Mathematical Models in Science
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La Science

On dit souvent que nous vivons dans l'ére scientifique, que notre économie, et notre culture sont devenu dépendants des recherches scientifique et que même notre manière de penser est imprégné par les méthode scientifique. On trouve presque tout les mois, des livres intéressantes qui traitent de ces questions. Vous avez peut-étre vu le livre de Jean-Pierre Changeux et Alain Connes:"Matière à pensée", ou le petit livre édité par Étienne Klein et Michel Spiro: "Le Temps et sa Fleche". Le dernier est un de mes favoris. Dans le premier, on discute ce que cela veut dire quand on dit qu'on sais quelque choses. On discute les limites de notre savoir, Ils pose par éxemple la question (J-PC):

- Peut-on identifier la réalité extérieure à des idéalités mathématiques?
- Ces idéalitées décrivent-elles intégralement les phénomènes?

Phénoménologie

Et, la réponse de (AC) n'est pas tout à fait claire, mais quand même,

- Comprendre la matrice S ne veut... pas dire qu'on a compris ce qui se passe, mais qu'on dispose d'un modèle donnant des résultat adéquats â la réalité expérimentale
- (J-PC):C'est ce qu'on appelle une phénoménologie!
- (AC):Oui

Dans le livre de Klein et Spiro, on cherche la réalité, on veut savoir si le temps a une flêche!

- Le temps, est-il révérsible?
- A-t-il un commencement?
- Un fin?

On peut donc se demander:

C'est quoi le Temps?

Jean -Marc Lévy-Leblond: Ne vaudrait-il pas la peine alors de se reposer la question de la formalisation, ou plutôt des formalisations mathématiques de la temporalité, et de chercher des alternatives à sa représentation par l'ensemble des nombres dits réels? Ne serait-il pas utile, ne fût-ce que pour des raison phénoménologiques, dans tel ou tel domaine (des sciences de la vie, en particulier), d'intégrer au départ à une notion mathémathisée du temps certaines de ses propriétés que nous cherchons à lui rendre à l'arrivée? Peut-on imaginer une (des) mathémathisation(s) non-triviale(s) qui décrirai(en)t un temps par essence irréversible?un temps à instantanéité floue? un temps multiple et "épais"?
Après tout, on ne ferait ainsi que renouer avec Aristote, pour qui, comme il est bien connu, "le temps est le nombre du mouvement".
C'est ce que j'ai tenté de faire, avec la modestie, qui est la marque d'un dilettante. Et, pour cela, je doit premièrement m'intéresser à la notion:

Mathematical Models in Science I

In Physics we encounter the following notions

- Space
- Time
- Velocities
- Acceleration

Most people have an intuitive understanding of these notions, but most of you will use your mathematical experience and be able to give precise definitions of each of these terms.

- You will have a Mathematical Model

in which these terms can be interpreted in such a way that they fit together into a structure that you have accepted as relevant.

Mathematical Models in Science II

It is much more difficult to produce mathematical models for notions like forces; gravitation, electromagnetism, not to speak of the week force, the strong or color force.

And before trying, we should understand the notions

- Field
- Particle
- Molecules
- Quanta
- Mass
- Charge
- Spin
- Energy

There are, of course, lots of other subjects, in the Sciences, in which mathematical models are assumed to be of relevance, and in which they are developed, and used.

Think of:

- Chemistry: Big Molecules
- Biology: DNA-molecules and their geometrical structure
- Finance: Insurance, Derivatives.
- Sociology: Game-theory
- Peace Research!

So maybe we should first ask the questions,

- What is a mathematical model? And what is it modeling?

Modeling Natural Phenomena

If we want to study a natural phenomenon, called **P**, we must in the present scientific situation, (since to Galilei), describe **P** in some mathematical terms, say as a mathematical object, X, depending upon some parameters, and in particular on those conceived to give us an idea of the Location of **P**; in such a way that the changing aspects of **P** would correspond to altered parameter-values for X.

- This object, X, would be a Model for **P** if, moreover, X with any choice of parameter-values, would correspond to some, possibly occurring, aspect of **P**.

This piece of poetry is hardly convincing, without being put to test, in a real situation, whatever that might be. However, before this, let us expand the idea behind it, and put it into a more elaborate mathematical form, containing clues to our first basic notions:

- Space and Time

Moduli Spaces, Space and Time

Two mathematical objects X(1), and X(2), corresponding to the same aspect of **P**, would be called equivalent, and,

- The set, $\mathcal{P}$, of equivalence classes of the objects **P**, would correspond to (a quotient of) the *moduli space*, **M**, of the models, X.
- The parameters assumed to relate to the location of the phenomenon, would correspond to some form of coordinate functions defined on **M**, corresponding to our notion of Space
- The study of the natural phenomena **P**, and its changing aspects, would then be equivalent to the study of the *structure* of $\mathcal{P}$, and therefore to the study of the dynamics of the moduli space **M**.
- In particular, the notion of *time* would, in agreement with Aristotle and St. Augustin, correspond to some metric defined in **M**.

See, [1] and [16],

Modeling Dynamics

It turns out that to obtain a complete theoretical framework for studying the phenomenon **P**, or the model X, together with its dynamics, we should introduce the notion of idynamical structure, defined on the space, **M**. Assuming that **M** is an algebraic scheme of some sort, this is done via the construction of a universal non-commutative *Phase Space*-functor, $Ph(-) : Alg_k \rightarrow Alg_k$. It extends to the category of schemes, and its infinite iteration $Ph^\infty(-)$, is outfitted with a universal *Dirac derivation*, $\delta \in Der_k(Ph^\infty(-), Ph^\infty(-))$.

A dynamical structure defined on an associative k-algebra $A \in Alg_k$ is now a δ-stable ideal $\sigma \subset Ph^\infty(A)$, and the structure we are interested in is the *space* $\mathbf{U} := Ph^\infty(\mathbf{M})/\sigma$, corresponding to an affine covering of **M** by algebras of the type, $Ph^\infty(A)/\sigma$,

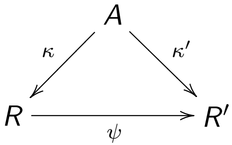
see [16], [17], see also [18].

Gauge Lie Groups and Lie Algebras

But now we observe that there may be an action of a Lie algebra $\mathfrak{g}$, a *gauge group*, on $\mathbf{U}$, such that the dynamics of $\mathcal{P}$, really, corresponds to that of the quotient $\mathbf{U}/\mathfrak{g}$.
To any *open* subset V, of $\mathbf{U}$, there would be associated a, not necessarily commutative, affine k-algebra, $A := O_{\mathbf{U}}(V)$, with an action of the Lie algebra $\mathfrak{g}$, containing the available information about the structure of O. An element of this algebra would be called an *observable*, and wishing to measure the *values* of an observable, leads to the study of representations of this algebra. Finally, the Dirac derivation and the gauge group $\mathfrak{g}$, will act on the moduli space of representations of A, inducing the dynamical laws we are interested in.

Phase spaces of associative algebras

Given an associative k-algebra A, denote by $A/k - \underline{alg}$ the category where the objects are homomorphisms of k-algebras $\kappa : A \to R$, and the morphisms, $\psi : \kappa \to \kappa'$ are commutative diagrams,



and consider the functor,

$$Der_k(A, -) : A/k - \underline{alg} \longrightarrow \underline{Sets}.$$

It is, see [1], et [2], representable by a k-algebra-morphism,

$$\iota : A \longrightarrow Ph(A),$$

with a universal family given by a universal derivation,

$$d : A \longrightarrow Ph(A).$$

The Universal family

This universal family,

$$d : A \longrightarrow Ph(A).$$

has the property that, for any A-module, $\rho : A \rightarrow End_k(V)$, and any derivation,

$$\xi : A \rightarrow End_k(V)$$

there exists a homomorphism,

$$\tilde{\rho} : Ph(A) \longrightarrow End_k(V).$$

such that,

$$\xi = d \circ \tilde{\rho}.$$

Tangents, Velocities, Momenta

Let $A = k[x_1, x_2, x_3]$, where k is either the reals or the complex numbers, and consider the A -module,

$$\rho : A \to End_k(V),$$

and suppose first that $V = k$, i.e. $dimV = 1$. Obviously, ρ is then defining a point $x = (\rho(x_1), \rho(x_2), \rho(x_3))$, in 3-space, and a tangent vector at this point is, by definition, a derivation,

$$\xi : A \to End_k(V) = k.$$

If we had a well defined notion of time and mass, as some length-function defined for tangents, we would have been able to make precise the notions, velocity and momenta, as such a derivation ξ, and therefore as the corresponding homomorphism,

$$\tilde{\rho} : Ph(A) \longrightarrow End_k(V).$$

such that,

$$\xi = d \circ \tilde{\rho}.$$

Tangents, Velocities, Momenta of Representations

In Quantum Theory the spaces of classical mechanics are replaced by associative, but not necessarily commutative, k-algebras A, where k is, still, either the reals or the complex numbers. The points of our spaces are replaced by representations of this algebra, i.e. by homomorphisms,

$$\rho : A \to End_k(V),$$

where V is any k-vectorspace.
And a tangent vector at this point, ρ is, still defined by a derivation,

$$\xi : A \to End_k(V).$$

Assuming we have, as above, a well defined notion of time and mass, the velocity and momenta, of the point, ρ would be defined by ξ, and therefore by the corresponding homomorphism,

$$\tilde{\rho} : Ph(A) \longrightarrow End_k(V).$$

such that,

$$\xi = d \circ \tilde{\rho}.$$

A universal derivation associated to an A-module

Clearly we have the identities,

$$d_* : Der_k(A, A) = Mor_A(Ph(A), A),$$

and,

$$d^* : Der_k(A, Ph(A)) = End_A(Ph(A)),$$

the last one associating d to the identity endomorphism of Ph. Let now V be a right A-module, with structure morphism

$$\rho : A \to End_k(V).$$

We obtain another universal derivation,

$$c : A \longrightarrow Hom_k(V, V \otimes_A Ph(A)),$$

defined by, $c(a)(v) = v \otimes d(a)$.

The Kodaira-Spencer class

Using the long exact sequence, of Hochschild cohomology,

$$0 \rightarrow Hom_A(V, V \otimes_A Ph(A)) \rightarrow Hom_k(V, V \otimes_A Ph(A)) \\ \xrightarrow{\iota} Der_k(A, Hom_k(V, V \otimes_A Ph(A))) \xrightarrow{\kappa} Ext^1_A(V, V \otimes_A Ph(A)) \rightarrow 0,$$

with,

$$c \in Der_k(A, Hom_k(V, V \otimes_A Ph(A))),$$

we obtain the non-commutative *Kodaira-Spencer class*,

$$c(V) := \kappa(c) \in Ext^1_A(V, V \otimes_A Ph(A)),$$

inducing, via the identity d_*, the *Kodaira-Spencer morphism*,

$$g : \Theta_A := Der_k(A, A) \longrightarrow Ext^1_A(V, V).$$

Connections

If $c(V) = 0$, then the exact sequence above proves that there exist an element, $\nabla \in Hom_k(V, V \otimes_A Ph(A))$ such that $c = \iota(\nabla)$. This is just another way of proving that if $c(V) = 0$, then c is given by a connection,

$$\nabla : Der_k(A, A) \longrightarrow Hom_k(V, V).$$

In particular, we deduce, from the corresponding long exact sequence,

$$\begin{aligned} 0 \rightarrow & Hom_A(V, V) \rightarrow Hom_k(V, V) \\ & \xrightarrow{\iota} Der_k(A, Hom_k(V, V)) \xrightarrow{\kappa} Ext^1_A(V, V) \rightarrow 0, \end{aligned}$$

the following elementary,

Hamiltonian

Theorem

Let $\rho : A \longrightarrow End_k(V)$, be an A-module, and let $\delta \in Der_k(A, Hom_k(V, V))$, map to 0 in $Ext^1_A(V, V)$, i.e. assume $\kappa(\delta) = 0$, then there exist an element, $Q_\delta \in Hom_k(V, V)$, the Hamiltonian, such that for all $a \in A$,

$$\rho(\delta(a)) = [Q_\delta, \tilde{\rho}(a)].$$

If V is a simple A-module, $ad(Q_\delta)$ is unique.

As is well known, in the commutative case, the Kodaira-Spencer class gives rise to a *Chern character* by putting,

$$ch^i(V) := 1/i!\ c^i(V) \in Ext^i_A(V, V \otimes_A Ph(A)),$$

and if $c(V) = 0$, the curvature $R(V)$ of ∇, induces a curvature class,

$$R_\nabla \in H^2(k, A; \Theta_A, End_A(V)).$$

The structure of Ph*

Iterating the Ph construction, we obtain a sequence, $\{Ph^n(A)\}_{1\leq n}$, defined inductively by

$$Ph^0(A) = A,\ Ph^1(A) = Ph(A), ..., Ph^{n+1}(A) := Ph(Ph^n(A)).$$

Let $i_0^n : Ph^n(A) \to Ph^{n+1}(A)$ be the canonical imbedding, and let $d_n : Ph^n(A) \to Ph^{n+1}(A)$ be the corresponding derivation. Since the composition of i_0^n and the derivation d_{n+1} is a derivation $Ph^n(A) \to Ph^{n+2}(A)$, there exist by universality a homomorphism $i_1^{n+1} : Ph^{n+1}(A) \to Ph^{n+2}(A)$, such that,

$$d_n \circ i_1^{n+1} = i_0^n \circ d_{n+1}.$$

Notice that we here compose functions and functors from left to right. Clearly we may continue this process constructing new homomorphisms,

$$\{i_j^n : Ph^n(A) \to Ph^{n+1}(A)\}_{0\leq j\leq n},$$

with the property,

$$d_n \circ i_{j+1}^{n+1} = i_j^n \circ d_{n+1}.$$

The Dirac derivation

Define, via the direct limit functor,

$$Ph^{\infty}(A) := lim_{0\leq j\leq n}\{i_j^n : Ph^n(A) \to Ph^{n+1}(A)\},$$

and put,

$$i_n : Ph^n(A) \to Ph^{\infty}(A).$$

The family of derivations $\{d_n\}$ induces a derivation, the Dirac derivation,

$$\delta : Ph^{\infty}(A) \longrightarrow Ph^{\infty}(A),$$

such that,

$$i_n \circ \delta = d_n \circ i_{n+1},$$

and it is easy to see that this is a universal construction, i.e. for any pair of a morphism,

$$i : A \longrightarrow B$$

and a derivation $\xi \in Der_k(B)$, $\iota \circ \xi$ factorizes via $Ph^{\infty}(A)$, and δ.

Preparation

Recall from deformation theory, that given a right A-module V, then any derivation $\delta \in Der_k(A, End_k(V))$ defines a class,

$$\xi(v) \in Ext^1_A(V, V) := Der_k(A, End_k(V))/Triv$$

i.e. a *tangent vector* of the *formal moduli* of the representation V, at the unique point.
The above implies that a representation,

$$\rho : Ph^\infty(A) \to End_k(V),$$

corresponds to a family of $Ph^n(A)$-module-structures on V, for $n \geq 1$, i.e.
To a tangent $\xi_0 \in Ext^1_A(V, V)$, of the deformation functor of $V_0 := V$, as A-module, an element $\xi_1 \in Ext^1_{Ph(A)}(V, V)$, i.e. a tangent of the deformation functor of $V_1 := V$ as $Ph(A)$-module, to an element $\xi_2 \in Ext^1_{Ph^2(A)}(V, V)$, i.e. a tangent of the deformation functor of $V_2 := V$ as $Ph^2(A)$-module, to etc.

The Toy Model

I have a favourite "Toy Model", of General Relativity, and Quantum Theory. It is the philosophically reasonable (?) Physical Model, of an Observer and an Observed in 3-dimensional space, mathematically modelled by the Hilbert scheme **H** of length 2 sub-schemes in $\mathbf{A}^3$. Consider the diagonal, $\underline{\Delta} \subset \mathbf{A}^3 \times \mathbf{A}^3 = \underline{H}$, and let $\underline{\tilde{H}}$ be the blow up of $\underline{H}$ in $\underline{\Delta}$. We find a diagram,

$$\begin{array}{ccccc} \underline{\tilde{\Delta}} & \longrightarrow & \underline{\tilde{H}} & \xrightarrow{Z_2} & \mathbf{H} \\ \downarrow & & \downarrow & & \\ \underline{\Delta} & \longrightarrow & \underline{H} & \longleftarrow & \tilde{\mathbf{A}}^3 \end{array}$$

where $\mathbf{H} = Hilb^2(\mathbf{A}^3) = \underline{\tilde{H}}/Z_2$. Coordinates for **H**: $(\underline{\lambda}, \underline{\omega}, \rho)$, with $\underline{\lambda} \in \underline{\Delta}$, $\underline{\omega}$ a spherical coordinate of the blow-up of $\underline{\Delta}$ in $\underline{H}$, and, ρ the length from $\underline{\Delta}$ along the line defined by $\underline{\omega}$.

Time-Space I

Measurable time, in this mathematical model, should be a metric ρ on the time-space, measuring all possible infinitesimal changes of *the state* of the objects in the family we are studying. This implies that the notion of relative velocity may be interpreted as an oriented line l in the tangent space of a point $\underline{t} \in \underline{\tilde{H}}$. The classical relative velocity comes out as,

$$v = sin(\theta), \theta = angle(l, \tilde{\Delta}).$$

Thus the space of velocities is compact.

Time-Space II

This lead to A "physics" where there are no infinite velocities, and where the principle of relativity comes for free. The affine group, acts on $\mathbf{A}^3$, and therefore on $\underline{\tilde{H}}$. The Abelian Lie-algebra of translations defines a 3-dimensional distribution, $\tilde{\Delta}$ in the tangent bundle of $\underline{\tilde{H}}$, corresponding to 0-velocities. Given a metric on $\underline{\tilde{H}}$, we defined the distribution $\tilde{c}$, corresponding to light-velocities, as the normal space of $\tilde{\Delta}$. We see now that the classical Space-Time can be thought of as a universal subspace, $\underline{\tilde{M}}(l)$, of $\underline{\tilde{H}}$, defined by a fixed line, $l \subset \mathbf{A}^3$. We also show that the generator $\tau \in Z_2$, above, is linked to the operators C, P, T in classical physics, such that $\tau^2 = \tau PT = id$.

Canonical Sub-Spaces of Θ

Moreover, we observed that the three fundamental gauge groups of current quantum theory $U(1)$, $SU(2)$ and $SU(3)$ are part of the structure of the fiber space,

$$\tilde{\mathbf{A}}^3 \longrightarrow \underline{\tilde{H}}.$$

In fact, for any point $\underline{t} = (o, x)$ in $\underline{H}$, outside the diagonal $\underline{\Delta}$, we may consider the line l in $\mathbf{A}^3$ defined by the pair of points $(o, x) \in \mathbf{A}^3 \times \mathbf{A}^3$. We may also consider the action of $U(1)$ on the "normal plane" $B_o(l)$, of this line, oriented by the normal (o, x), and on the same plane $B_x(l)$, oriented by the normal (x, o). Using "parallel transport" in $\mathbf{A}^3$, we find an isomorphisms of bundles,

$$P_{o,x} : B_o \to B_x, P : B_o \oplus B_x \to B_o \oplus B_x,$$

the *partition isomorphism*. Using P we may write, (v, v) for $(v, P_{o,x}(v) = P((v, 0))$.

Canonical Sub-Spaces of Θ

We also observe that a line, $l \subset \underline{H}$ defines a unique sub scheme $\underline{H}(l) \subset \underline{H}$. The corresponding tangent space at (o, x), is called $A_{(o,x)}$. Together this define a decomposition of the tangent space of $\underline{H}$,

$$T_{\underline{H}} = B_o \oplus B_x \oplus A_{(o,x)}.$$

Canonical Gauge Groups

This decomposition can also be extended to the complexified tangent bundle of $\underline{\tilde{H}}$. Clearly, $U(1)$ and $SU(2)$ and $SU(3)$ acts naturally on $\mathbf{C}B_o \oplus \mathbf{C}B_x$ and $\mathbf{C}\tilde{\Delta}$ respectively in such a way that their actions should be *physically* irrelevant. The groups, $U(1)$, $SU(2)$, $SU(3)$ are our elementary *gauge groups*, and we shall consider the corresponding Lie algebra,

$$\mathfrak{g} := \mathfrak{u}(1) \oplus \mathfrak{u}(2) \oplus \mathfrak{u}(3)$$

as the infinitesimal *gauge group*, in the sense of our philosophy, and according to Sophus Lie's own definition).

Kepler

Consider a metric, and therefore the notion of Time of the Model $\underline{\tilde{H}}$. Pick,

$$g = (\frac{\rho - h(\lambda)}{\rho})^2 d\rho^2 + (\rho - h(\lambda))^2 d\phi^2 + \kappa(\lambda) d\lambda^2,$$

for the simplified space, in which $\underline{\omega}$ is reduced to the angle ϕ, and the coordinates $\underline{\lambda}$ reduced to one parameter $\lambda = |\underline{\lambda}|$. This correspond to considering the sub-universe of $\mathbf{M}(BB)$, parametrized by (λ, ϕ, ρ). Now let $h(\underline{\lambda}) = h$ be constant, then the geodesics have the equations,

$$\frac{d^2\rho}{dt^2} = -(\frac{h}{\rho(\rho - h)})(\frac{d\rho}{dt})^2 + (\frac{\rho^2}{(\rho - h)})(\frac{d\phi}{dt})^2,$$

$$\frac{d^2\phi}{dt^2} = -2/(\rho - h)\frac{d\rho}{dt}\frac{d\phi}{dt}, \quad Kepler's\ 2.Law.$$

$$\frac{d^2\lambda}{dt^2} = 0,$$

Kepler and Newton

The definition of time gives us,

$$\rho^{-2}(\frac{d\rho}{dt})^2 = (\rho - h)^{-2}K^2 - (\frac{d\phi}{dt})^2.$$

where, $K^2 = (1 - (\frac{d\lambda}{dt})^2)$, is the kinetic energy.
Put this into the first equation above, and obtain,

$$\frac{d^2\rho}{dt^2} = -hK^2(\frac{\rho}{\rho - h})\frac{1}{(\rho - h)^2} + (\frac{\rho + h}{\rho - h})\rho(\frac{d\phi}{dt})^2.$$

Assume now $r := \rho - h \approx \rho$, we find,

$$\frac{d^2r}{dt^2} = -\frac{hK^2}{r^2} + r(\frac{d\phi}{dt})^2, \ \ Kepler's\ 1.Law.$$

The constant h, the radius of the exceptional fibre, is thus also related to mass. Recall that the Schwarzschild radius, the Einstein equivalent to h, is assumed to be, $r_s = 2GM/c^2$, where, $G =$ Newton's gravitational constant, $M =$ mass, $c =$ speed of light, which here, of course, is put equal to 1.

Time and Cosmos

What is meant by the notion Big Bang? All, or almost all religions try to explain the start of the Universe, even the start of Time. Somehow we humans cannot accept the notion of the Show not having a start nor an end. Therefore we have developed a formidable library of physical models, trying to explain a kind of beginning of it all, a kind of eternal cyclicality, sometimes including a notion of the Cosmos splitting up into any multitude of copies, acceptable by the formal possibilities of the favorite theory of the author.
Could we believe in the Toy Model, as a model for the Universe? Farfetched, but let us, nevertheless, try the questions:

- Why is our real space of dimension 3?
- Where in the Toy Model might we find the Big Bang?

Modeling the Big Bang

We may start the analysis by accepting the "fact" that we all, intuitively, in the last centuries have accepted a 3-dimensional Cartesian model of space. So, maybe there is some reason for this, stemming from the very beginning of Cosmos.
However, we have, above, made space and time dependent upon each other, so beginning of one is also the beginning of the other, and very few in physics would accept a theory contained the beginning of Time. Since I am, luckily, just playing with mathematical models, I am not concerned. Therefore, here is the story of Big Bang explained via Deformation Theory.

Deformations of associative algebras

We fix a field k. All algebras occurring, will be associative k-algebras.

Examples:

- $A = k[x_1, x_2, x_3]$ is the commutative coordinate ring of the affine 3-space.
- $U = A/(\underline{x})^2$ is, geometrically, a thick point in affine 3-space, but
- U is also a quotient of the free associative k-algebra, $F = k < x_1, x_2, x_3 >$,
- Let $\rho : F \rightarrow U$ be the quotient map, then the kernel, $ker(\rho) = (x_i x_j),\ \ i, j = 1, 2, 3$

Deformations of associative algebras

A deformation of U parametrized by the (commutative) k-algebra, B, is a flat k-algebra homomorphism,
$B \rightarrow B < x_1, x_2, x_3 > /(x_i x_j + \sum b^l_{i,j} x_l + b^0_{i,j})$

Examples:

- Put, $B = k[t], b^l_{i,j} = \epsilon_{i,j,l} t,\ \ b^0_{i,j} = \delta_{i,j}$ then the deformation of U along t for $t \neq 0$ is constant, equal to the Quaternions.
- Let $\underline{o} := (o_1, o_2, o_3)$, and $\underline{p} := (p_1, p_2, p_3)$ be sets of independent coordinates, and put,
  $B = H := k[o_1, o_2, o_3, p_1, p_2, p_3],\ \ b^l_{i,j} = -o_i x_l \delta_{l,j} - x_l p_j \delta_{l,i},\ \ b^0_{i,j} = o_i p_j$
  then,

  $$\tilde{\rho} : H \rightarrow \mathbf{U} := H < x_1, x_2, x_3 > /(x_i x_j - o_i x_j - x_i p_j + o_i p_j)$$

  is a deformation of U, parametrized by the 6-dimensional scheme, $\underline{H} := Spec(H) = \mathbf{A}^3 \times \mathbf{A}^3$

$\mathbf{U}$ extends to $\mathbf{H}$

$$
\begin{array}{ccccc}
\underline{\tilde{H}} & \longrightarrow & \underline{H} & \longleftarrow & \underline{\Delta} \\
\downarrow & & \uparrow & & \uparrow \\
\mathbf{H} & \longleftarrow & \mathbf{U} & \longleftarrow & U
\end{array}
$$

This follows from the relations: For $(o, p) \in \underline{H}$, for every $c \in \mathbf{A}^3$ and for any non-zero $\kappa \in k$, we have:

- $U(\kappa o, \kappa p) \simeq U(o, p)$
- $U(o, p) \simeq U(o - c, p - c)$
- $U(-p, -o) \simeq U(o, p)$

Denote by $\tilde{\Delta} \subset \tilde{\Theta}_{\underline{\tilde{H}}}$ the 3-dimensional distribution, generated by the translations $\{(o, p) \to (o + c, p + c),\ c \in \mathbf{A}^3\}$.

Gauge Groups

Consider the bundle of Lie algebras, defined on **H** by,

- $\mathfrak{g} := Der_{\mathbf{H}}(\mathbf{U})$
- $\mathfrak{g}(o,p) = Der_{k(o,p)}(U(o,p)), (o,p) \in \underline{H}$.

Any element $\delta \in \mathfrak{g}(o,p)$ must have the form,
$\delta(x_i) = \delta_i^0 + \delta_i^1 x_1 + \delta_i^2 x_2 + \delta_i^3 x_3$. Consider the 4-vectors,
$\delta_i = (\delta_i^0, \delta_i^1, \delta_i^2, \delta_i^3)$, $\overline{o} = (1, o_1, o_2, o_3)$, $\overline{p} = (1, p_1, p_2, p_3)$

Theorem

- $\delta \in \mathfrak{g}(o,p)$ *if and only if* $\delta_i.\overline{o} = \delta_i.\overline{p} = 0$,
- *If* $o \neq p$, *then,* $\mathfrak{g}(o,p) \simeq \begin{pmatrix} 0 & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{pmatrix}$
- $rad(\mathfrak{g}) = \{u, r_1, r_2\}$, $\mathfrak{g}/rad \simeq \mathfrak{sl}(2) \subset \mathfrak{g}$
- $h \in \mathfrak{h} \subset \mathfrak{g}$; *the generator of the Cartan algebra.*

Spin and Isospin

Denote by $\tilde{\Delta} \subset \tilde{\Theta}_{\underline{\tilde{H}}}$ the 3-dimensional distribution, generated by the translations $\{(o,p) \to (o+c, p+c),\ c \in \mathbf{A}^3\}$, and complexify all bundles.

- There exists a canonical action of $\mathfrak{g}_{\mathbf{C}}$ on $\Theta_{\tilde{H},\mathbf{C}}$, such that,
- $\mathfrak{g}(o,p)$ acts on the tangent space $T_{\mathbf{H},(o,p)} = T_{\mathbf{A}^3,o} \times T_{\mathbf{A}^3,p}$ killing the vector $p - o$, in both factors.
- There is an obvious action of $\mathfrak{su}(3) = \mathfrak{su}_g(\tilde{\Delta}_{\mathbf{C}})$ on $\tilde{\Delta}_{\mathbf{C}}$
- Consider moreover the regular representation of $\mathfrak{g}_{\mathbf{C}}$

Standard Model

One recognises the ingredients of the Standard Model, in the following canonical representations of, $\mathfrak{sl}(2) \subset \mathfrak{g} = Der_{\mathbf{H}}(\mathbf{U})$, and $\mathfrak{su}(3) := \mathfrak{su}_g(\tilde{\Delta}_{\mathbf{C}})$, respectively,

Ingredients of SM

- $B_o \overset{P}{\simeq} B_p \subset \Theta_{\underline{\tilde{H}}}$; Weyl spinors.
- $B_o \oplus B_p \subset \Theta_{\underline{\tilde{H}}}$; Dirac spinors.
- $\tilde{\Delta} \subset \Theta_{\underline{\tilde{H}}}$; Gell-Mann, 8-fold way, 3 colours.
- $d_3 \in \tilde{\Delta} \cap A_{o,p}$; up-quark
- $d_1, d_2 \in \tilde{\Delta}, < \ d_1, d_2 > \perp d_3$; left-right down-quark.

Consider the list of markers, $\mathfrak{h}_1$, $\mathfrak{h}_2$, $I_3^{\pm} = 3/4\mathfrak{h}_2 \pm 1/2\mathfrak{h}_1$, and $Y_W = 1/2\mathfrak{h}_2 \pm \mathfrak{h}_1$.

Markers	$1/2 \cdot h$	$\mathfrak{h}_1$	$\mathfrak{h}_2$	I_3	Y_W
d_1	$1/2$	$1/2$	$-1/3$	0	$-2/3$
d_2	$-1/2$	$-1/2$	$-1/3$	$-1/2$	$1/3$
d_3	0	0	$2/3$	$1/2$	$1/3$
$p_1 = d_1 d_3 d_3$	$1/2$	$1/2$	1	1	0
$p_2 = d_2 d_3 d_3$	$-1/2$	$-1/2$	1	1	0

Markers	$1/2 \cdot h$	$\mathfrak{h}_1$	$\mathfrak{h}_2$	I_3	Y_W
$n_1 = d_1 d_1 d_3$	1	1	0	$1/2$	-1
$n_2 = d_2 d_2 d_3$	-1	-1	0	$-1/2$	1
$n_{1,2} = d_1 d_2 d_3$	0	0	0	0	0
$e_L = d_1 d_1 d_1$	$3/2$	$3/2$	-1	0	-2
$e_R = d_1 d_1 d_2$	$1/2$	$1/2$	-1	$-1/2$	-1

Markers	$1/2 \cdot h$	$\mathfrak{h}_1$	$\mathfrak{h}_2$	I_3	Y_W
$W^+ = d_3 d_2^{-1}$	$1/2$	$1/2$	1	1	0
$\nu_L = d_1^{-1} d_2$	-1	-1	0	$-1/2$	1

To go from left to right handedness comes out by just exchanging d_1 and d_2. Here one may see that, $e_L + \nu_L = e_R$, and one easily find reasons for the decays,

$$n \to p + e + \nu_e, p^+ \to e^+ + \pi^0 \to e^+ + 2\gamma, d_1 \to d_3 + w^-, p^+ = e^- \to n + \nu_e.$$

Notice also that we may, in an obvious way, identify ν_L with the element $e + f$ in one coordinate, swaping d_1, d_2.

Big Bang Model 1.

Consider a metric, and therefore the notion of Time of the Model $\underline{\tilde{H}}$. Pick,

$$g = (\frac{\rho - h(\lambda)}{\rho})^2 d\rho^2 + (\rho - h(\lambda))^2 d\phi^2 + \kappa(\lambda) d\lambda^2,$$

for the simplified space, in which $\underline{\omega}$ is reduced to the angle ϕ, and the coordinates $\underline{\lambda}$ reduced to one parameter $\lambda = |\underline{\lambda}|$. This correspond to considering the sub-universe of $\mathbf{M}(BB)$, parametrized by (λ, ϕ, ρ). Computing the Force Laws, we find,

$$\begin{aligned}\frac{d^2\rho}{dt^2} &= -(\frac{h(\lambda)}{\rho(\rho - h(\lambda))})(\frac{d\rho}{dt})^2 \\ &\quad + (\frac{2}{(\rho - h(\lambda))})(\frac{dh}{d\lambda})(\frac{d\rho}{dt})(\frac{d\lambda}{dt}) + (\frac{\rho^2}{(\rho - h(\lambda))})(\frac{d\phi}{dt})^2, \\ \frac{d^2\phi}{dt^2} &= -2/(\rho - h(\lambda))\frac{d\rho}{dt}\frac{d\phi}{dt} + 2/(\rho - h(\lambda))(\frac{dh}{d\lambda})(\frac{d\phi}{dt})(\frac{d\lambda}{dt})\end{aligned}$$

BB Model 2.

Moreover we find,

$$\frac{d^2\lambda}{dt^2} = -(\frac{\rho - h(\lambda)}{\rho})(\frac{1}{\kappa(\lambda)})(\frac{dh}{d\lambda})(\frac{d\rho}{dt})^2$$
$$- (\rho - h(\lambda))(\frac{1}{\kappa(\lambda)})(\frac{dh}{d\lambda})(\frac{d\phi}{dt})^2$$
$$+ (\frac{dlog(\kappa)}{d\lambda})(\frac{d\lambda}{dt})^2$$

where t, is the time parameter of the model. From these formulas we see that the Gravitation is expanding inside the Horizon and contracting outside. Conservation of mass implies, $h(\underline{\lambda}) = h_0/\lambda$. From this follows that the Horizon at the BB, i.e. for $\lambda = 0$, is all of space. Interpreting λ as Cosmological time we find a striking cosmological model, complete with Inflation and Hubble formulas, $v = r/t$ and $v/\sqrt{1 - v^2} = r/\lambda$.

Quotients in Geometry

Let $\mathfrak{g} \subset Der_k(A)$, be a sub Lie-module, and consider first, in the commutative case, the scheme (algebraic variety), $X = Spec(A)$, and the Lie algebra $\mathfrak{g}$ as a Lie algebra of vector fields defined on X. The set of maximal integral subschemes,

$$X/\mathfrak{g} := \{M \subset X | \Theta_M = \mathfrak{g}|M\},$$

is called the quotient of X by $\mathfrak{g}$, and coincides, in good cases with the quotient of X by the group of automorphisms $\mathbf{G}$ acting on X, with $\mathfrak{lie}\mathbf{G} = \mathfrak{g}$, when this exists. In the classical, commutative case, one would identify,

$$X/\mathfrak{g} = Spec(A^{\mathfrak{g}}),$$

where, $A^{\mathfrak{g}} := \{a \in A | \forall \gamma \in \mathfrak{g},\ \gamma(a) = 0\}$. This is, however, only reasonable when $\mathfrak{g}$ is reductive and/or all orbits of $\mathbf{G}$ are closed, which they rarely are.

Quotients in Non-commutative Algebraic Geometry

In the non-commutative situation, this last definition of a quotient, has no meaning. The algebra A may have no 1-dimensional representations at all. The solution is to define the relevant points $Simp(A)$, of the geometry $\mathbf{A}$, defined by A, and then to seek out those points that should correspond to the points of the quotient $\mathbf{A}/\mathfrak{g}$.
Consider a representation $\rho : A \rightarrow End_k(V)$, and let for every $\gamma \in \mathfrak{g}$ the derivation, $\delta := \gamma \circ \rho \in Der_k(A, Hom_k(V, V))$, map to 0 in $Ext^1_A(V, V)$. This means that the representation ρ, is not moved by the action of $\mathfrak{g}$. As above, it follows from, $\kappa(\delta) = 0$, that there exist an element, $Q_\delta \in Hom_k(V, V)$, the Hamiltonian, such that for all $\gamma \in \mathfrak{g}$ and all $a \in A$,

$$\rho(\delta(a)) = Q_\delta \circ \tilde{\rho}(a) - \rho(a) \circ Q_\delta.$$

which can be written as,

$$Q_\delta(av) = \gamma(a)v + aQ_\delta(v), \forall v \in V,$$

i.e. Q is a $\mathfrak{g}$-connection on V.

Quotients in Non-commutative Algebraic Geometry

Now, we define ,

$$\mathbf{A}/\mathfrak{g} := Simp(\{\rho|\delta := \gamma \circ \rho = 0 \in Ext^1_A(V, V)\})$$

where *Simp* of course picks out those representations of this sort, with no such sub-representations.

The aim of, my kind, of non-commutative algebraic geometry is to to associate to any family of representations **V**, of the algebra A, the optimal extension-algebra,

$$\eta : A \rightarrow \mathbf{O}(\mathbf{V})$$

for which the family **V** becomes the set of simple $\mathbf{O}(\mathbf{V})$-representations.
Then **V** should be called: A Scheme for $\mathbf{O}(\mathbf{V})$.

Quotients in Geometry and Physics: Gauge Groups

As we shall see, later on, the notion of Gauge Group in physics, is intimately related to this non-commutative quotient structure.
We shall, in fact, show that there is an algebra H, representing our Cosmos, a Lie algebra, $\mathfrak{G}$ acting on $H(\sigma) := Ph(H)/([h, dh]|h \in H)$, the partially commutativization of $Ph(H)$, such that the main ingredients of the Standard Model, including representations like the Weyl and the Dirac Spinors pop up as points of the quotient,

$$H(\sigma)/\mathfrak{G},$$

suggesting that the standard Quantum Theory should be thought of as part of non-commutative algebraic geometry.

The Semi-cosimplicial Structure of Ph*

It is easy to see that,

$$
\begin{aligned}
i_p^n i_q^{n+1} &= i_{q-1}^n i_p^{n+1},\ \ p < q \\
i_p^n i_p^{n+1} &= i_p^n i_{p+1}^{n+1} \\
i_p^n i_q^{n+1} &= i_q^n i_{p+1}^{n+1},\ \ q < p,
\end{aligned}
$$

proved by composing with i_0^{n-1} and d_{n-1}. Thus, the $Ph^*(A)$ is a semi-cosimplicial algebra with a cosection onto A. Therefore, for any object,

$$\kappa : A \to R \in A/k - \underline{alg}$$

the semi-cosimplicial algebra above induces a semi-simplicial k-vectorspace,

$$Der_k(Ph^\star(A), R),$$

and one should be interested in its homology .

Relation to the de Rham complex I

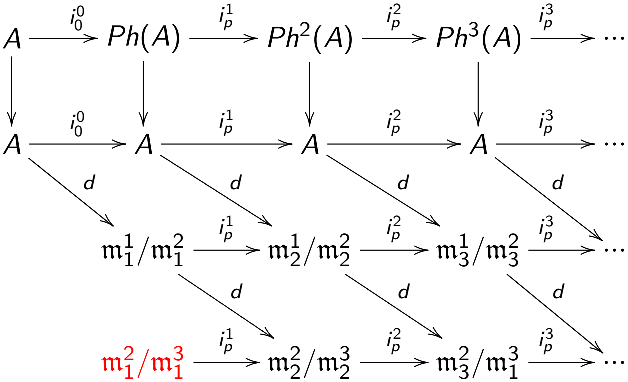
Consider now the diagram,

$$
\begin{array}{ccccccccc}
A & \xrightarrow{i_0^0} & Ph(A) & \xrightarrow{i_p^1} & Ph^2(A) & \xrightarrow{i_p^2} & Ph^3(A) & \xrightarrow{i_p^3} & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathfrak{m}_1^1 & \xrightarrow{i_p^1} & \mathfrak{m}_2^1 & \xrightarrow{i_p^2} & \mathfrak{m}_3^1 & \xrightarrow{i_p^3} &
\end{array}
$$

where, for each integer n, the symbol i_p^n, for $p = 0, 1, ..., n$ signify the family of A-morphisms between $Ph^n(A)$ and $Ph^{n+1}(A)$ defined above, and where $\mathfrak{m}_n^1$ is the ideal of $Ph^n(A)$ generated by $im(d)$, which is the same as the ideal generated by the family, $\{i_p^{n-1}(i_p^{n-2}(...(i_p^1(d(A))...))\}$, for all possible p. And, inductively, let $\mathfrak{m}_n^m$ be the ideal generated by $\mathfrak{m}_n^1\mathfrak{m}_n^{m-1}$.

Relation to the de Rham complex II

We find an extended diagram,



The diagonals are not necessarily complexes, but to kill all d^n, $n \geq 2$, it suffices to kill d^2, and for this it suffices to kill $d_1 d_0$, as one easily see, applying the edge homomorphisms to, $d_1(d_0(a))$ for all $a \in A$.

Curvature

Definition

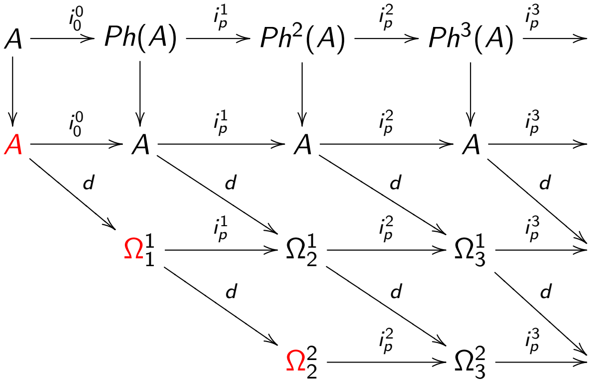
The curvature $R(A)$ of the associative k algebra, A, is the k-linear map composition of d_0 and d_1,

$$R(A) = d_0 d_1 : A \to \mathfrak{m}_2^2/\mathfrak{m}_2^3.$$

Now, kill the curvature $R(A)$, and all the terms under the first diagonal, beginning with $\mathfrak{m}_1^2/\mathfrak{m}_1^3$, together with all terms generated by the actions of the edge homomorphisms on these terms, and let, $\mathbf{\Omega}_n^m$ be the quotient of $\mathfrak{m}_n^m/\mathfrak{m}_n^{m+1}$, for $n \geq 0$. Clearly, $\Omega_n^0 = A$ for all $n \geq 0$, and we have got a graded semi co-simplicial A-module, with a k-differential d, such that $d^2 = 0$,

Generalized de Rham complex.

The diagram is now looking like,



It is therefore a graded complex, in two ways. First as a complex induced from the semi-cosimplicial structure, with differential of bidegree (1,0), and second, as complex with differential d, of bidegree (1,1).

The commutative case

Consider now the complex,

$$A \longrightarrow^{d} \Omega_1^1 \longrightarrow^{d} \Omega_2^2 \longrightarrow^{d} \Omega_3^3 \longrightarrow^{d} \ldots$$

.

Theorem

Suppose A is commutative, then there is a natural morphism of complexes of A-modules,

$$\Omega_A^{\star} \subset \Omega_{\star}^{\star},$$

with,

$$\Omega_A^n \simeq \Omega_n^n.$$

The proof

Let, $a_i \in A, i = 1, ..., r$, and compute in $\Omega^r_\star$ the value of, $d^r(a_1 a_2 ... a_r)$. It is clear that this gives the formula,

$$\sum d_{i_1}(a_1) d_{i_2}(a_2)..d_{i_r}(a_r) = 0,$$

the sum being over all permutation $(i_1, i_2, ..., i_r)$ of $(0, 1, ..., r-1)$. Here we consider A as a subalgebra of $Ph^n(A)$ via the unique compositions of the $i^s_0 : Ph^s(A) \subset Ph^{s+1}(A)$. In particular, we have,

$$d_0(a_1)d_1(a_2) + d_1(a_1)d_0(a_2) = 0,$$

for all $a_1, a_2 \in A$. This relation and the relation $d_0(a)d_1(b) = d_1(b)d_0(a)$, which follows from commutativity, $d(a)b = bd(a)$, forcing the left and right A-action on Ω_A to be equal, immediately give us, $d_0(a)d_1(b) = -d_0(b), d_1(a)$. It is now clear that the map that sends the element $da_1 \wedge da_2 \wedge ... \wedge da_r \in \Omega^r_A$ to $d_0(a_1)d_1(a_2)..d_{r-1}(a_r) \in \Omega^r_r$ is an isomorphism, and the rest should be clear.

Generalization to modules I

Let now, V be an A-module, and assume $c(V) = 0$, and pick a connection, $\nabla \in Hom_k(V, V \otimes_A Ph(A))$ with $c = \iota(\nabla)$. This imply that for $a \in A$ and $v \in V$ we have $\nabla(va) = \nabla(v)a + v \otimes d_0(a)$. Composing ∇ with the cosection, $o : Ph(A) \to A$, corresponding to the 0-derivation of A, we therefore obtain an A-linear homomorphism $P : V \to V$, *a potential*. Since $i_0^0 : A \to Ph(A)$ is a section of o, we find a k-linear map,

$$\nabla_0 := \nabla - P : V \to V \otimes \mathfrak{m}_1^1$$

Using the property,

$$d_n \circ i_{j+1}^{n+1} = i_j^n \, d_{n+1},$$

it is easy to find well defined k-linear maps,

$$\nabla_1 : V \to V \otimes \Omega_2^2, \ \nabla_2 : V \to V \otimes \Omega_3^3, ..., \nabla_n : V \to V \otimes \Omega_{n+1}^{n+1} \ \forall n \geq 0,$$

given by ,

$$\nabla_{n+1} := \nabla_n \circ i_1^{n+1}, \ n \geq 0.$$

Generalization to modules II

Fix the connection ∇. For all, $v \in V, \omega \in \Omega_n^n$, the formula,

$$\nabla_n(v \otimes \omega) = \nabla_n(v)\omega + v \otimes d_n(\omega).$$

makes sense, and defines *derivations*, also called d. We obtain a situation just like above,

$$\begin{array}{ccccccccc}
V & \xrightarrow{i_0^0} & V \otimes Ph(A) & \xrightarrow{i_p^1} & V \otimes Ph^2(A) & \xrightarrow{i_p^2} & V \otimes Ph^3(A) & \xrightarrow{i_p^3} & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
V & \xrightarrow{i_0^0} & V & \xrightarrow{i_p^1} & V & \xrightarrow{i_p^2} & V & \xrightarrow{i_p^3} & \\
& \searrow^{d} & & \searrow^{d} & & \searrow^{d} & & \searrow^{d} & \\
& & V \otimes \Omega_1^1 & \xrightarrow{i_p^1} & V \otimes \Omega_2^1 & \xrightarrow{i_p^2} & V \otimes \Omega_3^1 & \xrightarrow{i_p^3} & \\
& & & \searrow^{d} & & \searrow^{d} & & \searrow^{d} & \\
& & & & V \otimes \Omega_2^2 & \xrightarrow{i_p^2} & V \otimes \Omega_3^2 & \xrightarrow{i_p^3} &
\end{array}$$

Generalization to modules III

In general, there are no reasons for these $d's$ to define complexes, and we shall make the following definition,

Definition

The curvature $R(V,\nabla)$ of the connection ∇ defined on the right k A-module V, is the k-linear map composition of d_0 and d_1,

$$R(V,\nabla) = d_0 d_1 : V \to V \otimes \Omega_2^2.$$

The following result is then easily proved,

Theorem

Suppose A is commutative, and let $\nabla : \Theta_A \to End_k(V)$ be the connection corresponding to ∇_0. Suppose moreover that the curvature R of ∇ is 0, then $R(V) = 0$, implying that $d^2 = 0$, and so the diagonals in the diagram above, are all complexes.

Representations of $Ph^{\infty}(k[\underline{t}])$ I

Theorem

Given an r-dimensional $k[\underline{t}] := k[t_1, ..., t_d]$*-module, consisting of r points* $\{P_p = (\alpha_1^0(p), \alpha_2^0(p), ..., \alpha_d^0(p))\}_{p=1,...,r}$. *Assume given,* $\alpha_i^n(p) \in k,\ i = 1, ..., d,\ n \geq 0,\ p = 1, ..., r$, *and arbitrary coupling constants,* $\sigma_m(p, q) \in k$ *with* $\sigma_0 = 0$. *Put* $\alpha_i^n(p, q) = \alpha_i^n(p) - \alpha_i^n(q),\ i = 1, ..., d$, *and assume, for all* $n \geq 1$,

$$\sum_h \binom{n}{h} \sigma_{n-h}(p, q)(\alpha_i^h(p, q)\alpha_j^0(p, q) - \alpha_i^0(p, q)\alpha_j^h(p, q))$$

$$= \sum_{k,l,m,s} \frac{n!\sigma_{n-k-m}(p, s)\sigma_{k-l}(s, q)}{l!m!(k-l)!(n-k-m)!}(\alpha_j^m(p, s)\alpha_i^l(s, q) - \alpha_i^l(p, s)\alpha_j^m(s, q)),$$

Representations of $Ph^{\infty}(k[\underline{t}])$ II

Consider the matrix,

$$D_i^n := \begin{pmatrix} \alpha_i^n(1) & r_i^n(1,2) & \dots & r_i^n(1,r) \\ r_i^n(2,1) & \alpha_i^n(2) & \dots & r_i^n(2,r) \\ . & . & \dots & . \\ r_i^n(r,1) & r_i^n(r,2) & \dots & \alpha_i^n(r)) \end{pmatrix}$$

with,

$$r_i^0(p,q) = 0, \; r_i^n(p,q) = \sum_{l=0}^{n} \binom{n}{l} \alpha_i^l(p,q)\sigma_{n-l}(p,q),$$

Then $\rho(d^n t_i) = D_i^n$ define a representation,

$$\rho : Ph^{\infty}(k[\underline{t}])) \to M_r(k)$$

Proof

Let us, as above, consider the matrix,

$$X_i = \rho(exp(\tau\delta))(t_i) = \sum_{n\geq 0} \tau^n/n! D_i^n$$

Putting

$$\alpha_i(p) = \sum_{n=0}^{\infty} \tau^n/n!\ \alpha_i^n(p),\ \ \alpha_i(p,q) = \sum_{n=0}^{\infty} \tau^n/n!\ \alpha_i^n(p,q),$$

and, $\sigma(p,q) = \sum_{n=0}^{\infty} \tau^n/n!\ \sigma^n(p,q)$, we find the explicite formulas,

$$X_i = \begin{pmatrix} \alpha_i(1) & \sigma(1,2)\alpha_i(1,2) & ... & \sigma(1,r)\alpha_i(1,r) \\ \sigma(2,1)\alpha_i(2,1) & \alpha_i(2) & ... & \sigma(2,r)\alpha_i(2,r) \\ ... & ... & ... & ... \\ \sigma(r,1)\alpha_i(r,1) & \sigma(r,2)\alpha_i(r,2) & ... & \alpha_i(r) \end{pmatrix},\ \ i = 1, ..., d.$$

Now, compute, and put,

$$[X_i, X_j] = 0,$$

and see that the condition of the theorem emerges.

Formal Moduli of finite Representations of $Ph^{\infty}(k[\underline{t}]))$

We may consider the *space*

$$\mathbf{A}(r) = k[\alpha_i^n(p), \sigma_n(p,q)]/\mathfrak{a},$$

with coordinates $\{\alpha_i^n(p), \sigma_n(p,q), i = 1, ..., d, n \geqslant 0, p, q = 1, ..., r\}$, and where the ideal $\mathfrak{a}$ is generated by the equations above, as the versal base space for the versal family of the non-commutative deformation theory applied to the family of $Ph^{\infty}(k[\underline{t}]))$ modules defined by the object $\mathcal{P}$. Since $Ph^{\infty}(k[\underline{t}]))$ is infinitely generated, there is, strictly speaking, no such thing, but we shall see that in special cases, we can overcome this difficulty. In dimension $d = 3$ and order 1 the condition above reads:

$$\sigma_1(p,q)(\alpha^1(p,q)\times\alpha^0(p,q)) = -\sigma_1(p,s)\sigma_1(s,q)(\alpha^0(p,s)\times\alpha^0(s,q)), \forall p, s, q$$

This says that for any two of the three points in space, the relative momentum must sit in the plane defined by the three points, the length being determined by the 3 coupling constants. Moreover, the sum of all three relative momenta must be 0.

The tangent space $T^\star$ of the formal moduli of an associative algebra

For any associative k-algebra A, there is a formal moduli, i.e. a complete local k-algebra, $H(A)$, and a versal family μ of associative algebras,

$$\begin{array}{ccc} \mu : H(A) & \longrightarrow & \mathbf{A} \\ \downarrow & & \downarrow \\ B & \longrightarrow & A_B \end{array}$$

containing all infinitesimal deformations of A. The tangent space of $H(A)$, is calculated as:

$$T^\star = A^1(k, A; A) = Hom_F(ker(\rho), A)/Der$$

where, $\rho : F \rightarrow A$, is a surjective homomorphism of a free k-algebra F onto the given algebra A, Hom_F is the set of F-bilinear maps, and $Der \subset Hom_F$, denotes the restrictions to $ker(\rho)$, of the derivations $Der_k(F, A)$.

The formal moduli of U

It is easy to compute the tangent space of the formal moduli of U, the dimension of $Hom_F(ker(\rho), U)/Der$ turns out to be 27. Given two 3-vectors,

$$\overline{o} := (o_1, o_2, o_3), \overline{p} := (p_1, p_2, p_3)$$

the bi-linear homomorphisms,

$$\kappa : ker(rho) = (x_i x_j) \rightarrow U, \ \kappa(x_i x_j) := o_i x_j + x_i p_j$$

represents linearly independent elements in $A^1(k, U; U)$, and forms the tangent space, of the base $\underline{H}$ of the algebraic family

$$\tilde{\rho} : H \rightarrow \mathbf{U} := H < x_1, x_2, x_3 > /(x_i x_j - o_i x_j - x_i p_j + o_i p_j)$$

at the point U, corresponding to $\overline{o} := (0,0,0), \overline{p} := (0,0,0)$.

End Remark

So I have shown how the Time-space, **H**, can be thought of as an immediate product of a mathematical scenario incorporating A Big Bang Event, making this event mathematically sound.
This, however, should not be interpreted as if I actually propose this mathematical model, as a physical explanation of the universe we observe today, whatever that would mean. This talk has been about mathematics, and should be considered as a continuation of my struggle to understand the language of physics, as I have explained in [18]. In this struggle I have been comforted by Dirac, in particular by his "Lecture delivered on presentation of the James Scott prize, February 6, 1939", see [3], where he talks about The mathematical quality in Nature. It is this quality that fascinates me, and it is the realization of a relationship between *numbers and nature*, that goes back to the Pythagoreans, that has inspired me, see [6].

Time-Space and Space-Times

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