = X \times X$ and such that each $A_i$ admits a section of $\zeta_X$ over $A_i$. Note that if $\text{TC}\ X = k$ then $X \times X$ admits such a family $\{A_i\}_{i=0}^k$ (and even with open $A_i, i = 0, \ldots, k$).
However, this is not enough for our goals. In fact, the local sections \( s_i \) can overlap since, in general, we have \( A_i \cap A_i \neq \emptyset \). So, here we will not get a well-defined motion planning algorithm.

To cope with this inconvenience, it makes sense to enlarge the class of considering domains of continuity (by using not only open subsets but something more), while to keep good properties of \( \{A_i\} \)'s. This needs some expenses, such as restrictions on the configuration space \( X \), but this is enough for most applications. This program was successfully realized by Farber, who guessed to use Euclidean Neighborhood Retracts (ENRs). See [Do1995] concerning ENRs. From our point of view, the advantage of ENR is the property that, given two open subsets \( U \) and \( V \) of an ENR \( X \), the \( U \setminus V \) is also an ENR.

4.2. **Theorem** (Farber [Fa2008]). Let \( X \) be a polyhedron in \( \mathbb{R}^N \). If \( TC(X) = k \) then there exist a motion planning algorithm \( s : X \times X \to X^I \) and a partition \( X = F_0 \cup F_1 \cup \ldots F_k \) such that

- each \( F_i \) is an Euclidean Neighborhood Retract (ENR);
- for each \( i \) the restriction \( s|_{F_i} : F_i \to X \times X \) is continuous;
- \( F_i \cap F_j = \emptyset \) for \( i \neq j \).

Thus, if \( TC(X) = k \) then there exists a motion planning algorithm \( s : X \times X \to X^I \) that has \( k + 1 \) domains of continuity of \( s \), and each domain of continuity is an ENR.

5. **Higher Topological Complexity**

Let \( J_n \) denote the wedge of \( n \) copies of the closed interval \([0, 1]\), in all of which \( 0 \in [0, 1] \) is the base point. Given a space \( X \), every element \( \alpha \in X^{J_n} \) can be regarded as an \( n \)-tuple (multipath) \((\alpha_1, \ldots, \alpha_n)\) of paths in \( X \) all of which start at a common point.

Consider a fibration \( \zeta_n = \zeta_{n,X} = \{e_n : X^{J_n} \to X^n\} \), \( e_n(\alpha) = (\alpha_1(1), \ldots, \alpha_n(1)) \) where \( X \) is a path-connected CW space of finite type.

5.1. **Definition** (Rudyak [Ru2010]). A *higher topological complexity* (of order \( n \)) of a space \( X \) (denoted by \( TC_n(X) \)) is the sectional category of \( \zeta_n \). So, \( TC_n(X) := secat \zeta_n \).

Clearly, \( TC(X) = TC_2(X) \).

It is worthy to note that, given \( n \), the equality \( TC_n(X) = 0 \) holds if and only if \( X \) is contractible.

5.2. **Proposition.** If \( A \) is a retract of \( X \) then \( cat A \leq cat X \) and \( TC_n(A) \leq TC_n(X) \).

**Proof.** Obvious.  

5.3. **Remark.** In Sections 3, 4, 5, assume that \( X \) is a polyhedron. Then the values \( cat X \) and \( TC_n(X) \) do not change if, in the definitions, we assume that each \( A_i \) is an euclidean neighborhood retracts, not necessary an open subset of the base. It is proved for \( n = 2 \) in Theorem ??, and the general case can be proved similarly.

There is another interpretation of \( TC_n \). Consider a fibration

\[
\nu_n = \{u_n : X^I \to X^n\},
\]

\[
\nu_n(\alpha) = \left( \alpha(0), \alpha\left(\frac{1}{n}\right), \ldots, \alpha\left(\frac{n - 1}{n}\right), \alpha(1) \right).
\]
It is easy to check (and we will see it below) that $\zeta_n$ and $\upsilon_n$ have equal sectional categories.

Now you can see how $\text{TC}_n$ is related to motion planning theory. Indeed, $\text{TC}(X)$ is related to motion planning algorithm when a robot moves from a point to another point, while $\text{TC}_n(X)$ is related to motion planning problem whose input is not only an initial and final point but also an additional $n - 2$ intermediate points.

For more information on $\text{TC}_n$ see [BGRT2014, GLO2013, GLO15b, KL2012].

6. More on Sectional Category

Given two fibrations $\xi = \{ p : E \to B \}$ and $\xi' = \{ p' : E' \to B' \}$, consider their product

$$\xi \times \xi' = \{ p \times p' : E \times E' \to B \times B' \}.$$  

6.1. Theorem (Schwarz). We have

$$\text{secat}(\xi \times \xi') \leq \text{secat} \xi + \text{secat} \xi'.$$

In particular,

$$\text{cat}(X \times Y) \leq \text{cat} X + \text{cat} Y \text{ and } \text{TC}_n(X \times Y) \leq \text{TC}_n(X) + \text{TC}_n(Y).$$

Proof. For the proof, see [S1966, Prop. 21].

This theorem dates back to Bassi [B1937], who proved the similar inequality for Lusternik-Schnirelmann category.

Let $\xi = \{ p : E \to B \}$ be a fibration and $f : X \to B$ be a map. Consider the induced fibration $f^*\xi$ over $B$.

6.2. Proposition. We have $\text{secat} f^*\xi \leq \text{secat} \xi$.

Proof. Indeed, if $\xi$ has a local section over a subspace $A$ of $B$ then $f^*\xi$ has a local section over the subspace $f^{-1}(A)$ of $X$.\hfill \Box

Now we settle homotopy invariance of sectional category.

Consider two fibrations $\xi = \{ p : E \to B \}$ and $\eta = \{ p' : E' \to B \}$ and a commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow p & & \downarrow p' \\
B & \overset{\cong}{=} & B.
\end{array}
\]

6.3. Proposition. We have $\text{secat} \eta \leq \text{secat} \xi$. Furthermore, if $f$ is a fiber homotopy equivalence over $B$ then $\text{secat} \eta = \text{secat} \xi$.

Proof. Indeed, if $s : A \to E$ is a local section of $\xi$ over $A$ then $fs$ is a local section of $\eta$ over the same $A$. Hence, $\text{secat} \xi \geq \text{secat} \eta$. Furthermore, if $f$ is a fiber homotopy equivalence over $B$ then there exists a homotopy inverse $h : E' \to E$ over $B$ to $f$, and hence $\text{secat} \xi \leq \text{secat} \eta$. Thus, $\text{secat} \xi = \text{secat} \eta$.\hfill \Box
Consider two fibrations $\xi = \{p : E \to B\}$ and $\xi' = \{p' : E' \to B'\}$ and a commutative diagram

$$
\begin{array}{ccc}
E & \longrightarrow & E' \\
p \downarrow & & \downarrow p' \\
B & \longrightarrow & B'.
\end{array}
$$

6.4. Theorem. If $f$ is a fiber homotopy equivalence and $g$ is a homotopy equivalence then $\text{secat} \xi = \text{secat} \xi'$.

Proof. The bundle map $\xi \to \xi'$ can be decomposed as $\xi \to g^*\xi' \to \xi'$ where the correcting map $\xi \to g^*\xi'$ yields the identity map $1_B$ on bases. Now, $\text{secat} \xi = \text{secat} g^*\xi'$ by Proposition 6.3, while $\text{secat} g^*\xi' \leq \text{secat} \xi'$ by Proposition 6.2. Hence $\text{secat} \xi \leq \text{secat} \xi'$. Now, since $f$ is fiber homotopy equivalence, we can find a fiber homotopy equivalence $h : E' \to E$ that is fiber homotopy inverse to $f$ and prove that $\text{secat} \xi' \leq \text{secat} f$. □

6.5. Corollary. The invariant $\text{cat} X$, as well as $\text{TC}_n(X)$, is a homotopy invariant. □

6.6. Remark. Now you see the above-mentioned equality $\text{secat} e_n = \text{secat} u_n$. Indeed, both maps $e_n : X^J_n \to X^n$ and $v_n : X^i \to X^n$ are homotopy equivalent to the diagonal $d - n : X \to X^n$, and so the fibration $e_n$ and $v_n$ are fiber homotopy equivalent, like in Theorem 6.4. Thus, $\text{secat} e_n = \text{secat} u_n$.

7. Several Inequalities

7.1. Proposition. For any fibration $\xi = \{p : E \to B\}$ we have $\text{secat} \xi \leq \text{cat} B$.

Proof. This holds because, for any subset $A$ of $B$ that is contractible in $B$, the fibration $\xi$ admits a local section over $A$. □

7.2. Theorem. For every $n$ we have $\text{cat} X^{n-1} \leq \text{TC}_n(X) \leq \text{cat} X^n \leq \text{TC}_{n+1}(X)$.

Proof. The second inequality follows from the Proposition 7.1. For the inequality $\text{cat} X^{n-1} \leq \text{TC}_n(X)$, see [BGRT2014, Proposition 3.1]. (Note that Farber [Fa2008] considered the case $n = 2$.) □

7.3. Corollary. $\text{TC}_n(X) \leq \text{TC}_{n+1}(X)$. □

7.4. Open Problem. Does there exist a non-contractible space $X$ such that $\text{TC}_n(X) = \text{TC}_{n+1}(X)$?

7.5. Proposition. If $X$ is not contractible then $\text{TC}_n(X) \geq n - 1$

Proof. This is proved in [Ru2010, Proposition 3.5]. We present one more proof. Ganea and Hilton [GH1959] proved that $\text{cat} X^n \geq n$ for $X$ non-contractible. Now the proposition follows from the inequality $\text{TC}_n(X) \geq \text{cat} X^{n-1}$. □

7.6. Theorem. If $G$ is a path-connected $H$-space (e.g. a topological group) then $\text{TC}_n(G) = \text{cat} G^{n-1}$.

Proof. For a topological group and $n = 2$ this is proved in [Fa2004], for $n > 2$ see [BGRT2014]. For arbitrary $H$-spaces see [LuSh2013]. □
Note also the following difference between cat and TC.

We know that \( \text{cat}(X \vee Y) = \max\{\text{cat } X, \text{cat } Y\} \). This is not true for TC. Namely, \( \text{TC}(S^1) = 1 \) while \( \text{TC}(S^1 \vee S^1) = 2 \).

**7.7. Theorem** (Dranishnikov [Dr2014]). Let \( X, Y \) be two absolute neighborhood retracts. Then
\[
\max\{\text{TC}(X), \text{TC}(Y), \text{cat}(X \times Y)\} \leq \text{TC}(X \vee Y) \leq \text{TC}(X) + \text{TC}(Y) + 3.
\]

We know that if \( \widetilde{X} \to X \) is a cover map that \( \text{cat } \widetilde{X} \leq \text{cat } X \). This is not true for TC.

**7.8. Example** (Dranishnikov [Dr2014]). Let \( X = S^3 \times S^3 \vee S^1 \), and let \( \widetilde{X} \) be the universal cover of \( X \). Then \( \text{TC}(X) \leq 3 \) while \( \text{TC}(\widetilde{X}) \geq 4 \).

### 8. Topological Complexity of Discrete Groups

Let \( \pi \) be a discrete group. Define \( \text{TC}(\pi) := \text{TC}(B\pi) \) where \( B\pi \) denotes the classifying space of \( \pi \). Since the classifying space (assumed to be CW) is defined uniquely up to homotopy equivalence, and because of the homotopy invariance of TC, the invariant \( \text{TC}(\pi) \) is well-defined.

Note that the invariant \( \text{cat } \pi := \text{cat } (B\pi) \) has a known purely group-theoretical description. In fact, \( \text{cat } \pi \) is equal to the cohomological dimension \( \text{cd}(\pi) \) of \( \pi \), see [EG1957] for \( \text{cat } \pi \neq 2 \) and [St1968, Sw1969] for \( \text{cat } \pi = 2 \).

The situation for TC looks more complicated. We know that \( \text{cat } X \leq \text{TC}(X) \leq \text{cat } (X \times X) \) for all \( X \). The following proposition tells us that, in the class of \( K(\pi, 1) \)-spaces, the above mentioned inequality gets no new bounds. In other words, we have examples of two group \( \pi, \pi' \) such that \( \text{cat } \pi = \text{cat } \pi' \) while \( \text{TC}(\pi) \neq \text{TC}(\pi') \).

**8.1. Proposition** ([Ru2016]). For every natural \( k \) and every natural \( l \) with \( k \leq l \leq 2k \) there exists a discrete group \( \pi \) such that \( \text{cat } \pi = k \) and \( \text{TC}(\pi) = l \). In fact, we can put \( \pi = \mathbb{Z}^k \ast \mathbb{Z}^{l-k} \).

Because of the proposition, the following problem turns out to be essential.

**8.2. Open Problem** (Farber). Describe \( \text{TC}(\pi) \) in purely group-theoretical terms.

### 9. Homotopy-Theoretical Description of Sectional Category

Recall that the *join* \( X \star Y \) of two CW spaces \( X \) and \( Y \) is defined to be a quotient space \( (X \times I \times Y)/R \), where \( R \) is the equivalence generated by the equivalences \( (x, 0, y_1) \sim (x, 0, y_2) \) for all \( x \in X, y_1, y_2 \in Y \) and \( (x_1, 1, y) \sim (x_2, 1, y) \) for all \( x_1, x_2 \in X, y \in Y \).

Note also that \( X \star Y \) is the double mapping cylinder of the diagram
\[
\begin{array}{ccc}
X & \xleftarrow{p_1} & X \times Y & \xrightarrow{p_2} & Y.
\end{array}
\]

More generally, given two maps \( f : X \to Z \) and \( g : Y \to Z \), the *fiberwise join* of \( f \) and \( g \) is defined to be the map \( f \ast g : X \star_Z Y \to Z \) where
\[
X \star_Z Y = \{[x, t, y] \in X \star Y \mid f(x) = g(y)\}.
\]
and \((f \ast g)(x, t, y) = f(x)\).

Note that \(X \ast_Z Y\) turns into \(X \ast Y\) if \(Z\) is the point.

We can iterate the join construction. In particular, given a fibration \(\xi = \{p : E \to B\}\) we can form the fibration
\[
\xi^k := \{p \ast \cdots \ast p : \underbrace{E_B \ast \cdots \ast E_B}_{k \text{ times}} \to B\}.
\]

9.1. Theorem (Schwarz[1966]). The fibration \(\xi^k\) has a section iff \(\text{secat} \xi < k\). \(\square\)

In other words, \(\text{secat} \xi\) is the least value \(m\) such that \(\xi^{*(m+1)}\) admits a section.

For example, let \(\eta_X : p_X : PX \to X\) be the Serre path fibration. So, we have the iterated fiber join \((\eta_X)^m\) over \(X\).

9.2. Corollary (Ganea, Schwarz). We have: \(\text{cat} X < m\) iff \((\eta_X)^m\) has a section. \(\square\)

10. Dimension–Connectivity Relations

10.1. Proposition (Serre). Given a fibration \(\xi = \{p : E \to B\}\), for any two points \(b, b' \in B\) the fibers \(p^{-1}(b)\) and \(p^{-1}(b')\) are homotopy equivalent. \(\square\)

10.2. Definition. The homotopy fiber of a fibration \(\xi\) is defined to be the homotopy equivalence class of \(p^{-1}(b), b \in B\). The notion is well-defined because of Proposition 10.1.

10.3. Example. Consider the Serre path fibration \(\eta_X = \{p_X : PX \to X\}\). It is clear that \(p_X^{-1}(x_0)\) is the loop space \(\Omega(X, x_0)\). So, the homotopy fiber of \(\eta_X\) is (the homotopy class of) \(\Omega(X, x_0)\).

It is worthy to note that, generally, \(p_X^{-1}(x_1)\) for \(x_1 \neq x_0\) is homeomorphic neither to \(\Omega(X, x_0)\) nor to \(\Omega(X, x_1)\).

10.4. Theorem (Schwarz[1966]). Given a fibration \(\xi = \{p : E \to B\}\), take a point \(b \in B\) and put \(F = p^{-1}(b)\). Put \(\dim B = d\) and assume that \(\pi_k(F) = 0\) for \(k < s\) (i.e., the space \(F\) is \((s-1)\)-connected). Then
\[
\text{secat} \xi < \frac{d + 1}{s + 1}.
\]

10.5. Remark. It follows from Proposition 10.1 that the condition \(\pi_k(F) = 0\) does not depend on choice of \(b\), and we can express the condition as follows: the homotopy fiber of \(\xi\) is \((s-1)\)-connected.

10.6. Example. Let \(X\) be a \((s-1)\)-connected space with \(s > 0\) (that tells us that \(X\) is path connected). Put \(d = \dim X\). Evaluate \(\text{cat} X\). Recall the fibration \(\eta_X = \{p : PX \to X\}\). The homotopy fiber of \(\eta_X\) has the homotopy type of \(\Omega X\) (the loop space of \(X\)). Note that \(\Omega X\) is \((s-2)\)-connected. Thus, because of the Theorem 10.4, we have
\[
\text{cat} X = \text{secat}(\eta_X) < \frac{d + 1}{s}, \quad \text{or} \quad \text{cat} X \leq \frac{d}{s}.
\]
Similarly, \(\text{TC}(X) \leq 2d/s\) and \(\text{TC}_n(X) \leq dn/(s)\).
11. Theorem. We have the following estimate:

such that there exist \( u, \ldots, u_k \in \tilde{H}^*(X; R) \) with \( u \cup \cdots \cup u_k \neq 0 \).

11.1. Definition. Given a path-connected space \( X \) and a commutative ring \( R \), define the \textit{cup-length of} \( X \) with coefficients in \( R \) (denoted by \( \text{cl}_R(X) \)) to be the maximal number \( k \) such that there exist \( u_1, \ldots, u_k \in \tilde{H}^*(X; R) \) with \( u_1 \cup \cdots \cup u_k \neq 0 \).

11.2. Theorem. We have the following estimate: \( \text{cl}_R(X) \leq \text{cat} X \).

Proof (sketch). The idea of the proof is quite simple. Let \( \text{cat} X = n \). Take a covering \( \{A_0, A_1, \ldots, A_n\} \) by open and contractible in \( X \) sets. Suppose that \( \text{cl}_R(X) = k > n \) and take \( u_1, \ldots, u_k \) with \( u_1 \cup \cdots \cup u_k \neq 0 \). Now,

\[
    u_1|_{A_1} = 0, \ldots, u_n|_{A_n} = 0, u_{n+1}|_{A_0} = 0.
\]

Thus, \( u_1 \cup \cdots \cup u_{n+1} = (u_1 \cup \cdots \cup u_{n+1})|_X = u_1|_{A_1} \cup \cdots \cup u_{n+1}|_{A_0} = 0 \).

This is a contradiction. \( \square \)

11.3. Examples. 1. It is easy to see that \( \text{cl}_\mathbb{Z}(T^n) = n \). Hence, \( \text{cat} T^n \geq n \). Together with the inequality \( \text{cat} T^n \leq \dim T^n = n \) we conclude that \( \text{cat} T^n = n \).

2. We have \( H^*(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2[u]/u^{n+1}, \dim u = 1 \). Hence, \( \text{cl}_{\mathbb{Z}/2} = n \). So, \( \text{cat}(\mathbb{R}P^n) \geq n \), and we have \( \text{cat}(\mathbb{R}P^n) = n \) because \( \text{cat}(\mathbb{R}P^n) \leq \dim \mathbb{R}P^n = n \).

3. We have \( H^*(\mathbb{C}P^n) = \mathbb{Z}/[u]/(u^{n+1}), \dim u = 2 \). Hence, \( \text{cl}_{\mathbb{Z}}(\mathbb{C}P^n) = n \). So, \( \text{cat}(\mathbb{C}P^n) \geq n \), and \( \text{cat}(\mathbb{C}P^n) \leq \dim(\mathbb{C}P^n)/2 = 2n/2 = n \); the denominator 2 appears because \( \mathbb{C}P^n \) is simply connected. Thus, \( \text{cat}(\mathbb{C}P^n) = n \).

4. For completeness, note that \( \text{cat} S^m = 1 \) for all \( m > 0 \). Indeed, \( \text{cat} S^m > 0 \) because \( S^m \) is not contractible, while \( S^m \) can be covered by two contractible spaces (discs).

We leave it to the reader to check that \( \text{cat} S = 2 \) for all closed surfaces except \( S^2 \).

Now we pass to the general situation. Consider a fibration \( \xi = \{p : E \to B\} \).

11.4. Definition. Define the \textit{cup-length of} \( \xi \) with coefficients in \( R \) (denoted by \( \text{cl}_R(\xi) \)) to be the maximal number \( k \) such that there exist

\[
    u_1, \ldots, u_k \in \text{Ker}\{\tilde{H}^*(B; R) \to \tilde{H}^*(E; R)\}
\]

with \( u_1 \cup \cdots \cup u_k \neq 0 \).

11.5. Theorem (Schwarz). We have the following estimate: \( \text{cl}_R(\xi) \leq \text{secat} \xi \). \( \square \)

11.6. Remarks. 1. In a special case of the Serre fibration \( \eta_X = \{p : P_X \to X\} \) the space \( P_X \) is contractible. Therefore \( \text{cl}(\eta_X) = \text{cl}(X) \).

2. In the definition and application of cup-length, we can consider more general situation: to consider \( u_i \in H^*(B; A_i) \) for arbitrary coefficient groups (and even local coefficient systems) \( A_i \) with \( u_1 \cup \cdots \cup u_k \in H^*(B; A_1 \otimes \cdots \otimes A_k) \).

Consider the fibration $\zeta_n = \{e_n : X^{J_n} \to X^n\}$ and the homotopy commutative diagram

$$
\begin{array}{ccc}
X^{J_n} & \longrightarrow & X \\
\downarrow e_n & & \downarrow d \\
X^n & \longrightarrow & X^n
\end{array}
$$

where $d$ is the iterated diagonal map, $d_n(x) = (x, \ldots, x)$ and the top map has the form $(\alpha_1, \ldots, \alpha_n) \mapsto \alpha_1(0)$.

12.1. Definition (Farber [Fa2008]). A cohomology class $u \in H^*(X^n; R)$ is called a zero-divisor if $d_n^*(u) = 0$.

So, the cup-length of $\zeta_n$ can be reformulated as the maximal number $k$ such that there exist zero-divisors $u_1, \ldots, u_k$ with $u_1 \sim \cdots \sim u_k \neq 0$.

12.2. Theorem (Farber [Fa2008]). We have $TC(S^{2k+1}) = 1$ and $TC(S^{2k}) = 2$.

So, we have a remarkable contrast with the claim $\text{cat} S^m = 1$.

Proof. Prove that $TC(S^{2k-1}) = 1$. We must construct two continuous sections $s_i : A_i \to (S^{2k-1})^I$ where $A_0 \cup A_1 = S^{2k-1} \times S^{2k-1}$. In other words, cover $S^{2k-1} \times S^{2k-1}$ by subsets $A_0, A_1$ such that every two points $(x, y) \in A_i, i = 1, 2$ can be joined by an arc in $S^{2k-1}$, and the arc depends on $x, y$ continuously in each $A_i$. Put $A_0 = \{(x, y) | x, y \in S^{2k+1} \text{ with } x \neq -y\}$, and join $x$ to $y$ by the shortest geodesic.

Put $A_1 = \{(x, y) | x = -y\}$. To construct $s_1$, recall that $S^{2k-1}$ possesses a non-vanishing continuous tangent vector field $v$. Now, given $x \in S^{2k-1}$, join $x$ to $y = -x$ by the geodesic. Prove that $TC(S^{2k}) = 2$. Take a generator $u \in H^2(S^{2k})$ and consider the element

$$v := u \otimes 1 - 1 \otimes u \in H^2k(S^{2k} \times S^{2k}) = H^2k(S^{2k}) \otimes H^2k(S^{2k})$$

Note that $v$ is a zero-divisor. Indeed, $d_n^*(u \otimes 1) = u = d_n^*(1 \otimes u)$, and so $d_n^*(v) = 0$. Furthermore, since dim $v = 2k$ is even, we have $v \sim v = -2u \otimes u \neq 0$. Indeed

$$v \sim v = ((u \otimes 1) - (1 \otimes u)) \sim ((u \otimes 1) - (1 \otimes u))$$
$$= -(u \otimes 1) \sim (1 \otimes u) - (1 \otimes u) \sim (u \otimes 1)$$
$$= -2u \otimes u, \text{ since dim}(1 \otimes u) = \dim(u \otimes 1) \text{ is even.}$$

So, $v \sim v \neq 0$, and hence $\text{cl}(\zeta_{S^{2k}}) \geq 2$. So, $TC(S^{2k}) \geq 2$. Furthermore, $TC(S^{2k}) \leq 2$ because of the dimension-connectivity relation, and thus $TC(S^{2k}) = 2$.

12.3. Theorem (Rudyak [Ru2010]). We have $TC_n(S^{2k+1}) = n - 1$ and $TC(S^{2k}) = 2n$.

Proof. First, we prove that $TC_n(S^{2k+1}) = n - 1$. Consider a unit tangent vector field $v$ on $S^{2k+1}$. Given $x, y \in S^{2k+1}, y = -x$, denote by $[x, y]$ the path determined by the geodesic semicircle joining $x$ to $y$ and such that the $v(x)$ is the direction of the semicircle at $x$.

If $x \neq -y$, denote by $[x, y]$ the path determined by the shortest geodesic from $x$ to $y$.

Determine a (non-continuous) function

$$\varphi : (S^{2k+1})^n \to (S^{2k+1})^J, \quad \varphi(x_1, \ldots, x_n) = \{[x_1, x_1], \ldots, [x_1, x_n]\}$$
For each \( j = 0, \ldots, n - 1 \) consider the submanifold (with boundary) \( U_j \) in \( S^{2k+1} \) such that each \( n \)-tuple \((x_1, \ldots, x_n)\) in \( U_j \) has exactly \( j \) antipodes to \( x_1 \). Then \( \varphi_{|U_j} : U_j \to (S^{2k+1})^J_U \) is a continuous section of \( \zeta_{n, S^{2k+1}} \). Hence, \( TC_n(S^{2k+1}) \leq n - 1 \), and thus \( TC_n(S^{2k+1}) = n - 1 \).

Now we prove that \( TC_n(S^{2k}) = n \). Take a generator \( u \in H^{2k}(S^{2k}) \) and consider the element

\[
w = \left( \sum_{i=1}^{n-1} 1 \otimes \cdots \otimes 1 \otimes \text{(ith place)} \otimes 1 \cdots \otimes 1 \right) - 1 \otimes \cdots \otimes 1 \otimes (n - 1)u
\]

Note that \( w \) is a zero-divisor class. Furthermore, \( w^{n-1} = (1 - n)n!(u \otimes \cdots \otimes u) \) (since \( \dim S^{2k} \) is even). Hence \( TC_n(S^{2k}) \leq n \) by the cup-length argument, and thus \( TC_n(S^{2k}) = n \) by the dimension-connectivity argument.

Note also the following fact.

12.4. Theorem (Grant-Lupton-Oprea\cite{GLO2013}). If \( TC(X) = 1 \) then \( X \cong S^{2n+1} \).

Generally, for \( n > 2 \) we do not know if the equality \( TC_n(X) = n - 1 \) implies that \( X \cong S^{2k+1} \). This is true for many cases (for example, if \( X \) is a simply connected space), but it is an open question in general.

12.5. Open Problem. Does the equality \( TC_n(X) = n - 1 \) imply the homotopy equivalence \( X \cong S^{2k+1} \)?

13. Surfaces

In this section, for brevity we wrote \( xy \) for \( x \sim y \) for \( x, y \in H^*(X) \).

For orientable closed surfaces, we have the following

- \( TC(S^2) = 2 \)
- \( TC(T^2) = \text{cat} T^2 = 2 \), since \( T^2 \) is a group.
- \( TC(S_g) = 4 \) for any closed orientable surface \( S_g \) of genus \( g > 1 \) \cite{Fa2008}. Indeed, take \( a_1, a_2, b_1, b_2 \in H^1(S_g) \) such that

\[
a_1a_2 = b_1b_2 = a_1b_2 = a_2b_1 = a_1^2 = a_2^2 = b_1^2 = b_2^2 = 0
\]

and that \( a_1b_1 = a_2b_2 \in H^2(S_g) \) is a non-zero element. Now, we can see the non-zero product of zero-divisors

\[
\prod_{i=1}^2 (a_i \otimes 1 - 1 \otimes a_i)(b_i \otimes 1 - 1 \otimes b_i).
\]

Hence, \( TC(S_g) \geq 4 \), and we get the equality \( TC(S_g) = 4 \) by the dimension-connectivity relation.

Now about non-orientable surfaces \( N_g \) (\( N_1 = \mathbb{R}P_2 \), \( N_2 \) is the Klein bottle).

First, \( TC(\mathbb{R}P^2) = 3 \). Indeed, we have \( H^2(\mathbb{R}P^2) = \mathbb{Z}/2[u]/u^3 = 0 \), and \( u \otimes 1 + 1 \otimes u \) is a zero-divisor in \( H^*(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}/2) \). Now

\[
(u \otimes 1 + 1 \otimes u)^3 = u^2 \otimes u + u \otimes u^2 \neq 0.
\]

So, \( TC(\mathbb{R}P^2) \geq 3 \). Now we can see the inequality \( TC(\mathbb{R}P^2) \leq 3 \) geometrically, \cite{Fa2008}.

We do not know \( TC(N_g) \) for \( g = 2, 3 \). (It is clear, however, that \( 3 \leq N_g \leq 4 \)). Recently it was proved (Dranishnikov \cite{Dr2015}) that \( TC(N_g) = 4 \) for \( g > 3 \).
Concerning $T_{C_n}$.

We have $T_{C_n}(S^2) = n$

We have $T_{C}(T^2) = \text{cat}((T^2)^{n-1}) = 2n - 2$, since $T^2$ is a group.

We note the following surprising fact:

13.1. **Theorem** ([GGGL2015]). If $n > 2$ then $T_{C_n}(F) = 2n$ for all other surfaces $F$, no matter whether $F$ is orientable or not.

Why surprising? Two exciting moments.

First, $T_{C}(\mathbb{R}P^2) = 3$ while $T_{C_n}(\mathbb{R}P^2) = 2n$ for $n > 2$.

Second, for the Klein bottle $K$ (and $K\# \mathbb{R}P^2$), we do not know the value of $T_{C}(K)$ while we know that $T_{C_n}(K) = 2n$ for $n > 2$.

14. **Some High-Dimensional Examples**

14.1. **Theorem** ([BGRT2014]). For any path-connected space $X$ and positive integers $n$ and $k$ we have $\text{cl}(\zeta_{n,X\times S^k}) \geq \text{cl}(\zeta_{n,X}) + n - 1$. This inequality can be improved to $\text{cl}(\zeta_{n,X\times S^k}) \geq \text{cl}(\zeta_{n,X}) + n$ provided $k$ is even and $H^*(X)$ is torsion-free.

14.2. **Corollary.** $T_{C_n}(S^{k_1} \times S^{k_2} \times \cdots \times S^{k_m}) = m(n - 1) + l$ where $l$ is the number of even dimensional spheres.

*Proof.* This follows from theorems 6.1 and 14.1.

14.3. **Corollary.** $T_{C_n}(T^k) = k(n - 1)$.

*Proof.* This is a consequence of either Corollary 14.2 or Theorem 7.6.

14.4. **Theorem** ([BGRT2014]). Let $X$ be a CW complex of finite type, and $R$ a principal ideal domain. Take $u \in H^d(X; R)$ with $d > 0$, $d$ even, and assume that the $n$-fold iterated self $R$-tensor product $u^m \otimes \cdots \otimes u^m \in (H^{md}(X; R))^\otimes_n$ is an element of infinite additive order. Then $T_{C_n}(X) \geq mn$.

14.5. **Corollary.** For every closed simply connected symplectic manifold $M^{2m}$ we have $T_{C_n}(M) = mn$.

Note also the following nice and interesting result.

14.6. **Theorem** (Farber-Tabachnikov-Yuzvinsky [FTY2003]). For any $n \neq 1, 3, 7$ the number $T_{C}(\mathbb{R}P^n)$ is one less than the smallest $k$ such that the $\mathbb{R}P^n$ admits an immersion into $\mathbb{R}^{k-1}$. Furthermore, for $n = 1, 3, 7$ we have $T_{C}(\mathbb{R}P^n) = n$.

15. **The Sequence** $\{T_{C_n}(X)\}_{n=2}^{\infty}$ **as an Invariant of** $X$.

When the concept of higher topological complexity appeared, the following question arose: Do the invariants $T_{C_n}(X)$ give us really more information on $X$ than $T_{C}(X)$? In other words, is it true or not that the sequence $\{T_{C_n}(X)\}$ is completely determined by $T_{C}(X)$? The following example shows that the sequence $\{T_{C_n}(X)\}$ contains more information of $X$ than merely $T_{C}(X)$.

15.1. **Example.** $T_{C}(S^2) = T_{C}(T^2) = 2$, $T_{C_n}(S^2) = n$, $T_{C_n}(T^2) = 2n - 2$. 
More generally, what can we say about the behavior of the sequence \( \{TC_n\} \)? As an example, we note the following fact.

**15.2. Proposition.** For every CW space \( X \) of finite type, the sequence \( \{TC_n(X)\} \) has linear growth with respect to \( n \).

*Proof.* This follows from the inequalities \( TC_n(X) \leq \text{cat } X^n \leq n \text{ cat } X \). □

Given \( X \), we (can) introduce the power series \( \sum_{n=0}^{\infty} TC_{n+2}(X)z^n \) and ask about analytical properties of them.

**15.3. Example.** For \( X = S^{2k+1} \) we have

\[
\sum_{n=0}^{\infty} TC_{n+2}(S^{2k+1})z^n = \sum_{n=0}^{\infty} (n+1)z^n = (1 - z)^{-2} + 2(1 - z)^{-1}.
\]

Generally, we have the following fact

**15.4. Proposition.** For every CW space \( X \) of finite type, the radius of convergence of the series \( \sum TC_{n+2}(X)z^n \) is equal to 1.

*Proof.* Put \( a_n = TC_n(X) \) and note that \( a_n \leq \text{cat } X^n \leq n \text{ cat } X \). Hence \( a_n/n \leq \text{cat } X \). Hence the upper limit \( \lim(a_n/n) \) exists, and it is positive because \( \{a_n\} \) is an increasing sequence of positive numbers. This implies that

\[
\lim \frac{a_n}{a_{n+1}} = \lim \frac{n+1}{n} = 1.
\]

And an open question.

**15.5. Open Problem.** Do the power series \( \sum \text{cat}(X^n)z^n \) and \( \sum TC_{n+2}(X)z^n \) represent rational functions?

### 16. Monoidal topological complexity

Consider robot motion planning with the following property: if the initial position of a robot in the configuration space \( X \) coincides with the terminal position, then the algorithm keeps the robot still. This leads to the notion of monoidal topological complexity, \([?, ?]\).

**16.1. Definition.** For a CW space \( X \), the monoidal topological complexity \( TC^M(X) \) is the least number \( m \) such that there exists a cover of \( X \times X \) by \( m+1 \) open subsets \( A_i \), \( i = 0, 1, \ldots, m \) and, for each \( A_i \), a local section \( s_i : U_i \to PX \) for \( \zeta_X = \{ \pi : X'X \times X \} \) with the following property: \( s_i(x,x) \) is the constant path at \( x \) for all \( x \in X \).

**16.2. Remark.** Iwase and Sakai \([?]\) require that each \( A_i \) contains the diagonal \( d(X) \subset X \times X \). However, this definition agrees with the our one, cf. [Dr2014, p.1].

**16.3. Open Problem.** Is it true that \( TC^M(X) = TC(X) \) for all \( X \)?

In fact, Iwase and Sakai proclaimed the equality \( TC^M(X) = TC(X) \) in \([?]\) and then withdrew the claim in \([?]\).

**16.4. Proposition.** For any CW space we have \( TC(X) \leq TC^M(X) \leq TC(X) + 1 \).

*Proof.* See \([?, Dr2014]\) □
16.5. **Theorem.** The equality $\text{TC}(X) = \text{TC}^M(X)$ holds for all $k$-connected simplicial complexes $X$ with

$$(k + 1)\text{(TC}(X) + 2) \geq \dim X + 1.$$  

**Proof.** [Dr2014, Theorem 2.5].

Note also the equalities $\text{TC}(X) = \text{TC}^M(X)$ is $X$ is a sphere $S^n$ or a connected Lie group $G$,[Dr2014].

The following theorem refines Theorem 16.5.

16.6. **Theorem** (Dranishnikov [Dr2014]). Let $X, Y$ be two absolute neighborhood retracts. Then

$$\max\{\text{TC}(X), \text{TC}(Y), \text{cat}(X \times Y) \leq \text{TC}(X \lor Y) \leq \text{TC}^M(X \lor Y)$$

$$\leq \text{TC}^M(X) + \text{TC}^M(Y) + 1 \leq \text{TC}(X) + \text{TC}(Y) + 3.$$

□

17. **Symmetric topological complexity**

This section is an extract from [BGRT2014].

In discussing on robotics, it is natural to consider motion planning so that a path $\alpha$ from $A$ to $B$ is equal to the inverse one $\alpha^{-1}$ of the path from $B$ to $A$. This leads to symmetric versions of topological complexity.

We discuss two symmetric versions of $\text{TC}_n$. One of them, $\text{TC}^{\Sigma}_n$, has the advantage of being a homotopy invariant. The other, $\text{TC}^S_n$, is better for calculations and is a natural generalization of the symmetric topological complexity studied by Farber and Grant in [FG2007]. We begin with the $n = 2$ case of the homotopically well-behaved version.

Consider the involutions $\tau : X \times X \to X \times X$ and $\pi : X \to X$ defined by $\tau(x, y) = (y, x)$ and $\pi(\alpha)(t) = \alpha(1 - t)$, for $(x, y) \in X \times X$ and $\alpha \in X$. We work with symmetric subsets $A \subseteq X \times X$ (i.e. those for which $\tau A = A$), and equivariant maps $s : A \to X$ (i.e. those satisfying $\pi(s(a)) = s(\tau(a))$ for all $a \in A$).

17.1. **Definition.** $\text{TC}^{\Sigma}(X)$ is the least integer $k$ such that $X \times X = A_0 \cup A_1 \cup \cdots \cup A_k$ where each $A_i$ is open, symmetric, and admits a continuous equivariant section $s_i : A_i \to X$ of the map $\varepsilon_2$.

To define Farber–Grant symmetric complexity $\text{TC}^S$, consider the subspace $C_2 = X^2 \setminus -d(X)) \subset X \times X$ of ordered pairs of distinct points in $X$. The map $\pi : X \to X \times X$ restricts to a map $\pi' : \pi^{-1}(C_2(X)) \to C_2(X)$ that is a $\mathbb{Z}/2$-equivariant map with free $\mathbb{Z}/2$-actions on its domain and range. So, the quotient map

$$\varepsilon_2 := \pi'/(\mathbb{Z}/2) : \pi^{-1}(C_2(X))/(\mathbb{Z}/2) \to C_2(X)/(\mathbb{Z}/2)$$

is a fibration.

17.2. **Definition.** $\text{TC}^S_2(X) = 1 + \text{secat}(\varepsilon_2)$. 


17.3. Proposition. For each ENR we have
\[TC_2^S(X) - 1 \leq TC_2^\Sigma(X) \leq TC_2^S(X).\]

17.4. Example. For \(X\) contractible we have \(TC_2(X) = TC_2(X) = 0\) while \(TC_2^S(X) = 1\). In particular, \(TC_2^S\) is not a homotopy invariant.

17.5. Example. The numbers \(TC_2^S(S^k)\) and \(TC_2(S^k)\) have been computed in [FG2007, Corollary 18] and [Fa2003], respectively. Here we use the inequalities \(TC_2 \leq TC_2^\Sigma \leq TC_2^S\) together with the fact that \(TC_2^S(S^k) = 2 = TC_2(S^{2k})\) to deduce \(TC_2^\Sigma(S^{2k}) = TC_2^S(S^{2k}) = 2\) for all \(k\). On the other hand, since \(TC_2(S^{2k+1}) = 1\), the above argument only gives \(1 \leq TC_2^\Sigma(S^{2k+1}) \leq TC_2^S(S^{2k+1}) = 2\). Incidentally, note that the construction in [Fa2008, Example 4.8] gives an open covering \(S^{2k+1} \times S^{2k+1} = A_0 \cup A_1\) by symmetric sets \(A_i\), and continuous sections of \(e_2\) over each \(A_i, i = 0, 1\). However, one of these sections is not equivariant, which prevents us from deducing \(TC_2^\Sigma(S^k) = 1\).

17.6. Open Problem. Evaluate \(TC_2^\Sigma S^1\).

17.7. Remark. We do not know example with \(TC \neq TC_2^S\).

We next define higher analogues of \(TC_2^S\). Recall that for a given \(n\), the symmetric group \(\Sigma_n\) acts on the right of \(X^n\) and \(X^{J_n}\) by permuting coordinates and paths, respectively. Further, the fibre \(e_n\) in Definition ?? is \(\Sigma_n\)-equivariant. We now work with symmetric subsets \(A \subseteq X^n\) (i.e., those for which \(A \sigma = A\) for all \(\sigma \in \Sigma_n\)), and equivariant maps \(s : A \to X^{J_n}\) (i.e., those satisfying \(s(a) \sigma = s(a \sigma)\) for all \(a \in A\) and \(\sigma \in \Sigma_n\)). Definition 16.1 can now be extended to:

17.8. Definition. \(TC_2^\Sigma_n(X)\) is the least integer \(k\) such that \(X^n = A_0 \cup A_1 \cup \cdots \cup A_k\) where each \(A_i\) is open, symmetric and admits a continuous equivariant section \(s_i : A_i \to X^{J_n}\) for the map \(e_n\).

17.9. Theorem. If the spaces \(X\) and \(Y\) are homotopy equivalent then \(TC_2^\Sigma_n(X) = TC_2^\Sigma_n(Y)\).

Now we present the higher analog of \(TC_2^S\). Let \(C_n(X)\) stand for configuration space of \(n\) ordered distinct points in \(X\). The symmetric group \(\Sigma_n\) acts on \(X^{J_n}\) and \(X^n\) in an obvious way, and \(e_n : X^{J_n} \to X^n\) is an \(\Sigma_n\)-equivariant map. The \(\Sigma_n\)-actions are free on both domain and range of \(e_n\). Thus, at the level of orbit spaces we get a fibration
\[\varepsilon_n^X = \varepsilon_n : Y_n(X) \to C_n(X)/\Sigma_n\]
where \(Y_n := e_n^{-1}(C_n(X))/\Sigma_n\).

17.10. Theorem. If \(X\) is an ENR then
\[\text{secat}(\varepsilon_n) \leq TC_2^\Sigma_n(X) \leq \text{secat}(\varepsilon_n) + \cdots + \text{secat}(\varepsilon_2) + n - 1.\]

In view of previous inequality \(TC_2^S(X) - 1 \leq TC_2^\Sigma(X) \leq TC_2^S(X)\), it hints the following definition:

17.11. Definition. For \(n \geq 2\) set \( TC_2^S(X) = \text{secat}(\varepsilon_n) + \cdots + \text{secat}(\varepsilon_2) + n - 1 \).

18. Topological Complexity in Presence of Groups Actions

When an interesting topological concept appears, people consider topological groups \(G\) and do \(G\)-equivariant (\(G\)-invariant) generalization of the concept. Topological Complexity is not an exception. I am not able to discuss here different \(G\)-versions of Topological Complexity: the interested reader is referred to the papers [BK2015, CG2012, LM2015].
References


Department of Mathematics, University of Florida
358 Little Hall, Gainesville, FL 32611-8105, USA

E-mail address: rudyak@ufl.edu