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STRING TOPOLOGICAL ROBOTICS

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ABSTRACT

We claim to link two well known theories; namely the *string topology* (founded by M. Chas and D. Sullivan in 1999) and the *topological robotics* (founded by M. Farber some few years later, in 2003). For our purpose, we consider G a compact Lie group acting on a path-connected n-manifold X. On the set $\mathcal{M}_{\mathrm{LP}}(X)$ of the so-called *loop motion planners* (LMP for short), we define and discuss a string homology product. Firstly, we define *transversely* an *intersection LMP-product* at level of chains of $\mathcal{M}_{\mathrm{LP}}(X)$. Secondly, we define a boundary operator on the chains of $\mathcal{M}_{\mathrm{LP}}(X)$ and extend the intersection LMP-product, at level of homology of $\mathcal{M}_{\mathrm{LP}}(X)$, to a *string LMP-product*. Finally, we show that it induces on the shifted string LMP-homology $\mathbb{H}_*(\mathcal{M}_{\mathrm{LP}}(X)) := H_{*+2n}(\mathcal{M}_{\mathrm{LP}}(X))$ a structure of an associative and commutative graded algebra (acga). By the end, following Chas-Sullivan approach, we ask how one may extend this acga-structure to a structure of Gerstenhaber algebra or that of a Batalin-Vilkovisky algebra. Some ideas will be suggested.

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STRING TOPOLOGY

D. Sullivan and M. Chas (1999)

Context : X closed and orientable n-manifold.

Goal : Extend the intersection product at level of loops in $LX := X^{S^1}$.

Machinery :

- Set $\begin{array}{cccc} ev_0 : & C_i(LX;\mathbb{Z}) & \longrightarrow & C_i(X;\mathbb{Z}) \\ & \Sigma & \longmapsto & \Sigma(-)(0) \end{array}$ to link i-simplices in LX to that in X ;
- A bi-simplex $\Sigma\times\Theta : \Delta^i\times\Delta^j \to LX\times LX$ is said to be transverse if $\sigma := ev_0(\Sigma)$ and $\theta := ev_0(\Theta)$ intersect transversally in X ;
- Compute the intersection product $\sigma.\theta$ at each pair $(\delta,\delta')\in\Delta^i\times\Delta^j$ satisfying $\Sigma(\delta)(0) = \Theta(\delta')(0)$
- Perform the composition of the loops $\Sigma(\delta)$ and $\Theta(\delta')$ to obtain a $(i+j-n)$-simplex
$$\Sigma\cdot\Theta : \Delta^{i+j-n} \to LX.$$
- Put $\mathbb{H}_i(LX) = H_{i+n}(LX)$ and extend the intersection product, at level of the homology, to get a *string homology product*,
$$\mathbb{H}_i(LX)\otimes\mathbb{H}_j(LX) \xrightarrow{\bullet} \mathbb{H}_{i+j}(LX)$$
well defined by setting
$$[\Sigma]\bullet[\Theta] := [\Sigma\cdot\Theta].$$
- $(\mathbb{H}_*(LX),\bullet)$ is an associative and commutative graded algebra ;

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Context :

- X n-manifold, compact or not, orientable or not;
- G a compact Lie group acting on X.

Step 1 : Loop Motion Planner (LMP) : That is any $G\times G$-homotopic section of the loop bi-evaluation
$$\begin{array}{cccc} ev^{\mathrm{LP}} : & LX\times_{X/G} LX & \longrightarrow & X\times X \\ & (\gamma,\tau) & \longmapsto & (\gamma(0),\tau(1/2)) \end{array}.$$
Here $LX\ \times_{X/G}\ LX$ denotes the set of pairs
$$(\gamma,\tau)\in LX\times LX,\ G.\gamma(1/2) = G.\tau(0)$$
and $\mathcal{M}_{\mathrm{LP}}(X)$ denotes the set of all LMP.

Step 2 : Loop motion planners product For any $s_1, s_2 \in \mathcal{M}_{\mathrm{LP}}(X)$, we associate the natural concatenation
$$\begin{array}{rcll} \mu(s_1,s_2)(x,y)(t) & = & s_1(x,y)(t) & 0\le t\le \frac{1}{2} \\ & = & s_1(x,y)(3t-1) & \frac{1}{2}\le t\le\frac{2}{3} \\ & = & s_2(x,y)(3t-2) & \frac{2}{3}\le t\le 1 \end{array}$$

Step 3 : Intersection LMP-product

- Fix in $\mathcal{M}_{\mathrm{LP}}(X)$ a bi-simplex
$$\Sigma\times\Theta : \Delta^i\times\Delta^j \to \mathcal{M}_{\mathrm{LP}}(X)\times\mathcal{M}_{\mathrm{LP}}(X)$$
- Link it in X^2 to the bi-simplex
$$\sigma\times\theta := (ev^{\mathrm{LP}}\Sigma)\times(ev^{\mathrm{LP}}\Theta)$$
- Equip X^2 with a chart $\mathcal{A}$;
- Σ is said to be *small* when there exists $U(\Sigma)\in\mathcal{A}$ (chosen once for all) such that $\sigma(\Delta^i)\subset U(\Sigma)$;
- $\Sigma\times\Theta$ is said to be *transverse* if both Σ and Θ are small, with the additional condition that $\sigma\times\theta$ and all its faces are transverse in X^4 to the diagonal map ;
- Put $W := (\Delta_{X^2}\circ(\sigma\times\theta))^{-1}(X^4) \simeq \Delta^{i+j-2n}$;
- Put $\Sigma.\Theta := \mu\circ(\Sigma\times\Theta)_{|W}$;
- Note that $\Sigma.\Theta : W \to \mathcal{M}_{\mathrm{LP}}(X)\times\mathcal{M}_{\mathrm{LP}}(X)$.

Step 4 : String homology LMP-product

- Define a *boundary operator* on $\mathcal{M}_{\mathrm{LP}}(X)$:
$$\partial\Sigma := \sum_{k=0}^{i}\varepsilon_k(-1)^k F_k\Sigma$$
where ε_k is the sign of the Jacobian of the coordinates change $U(F_k\Sigma)\to U(\Sigma)$.
- The small chains with this boundary form a chain complex whose integral coefficients homology is $H_*(\mathcal{M}_{\mathrm{LP}}(X);\mathbb{Z})$;
- Extend the intersection LMP-product at level of homology by setting
$$[\Sigma]\bullet[\Theta] := [\Sigma.\Theta]$$
- Prove that it is well defined, in the sens that
  – Any bi-simplex can be represented by a small simplex, up to a preserving homotopy
  – $[\Sigma.\Theta]$ does not depend on the homological class representant.
- Prove that $(\mathbb{H}_*(\mathcal{M}_{\mathrm{LP}}(X)),\bullet)$ is an associative and commutative graded algebra.

INTERSECTION PRODUCT

Context : X closed and orientable n-manifold.

Definition : Y, Z (submanifolds and orientable, $\dim Y = i$, $\dim Z = j$) are said to *intersect transversaly* in X if for all $x\in Y\cap Z$, one have $T_xY + T_xZ = T_xX$. Therefore $Y\cap Z$ is dimension $i+j-n$ and orientable.

intersection product $[Y].[Z] := [Y\cap Z] \in H_{i+j-n}(X;\mathbb{Z})$ where $[-]$ denotes the homology fundamental class of the named submanifold.

Set $\mathbb{H}_*(X) := H_{*+n}(X;\mathbb{Z})$, then
$$\cdot : \mathbb{H}_i(X)\times\mathbb{H}_j(X)\to\mathbb{H}_{i+j}(X).$$

Conclusion : $(\mathbb{H}_*(X),\cdot)$ is an associative and commutative graded algebra.

TOPOLOGICAL ROBOTICS

M. Farber (2003)

Context : X path-connected topological space

Motion Planner Algorithm : Any continuous section $s : X^2 \to PX$ of the bi-evaluation map
$$\begin{array}{cccc} ev : & PX := X^{[0,1]} & \longrightarrow & X^2 \\ & \gamma & \longmapsto & (\gamma(0),\gamma(1)) \end{array}$$

Input : A pair $(x,y)\in X\times X$.

Output : Suggest to the robot a path, $s(x,y) = \gamma$ (a motion), from x to y.

Topology through Robotics : Continuity of s means that close pairs (x_1,y_1) and (x_2,y_2) produce close paths $\gamma_1 = s(x_1,y_2)$ and $\gamma_2 = s(x_2,y_2)$. In other words this interprets, how topology interferes in the stability of a robot's motion.

REFERENCES AND AKNOWLDGEMENTS

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EVENT : 5TH-GETOPHYMA, RABAT (MOROCCO) - JULY 11-21, 2016

Celebrating Jim Stasheff and Dennis Sullivan for their respective 80th and 75th anniversary.

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