Geometry and Topology via Group theory : How to put geometric structure on topological spaces;

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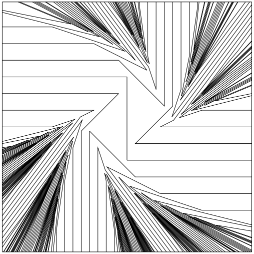
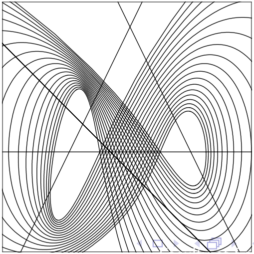
Faculty of Sciences, Meknès, 2 June 2014





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- If G acts *transitively,* that is, given any two points $x, y \in X$, some transformation $g \in G$ takes x to $g(x) = y$, then the work of Lie, Engel, Killing, Cartan et al "classifies" these *classical geometries.*

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- Parallelism.

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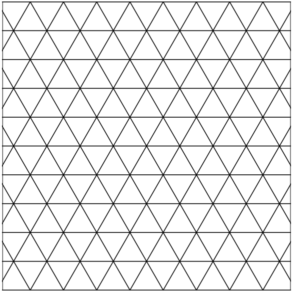
- Equilateral triangles (all angles 60^o) tile the plane.



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- For any $n \geq 3$, the sphere is tiled by $4n$ triangles, each with 2 right angles.

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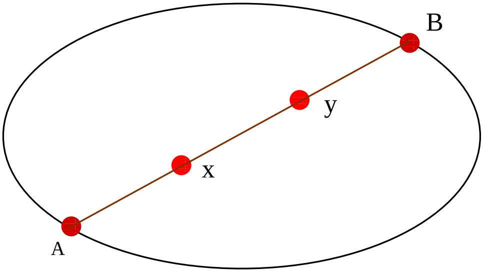

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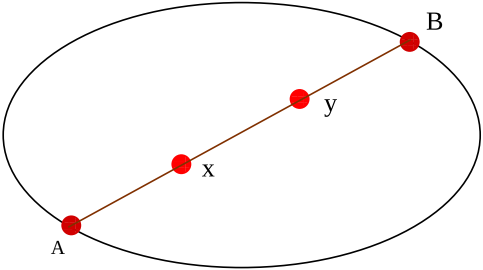

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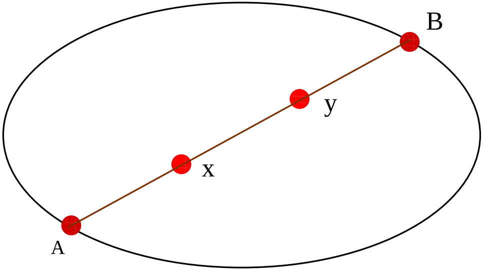

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- Exactly *two* lines through p are *asymptotic* to l; all others share a common perpendicular with l.

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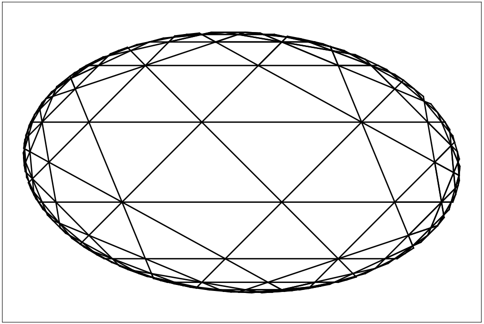
- The sum of the angles in a triangle is $< 180^o$.
- Here is a tiling of H^2 by triangles with angles $(60^o, 60^o, 45^o)$.



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  Given a *topology* Σ, that is, a loosely organized collection of points, and a *geometry* (G, X), *how many ways (if any) can one put the local geometry on* Σ*?*

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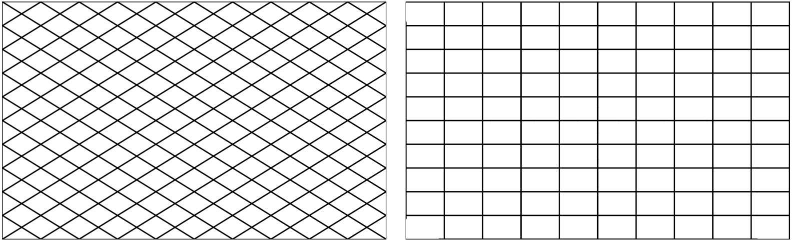
- *Example:* The S^2 admits no Euclidean structure:
  There exists no metrically accurate world atlas.

Example: Euclidean structures on the torus

- The 2-torus admits *many* Euclidean structures



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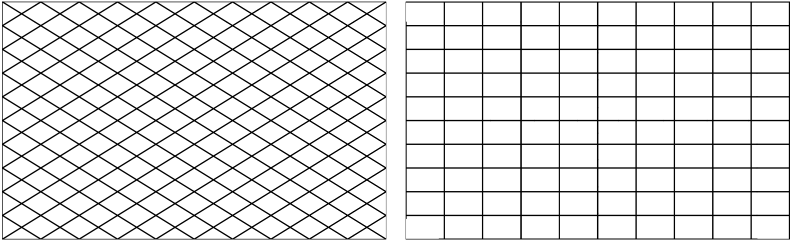

- Different Euclidean geometries on a torus correspond to *lattices* in $\mathbb{C}$, that is, embeddings $\mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathbb{C}$.

Moduli space for Euclidean structures on a torus

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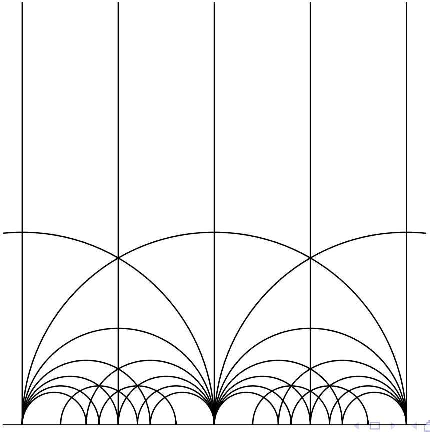

Moduli space for Euclidean structures on a torus

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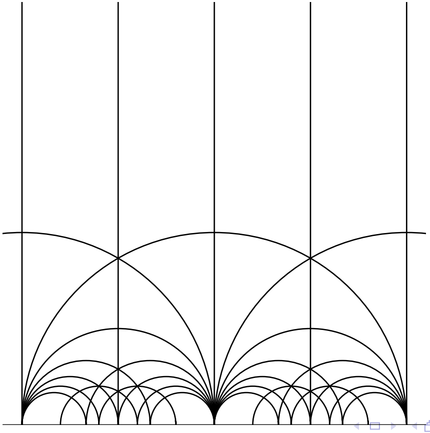
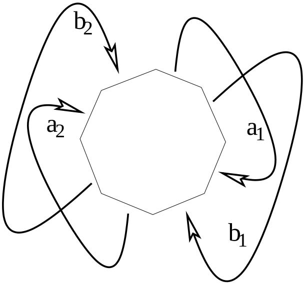
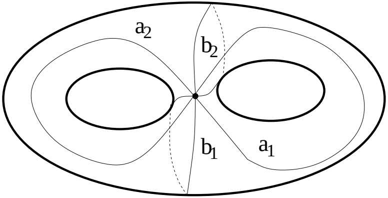
- This polygon has one right angle, one 60^o angle, and two sides which are *asymptotic,* (angle 0).



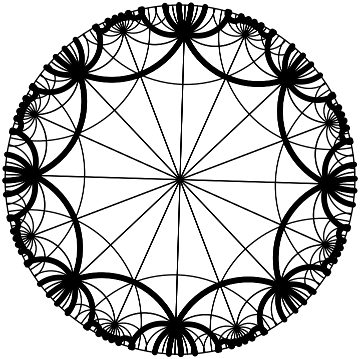
Example: A hyperbolic structure on a surface of genus two

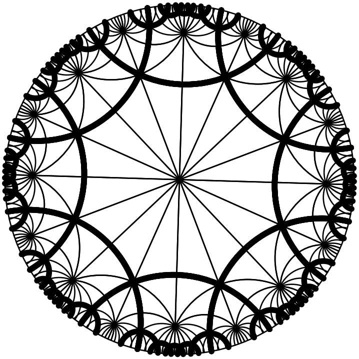
Identify sides of an octagon to form a closed genus two surface.





Two kinds of regular octagons in the hyperbolic plane





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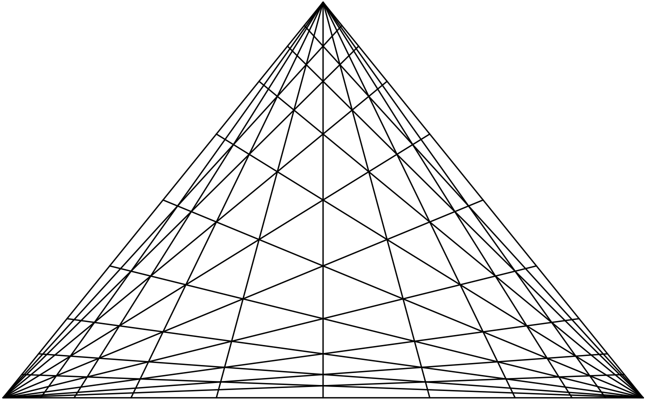
- *Locally homogeneous Riemannian geometries,* modeled on $X = G/H$, H compact.

Geometrization in 3 dimensions

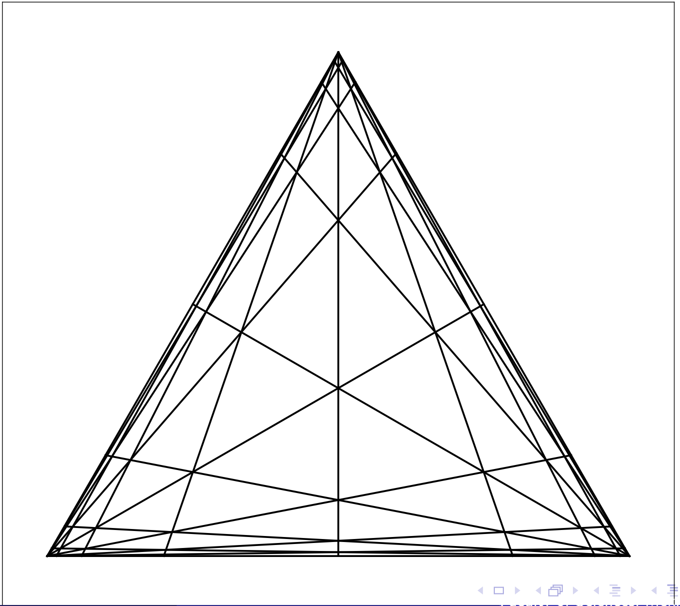
(Thurston 1976): 3-manifolds canonically decompose into *locally homogeneous Riemannian* pieces (8 types). (proved by Perelman)

Another example: Projective tiling of $\mathbb{RP}^2$ by equilateral 60^o-triangles



This tesselation of the open triangular region is equivalent to the tiling of the Euclidean plane by equilateral triangles.

Varying the projective triangle tiling



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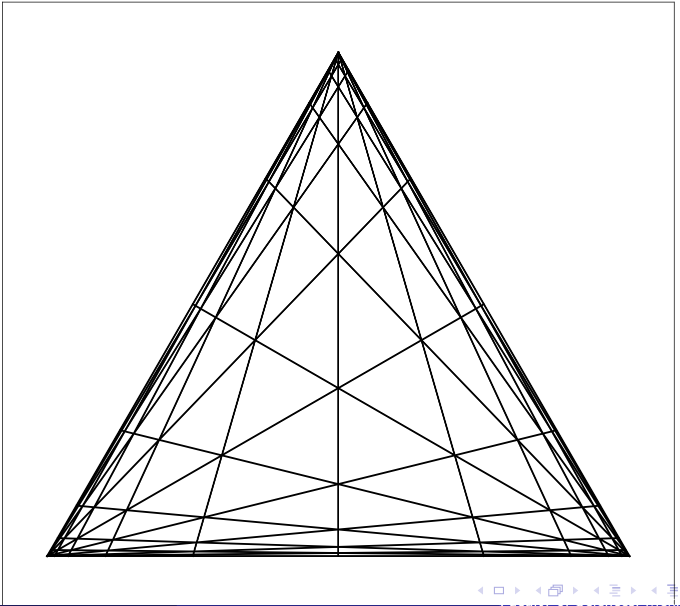

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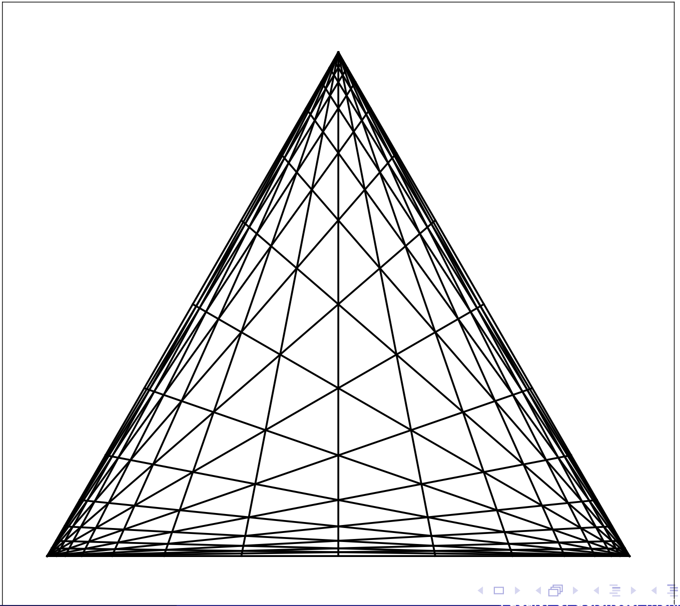

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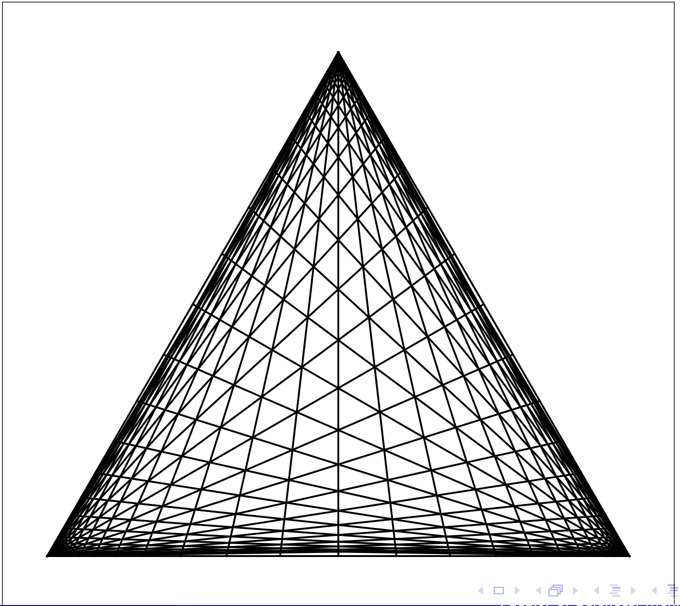

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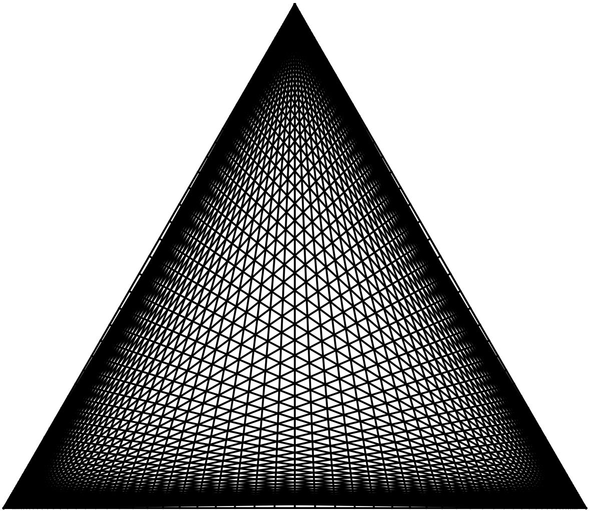
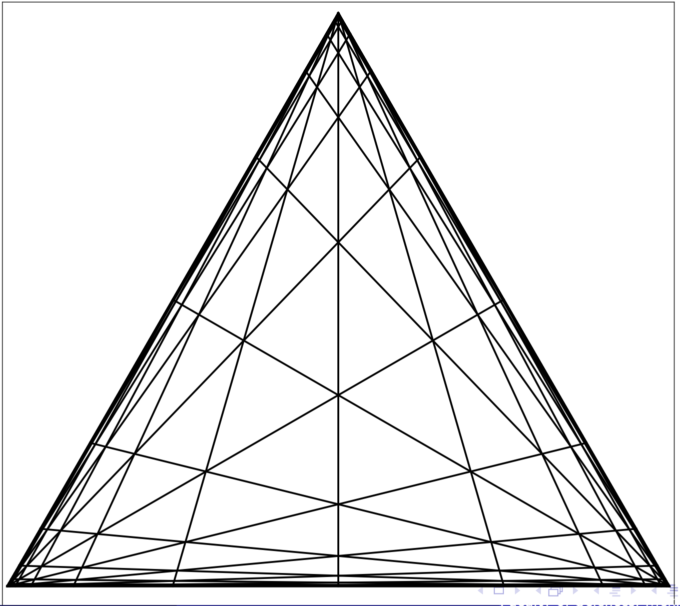


Limiting on the Euclidean tiling



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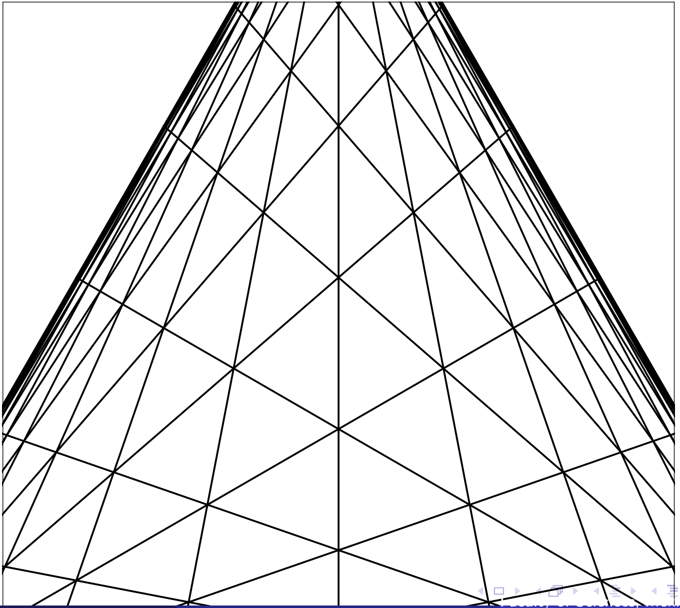

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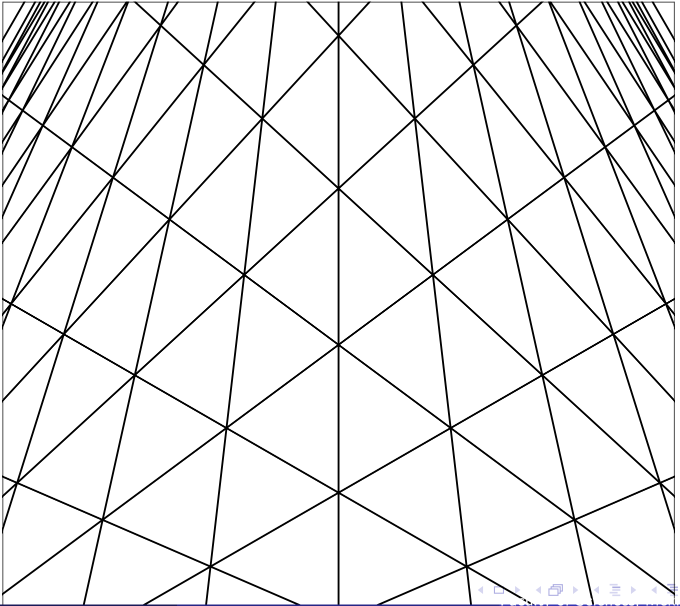

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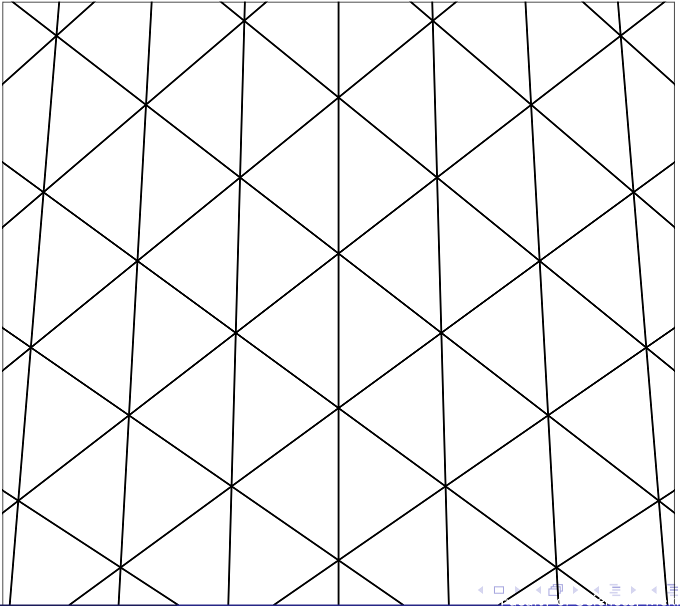
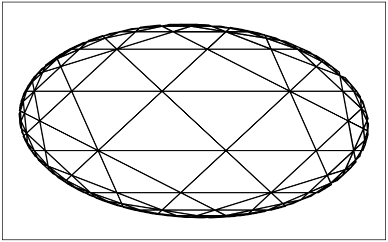
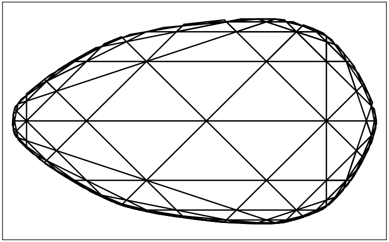


Example: A projective deformation of a tiling of the hyperbolic plane by $(60^o, 60^o, 45^o)$-triangles.





Both domains are tiled by triangles, invariant under a Coxeter group $\Gamma(3,3,4)$. First domain bounded by a conic (hyperbolic geometry), second domain bounded by $C^{1+\alpha}$-convex curve where $0 < \alpha < 1$.