String topology

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Références

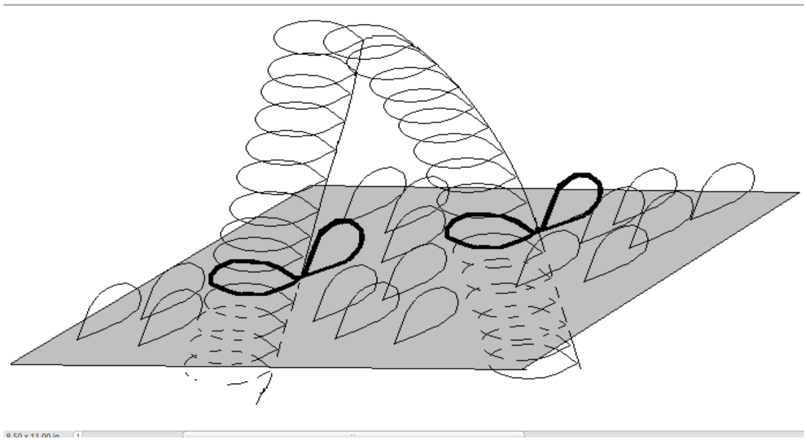
[CS99] M. Chas, D. Sullivan, *String Topology*, arXiv :math/9911159 [math.GT].

[L10] F. Laudenbach, *A note on the Chas-Sullivan product*, L'Enseignement Mathématique, Vol. **57**, Issue 1/2 (2011), 3-21.

1 Chas-Sullivan ideas

Let $X = M^n$ a n-dimensional orientable and compact manifold. Through this talk, $LX := X^{S^1}$ will denote its associated *free loop space*. The main goal of Chas-Sullivan's works is to extend the intersection product initialy defined on X to a similar one defined on LX. Their next aim was to study the algebaraic structure of $\mathbb{H}_*(LX)$ and relate it to a topological quantum field theories (TQFT).

Transeversality in LX is a little specific. As we know the usual notion of a tangent vector in a manifold is the velocity vector to a curve. But here we consider a path in LX, which is a path of loops, and a velocity vector in this case is a vector field.



An i-chain in LX is a linear combination of oriented i-dimensional simplices of loops in X. Such a chain naturally gives rise to a i-chain in X of associated marked points, the image of $1 \in S^1$.

The *loop homology product* denoted here by $\bullet$, is a combination of the intersection product an the product given by the concatenation of loops. It is defined transversally at the chain level of loops.

A natural approach (see [CS99]) to define the loop product is :

— consider two families of based loops (x, an i-chain of loops in X, and, y, an j-chain of loops in X ;

— intersect transversally in X the respective i-chain and j-chain of associative marked points ;

— obtain an $(i+j-n)$-chain z in X along which the marked points of x coincide with that of y.

— create a new family $(i+j-n)$-chain $x \bullet y$ of loops by concatenating at each point of z, loops that go around the loops of x and then around the loops of y.

Mathematically speaking, the loop product works as follows :

— consider $x : \Delta^i \longrightarrow X^{S^1}$ and $y : \Delta^j \longrightarrow X^{S^1}$ some simplices of X^{S^1} that $x(1) : \Delta^i \longrightarrow X$ and $y(1) : \Delta^j \longrightarrow X^{S^1}$ intersect transversally in X ;

— compute the intersection product $x(1).y(1)$ at each point $(s,t) \in \Delta^i \times \Delta^j$ such that $x(1)(s) = y(1)(t)$;

— perform the composition of the loops $x(s)$ and $y(t)$ to obtain at LM, the $i+j-n$-chain $x \bullet y$.

Set

$$\mathbb{H}_*(X^{S^1}) := H_{*+n}(X^{S^1}; \mathbb{Z}),$$

the so called *loop homology* of X. Then, the loop product passes to homology, following Lemma 2.3 in [CS99], and defines the *loop homology product*

$$\mathbb{H}_i(X) \otimes \mathbb{H}_j(X) \xrightarrow{\bullet} \mathbb{H}_{i+j}(X) ,$$

which endows $\mathbb{H}_*(X)$ with a structure of associative graded commutative algebra (see Theorem 2.2 in [CS99]). They defined also on $\mathbb{H}_*(X^{S^1})$ a string bracket $[-,-]$ which satisfies Jacobi and endows $\mathbb{H}_*(X^{S^1})$ with a structure of Lie graded algebra. Finaly, they defined a derivation $\Delta : \mathbb{H}_*(X^{S^1}) \longrightarrow \mathbb{H}_*(X^{S^1})$ of degree 1, which make the loop homology $\mathbb{H}_*(X^{S^1})$ into a Batalin Vilkovisky algebra. That means

— $(a,b) \mapsto (-1)^{|a|}\Delta - (a \bullet b) - (-1)^{|a|}\Delta a \bullet b - a \bullet \Delta b$ Δ is a derivation of each variable.

— The loop bracket $\{-,-\}$ on the loop homology is the deviation of Δ :

$$\{a,b\} = (-1)^{|a|}\Delta(a \bullet b) - (-1)^{|a|}\Delta a \bullet b - a \bullet \Delta b$$

2 Laudenbach generalization

Let now $X = M^n$ be a n-dimensional manifold, orientable or not, compact or not. We equip it with a chart $\mathcal{A}$. A p-simplex $\sigma : \Delta^p \longrightarrow X$ is said to be *small* if and only if $\exists U(\sigma) \in \mathcal{A}$ such that $\sigma(\Delta^p) \subset U(\sigma)$. $U(\sigma)$ is chosen once for all. We mean by *small p-chain*, we mean any linear combination $\sum n_i \sigma_i$ of many finitely p-small simplices σ_i, with integer coefficients $n_i ß\mathbb{Z}$.

A bi-simplex $\sigma \times \gamma : \Delta^p \times \Delta^q \longrightarrow X \times X$ is said to be *small* if and only if $\sigma : \Delta^p \longrightarrow X$ and $\gamma : \Delta^q \longrightarrow X$ are both small. A small bi-simplex $\sigma \times \gamma : \Delta^p \times \Delta^q \longrightarrow X \times X$ is said to be *transverse* when the map $\sigma \times \gamma$ and all its faces are transverse, in the usual context, to the diagonal Δ_X. Both of the two notions, small and transverse, extend naturally and linearly at the level of the bi-chains. The advantage to work with a transverse bi-simplex $\sigma \times \gamma$ is that $W = (\sigma \times \gamma)^{-1}(\Delta_X)$ is an orientable $p+q-n$-dimensional sub-manifold of $\Delta^p \times \Delta^q$ with corners. This yields to a $p+q-n$-small simplex

$$(\sigma \times \gamma)_{|W} : \Delta^{p+q} \longrightarrow \Delta_X \simeq X,$$

which extends naturally and linearly at the level of the bi-chains to *transverse product*.

A boundary operator at level of simplices chain, ∂ is well defined small chains, which tends naturally to a total boundary operator at the level of bi-chains by

$$D(\sigma \times \gamma) := \partial\sigma \times \gamma + (-1)^p \sigma \times \gamma.$$

Lemma 2.6 in [L10], states that small bi-cycle can be represented by a transverse small bi-cycle. Lemma 2.7 in [L10], states that it does not depend of the choice of the homological representants. Lemma 2.7 in [L10], claims that homological class of any transverse product does not depend on the the choice of the homological representants, and that the transverse product extends to a well defined *homological transverse* commutative and associative graded product :

$$\mathbb{H}_p(X) \otimes \mathbb{H}_p(X) \longrightarrow \mathbb{H}_{p+q}(X),$$

where $\mathbb{H}_*(X) := \mathbb{H}_{*+n}(X)$.