spaces of complexity two

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1. Definitions

2. About spaces of $TC_n(X) = n$

We recall that $cat(X)$ is the smallest n for which there is an open covering $\{U_0, ..., U_n\}$ by $(n+1)$ open sets, each of which is contractible in X.

The sectional category of a fibration $p : E \longrightarrow B$, denoted by $secat(p)$, is the smallest number n for which there is an open covering $\{U_0, ..., U_n\}$ of B by $(n+1)$ open sets, for each of which there is a local section $s_i : U_i \longrightarrow E$ of p, so that $p \circ s_i = j_i : U_i \longrightarrow B$, where j_i denotes the inclusion.

Let PX denote the space of (free) paths on a space X. There is a fibration $P_2 : PX \longrightarrow X \times X$, which evaluates a path at initial and final point : for $\alpha \in PX$, we have $P_2(\alpha) = (\alpha(0), \alpha(1))$. This is a fibrational substitute for the diagonal map $\Delta : X \longrightarrow X \times X$. We define the topological complexity $TC(X)$ of X to be the sectional category $secat(P_2)$ of this fibration. That is, $TC(X)$ is the smallest number n for which there is an open cover $\{U_0, ..., U_n\}$ of $X \times X$ by $(n+1)$ open sets, for each of which there is a local section $s_i : U_i \longrightarrow PX$ of P_2, i.e., for which $P_2 \circ s_i = j_i : U_i \longrightarrow X \times X$, where j_i denotes the inclusion.

More generally, let $n \geq 2$ and consider the fibration

$$P_n : PX \longrightarrow X \times X \times \cdots \times X = X^n$$

defined by dividing the unit interval $I = [0, 1]$ into $(n-1)$ subintervals of equal length, with n subdivision points $t_0 = 0, t_1 = 1/(n-1), \ldots, t_{n-1} = 1$ (thus $(n-2)$ subdivision points interior to the interval), and then evaluating at each of the n subdivision points, thus :

$$P_n(\alpha) = (\alpha(0), \alpha(1), \ldots, \alpha(t_{n-2}), \alpha(1))$$

for $\alpha \in PX$. This is a fibrational substitute for the n-fold diagonal $\Delta_n : X \longrightarrow X^n$. Then the higher topological complexity $TC_n(X)$ is defined as $secat(P_n)$.

$f : X \longrightarrow Y$ is called a weak homotopy equivalence if it induces isomorphisms $\pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$ for all $n \geq 0$

whitehead theorem

A weak homotopy equivalent between C.W. complexes is a homotopy equivalence.

Proposition

$TC_n(S^k) = n - 1$ for k odd and $TC_n(S^k) = n$ for k even.

Questions

if X is a C.W. complexe *simply connected* integral homology sphere and $TC_n(X) = n$. Would we have X is of the homotopy equivalent to some sphere S^k for k even. ? ?

Questions

In particular, it's true if X is a weak homotopy equivalent to S^k, for k even, by whitehead theorem.

Definition

A space Y is called an H-space if there exists a map $m : Y \times Y \longrightarrow Y$ s.t $m \circ i_1 \simeq m \circ i_2 \simeq Id_Y$ where $i_1, i_2 : Y \longleftarrow Y \times Y$ are the maps defined by $i_1(y) = (y, y_0)$, $i_2(y) = (y_0, y)$ (Y based space at y_0)

if we consider a space X s.t. $Cat(X) = 1$ and $X \ncong S^{odd}$ would we have $TC(X) = 2??$.
for $X = \Sigma Y$ we have $TC(X) = 2$.
And in the case of X is an H-space and $Cat(X) = 2$ also we have $TC(X) = 2$

suppose now that $Cat(X) = TC(X) = 2$ would we have X as an H-space ? ?