Moduli spaces from a homotopy point of view

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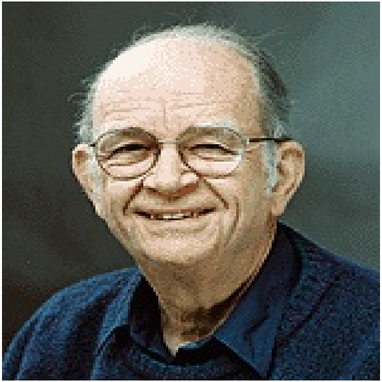


Introduction

Assalamu alaikum !

Moduli Spaces and Configuration Spaces are intimately related.

Start with the space

$$\mathcal{C}onf(\mathbf{R}^k, n)$$

of configurations of n distinct points in $\mathbf{R}^k$. Consider the action of the affine group on such configurations and define

$$\mathcal{M}(\mathbf{R}^k, n)$$

to be the quotient.

Basics

Riemann in 1857 observed:

Isomorphism classes of compact Riemann surfaces of genus g depend on $3g - 3$ continuous variables to be called Moduli.

Algebraic geometry: the compact Riemann surfaces may have simple nodes and marked points away from the nodes.

The moduli are regarded as parameterizing a moduli space

A set of isomorphism classes of interesting objects may often be treated as a ‘nice’ space parameterized by moduli.

Thoughts

In particularly good cases, the moduli space is again a space of the same type as is being classified,

but be prepared for more singular moduli spaces.

Think of a modulus as a parameter for
a family of the objects of interest.

Think of when two objects are near each other.

Think of how the objects can
modulate or vary or deform
throughout the family.

Homotopy pov

From a homotopy point of view, there are two main aspects of moduli spaces that stand out:

- the moduli spaces of rational homotopy types with given cohomology algebra $\mathcal{H}$
- classical moduli spaces giving rise to operads.

Classical algebraic geometry

Algebraic geometry is written in
"a foreign tongue that was not sung at my cradle" (Weyl).

Consider not only equivalence classes of objects
and their variation in families
but equivalence of families as well.

A fine moduli space supports a universal family of the objects;
any family over some base space is equivalent to
the pullback of the universal family via
a unique map of the base space into the moduli space.

Compare the notion of classifying space for bundles
or fibrations.

Classification of rational homotopy types

Rational homotopy theory regards **rational homotopy equivalence** of two simply connected spaces as the equivalence relation generated by the existence of a map $f : X \to Y$ inducing an isomorphism

$$f^* : H^*(Y; \mathbf{Q}) \to H^*(X, \mathbf{Q}).$$

An obvious invariant is the cohomology algebra $H^*(X; \mathbf{Q})$.

Halperin and I showed that all simply connected spaces X with

fixed cohomology algebra $\mathcal{H}$

of finite type over $\mathbf{Q}$
can be described (up to rational homotopy type) as follows:

Classification of rational homotopy types

Denote by SZ the free graded symmetric algebra on a graded vector space Z.

Resolve $\mathcal{H}$ by a **d**(ifferential) **g**(raded) **c**(ommutative) **a**(lgebra) (SZ, d) with a map $(SZ, d) \to \mathcal{H}$ of dgcas inducing an isomorphism

$$H(SZ, d) \simeq \mathcal{H}.$$

Here $H(SZ, d)$ denotes the cohomology of (SZ, d) with respect to the differential d.
For example, let $(SZ, d) \to \mathcal{H}$ be the Sullivan minimal model.

The construction proceeds by induction with respect to a resolution degree.

Perturbations

Given (SZ, d), a perturbation is a derivation of SZ of degree 1
which lowers resolution degree by at least 2
such that $(d+p)^2 = 0$.

The classification will be in terms of
the Lie algebra of perturbations as derivations of SZ.

Classification of rational homotopy types

Let $A^*(X)$ denote a differential graded commutative algebra of "differential forms over the rationals" for the space X,
e.g. Sullivan's version of the deRham complex.

Theorem

Given an isomorphism $i : \mathcal{H} \stackrel{\simeq}{\to} H^*(X)$, there is a perturbation p and a map of dgca's

$$(SZ, d + p) \to A^*(X)$$

which is a rational homotopy equivalence.

Classification as in algebraic deformation theory

Schlessinger and I look at the classification of rational homotopy types from two points of view:

- as a problem in algebraic deformation theory
- as analogous to the classification of fibrations.

Rational homotopy types are classified by a “moduli” space — a certain quotient V/G of an algebraic variety V of perturbations.

Classification as for fibre spaces

Fibres spaces $F \to E \stackrel{p}{\to} B$ with given fibre F
are classified in terms of
homotopy classes of maps of the base B
into a classifying space constructed from $Aut(F)$,
the monoid of homotopy equivalences of F to itself.

We classify perturbations (and therefore homotopy types) by the “path components” of a universal example which is a

cocomplete dgc coalgeba.

First consider the dg Lie algebra

$$L \subset \; Der\, SZ$$

consisting of the weight decreasing derivations.

Classification as for fibre spaces

Next apply the universal construction $\mathcal{C}$ from dg Lie algebras to cocomplete dgc coalgebas.

Theorem

Let $(SZ, d) \to \mathcal{H}$ be as above. The set of augmented homotopy types of dgca's with $i : \mathcal{H} \simeq H^*(X)$ is in 1–1 correspondence with the path components of $\mathcal{C}(L)$.

There is a subtlety here if we start with the rationalization of simply connected topological spaces:

the constructed classifying dgca need not correspond to the rationalization of a topological space.

Operads from moduli spaces

Consider the moduli space $\mathcal{M}(\mathbf{R}^k, n)$, taken to be the space $\mathcal{C}onf(\mathbf{R}^k, n)$ of configurations of n distinct points in $\mathbf{R}^k$ modulo the action of the affine group.

Getzler and Jones showed that as n varies suitable compactifications provide a natural structure of a

topological operad.

For $k = 1$, the space $\mathcal{M}(\mathbf{R}, n)$ can be identified as the open $n - 2$ simplex by numbering the points $x_1 < x_2 < \cdots < x_n$ and using the affine group to choose the representative configuration with

$$x_1 = 0, \ x_n = 1.$$

Homotopy associativity

For $n = 2$, the space $\mathcal{M}(\mathbf{R}, 2)$ consists of a single point, so is compact and will be called K_2.
Consider $K_2 \times X \times X \to X$ as a multiplication denoted

$$a \times b \mapsto ab.$$

For n=3, K_3, the compactification of the open 1-simplex, is the unit interval with the faces
$t = 0$ corresponding to $(ab)c$ and
$t = 1$ corresponding to $a(bc)$
and the general t parameterizing an associating homotopy.

That is, the homotopy is a map

$$K_3 \times X \times X \times X \to X.$$

For general k, the suitable compactifications are given either by

blowing up or truncating

certain faces of the closed simplex. One image is that of applying a magnifying glass to tease out additional structure as points approach collision. The resulting compactification is

the associahedron K_n.

The reason for the name is that the set of vertices of K_n is isomorphic to the set of full bracketings of a word of length n.

The reason they were invented (50+ years ago!) was to keep track of the higher homotopies of multiplication in a based loop space.

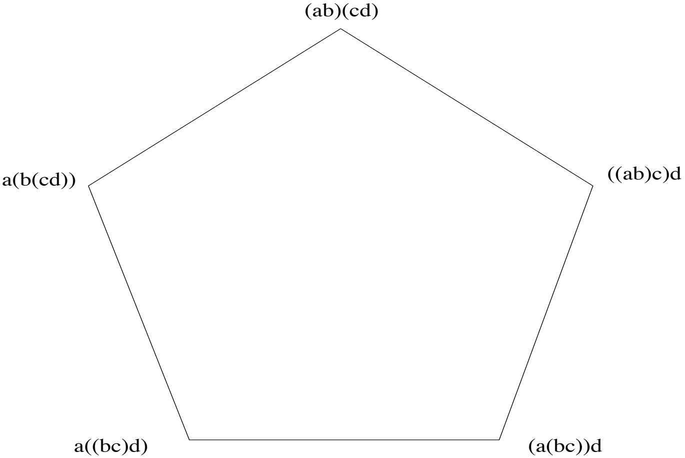
The facets of K_n are of the form $K_r \times K_s$ with $r + s = n + 1$ corresponding to inserting one bracketing in another.

For $k = 4$, we have 5 ways to approach full collision, giving the vertices of the pentagon K_4:

- $x_2 \to 0, x_3 \to 1$ - think $(ab)(cd)$
- $x_2 \to 0$ faster than $x_3 \to 0$ - think $((ab)c)d$
- $x_3 \to 1$ faster than $x_2 \to 1$ - think $a(b(cd))$
- $x_2, x_3 \to t \notin \{0, 1\}$ with $t \to 0$ - think $(a(bc))d$
- $x_2, x_3 \to t \notin \{0, 1\}$ with $t \to 1$ - think $a((bc)d)$

Edges correspond to uses of the associating homotopy.

The pentagon



Associahedra

The collection $\{K_n\}$ together forms the ‘operad’, that is, a machine for keeping track of all the higher homotopies for

strong homotopy algebras,

also known as

A_∞-algebras.

These higher homotopies were first identified in trying to characterize spaces of the homotopy type of a based loop space.

Since the ‘discovery’ of the associahedra, permutations of the variables have been added, leading to the general notion of an operad by Peter May.

Operads, and their generalizations, for a great variety of algebraic structures with higher homotopies have since been introduced by many people, some of which involve novel “hedra” polytopes, but that takes us well beyond moduli spaces.

A la prochaine

I hope many of you will find this brief survey tantalizing enough to jump in.

Don't hesitate to get in touch.

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Assalamu alaikum!