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On the Gorenstein spaces

We assume that the space is 1-connected with the cohomology is of finite type.

Let $(A, \mathfrak{m})$ be a local ring, $\mathbb{K} = A/\mathfrak{m}$ its residue field. Let n the Krull dimension of A. (A noetherian). Then A is called a Gorenstein ring if

$$\dim_{\mathbb{K}} \mathrm{Ext}_A^i(\mathbb{K}, A) = \begin{cases} 0 & i \neq n \\ 1 & i = n \end{cases}$$ or equivalently

$\mathrm{Ext}_A^i(\mathbb{K}, A) = 0$ for $i \neq n$ and $\mathrm{Ext}_A^n(\mathbb{K}, A) = \mathbb{K}$.

Now, let H be a noetherian (finitely generated) cdga. An element f in H is said regular in H is $\mathrm{Ann}(f) = \{0\}$

Recall. $\mathrm{Ann}_R M = \{ r \in R;\ \forall x \in M : rx = 0 \}$

It is obvious that $|f| =$ even.

A sequence $(f_1, \ldots, f_n)$ of H is said regular, if for every i, f_i is regular in $H/(f_1, \ldots, f_{i-1})$

* The Krull dimension of H is n if $\mathbb{Q}[x_1, \ldots, x_n] \subset H$

H is called Cohen-Macaulay alg if there exists a regular sequence $(f_1, \ldots, f_n)$ s.t. $\dim H/(f_1, \ldots, f_n) < \infty$

Then $\mathrm{Kdim}\, H = n$.

In particular, if there exists a regular sequence $(f_1, \ldots, f_n)$ s.t. $H/(f_1, \ldots, f_n)$ satisfies the Poincaré duality, we say that H is a <u>Gorenstein algebra</u>.

In [1], Félix, Halperin and Thomas introduced the Gorenstein spaces and gave the following

<u>Definition</u> A topological space X is called Gorenstein over a field $\mathbb{K}$ if

$\dim_{\mathbb{K}} \mathrm{Ext}_{(A,d)}(\mathbb{K}, (A, d)) = 1$, where $A = C^*(X; \mathbb{K})$

Note that to a top. space X, we can associate gda $(C_*(\Omega X, \mathbb{K}), \delta)$ and we have the following

<u>Theorem [1]</u>

$$\mathrm{Ext}_{(C^*(X;\mathbb{K}),d)}(\mathbb{K}, (C^*(X;\mathbb{K}),d)) \cong \mathrm{Ext}_{(C_*(\Omega X,\mathbb{K}),\delta)}(\mathbb{K}, C_*(\Omega X, \mathbb{K}), \delta)$$

The Ext = Ext and in general, the Eilenberg-Moore s.s converges

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{H^*(A,\mathbb{K})}(\mathbb{K}, H^*(A,\mathbb{K})) \Longrightarrow \mathrm{Ext}_A^{p+q}(\mathbb{K}, A)$$

<u>Recall</u>. Def of $\mathrm{Ext}^n(A,B)$

Let A, B be R-modules, let $A \xleftarrow{\varepsilon} P_*$ be a projective resolution of A and $B \xrightarrow{\eta} I^*$ an injective resolution of B. We can form then a double complex $\mathrm{Hom}_R(P_*, I^*)$

The morphisms ε and η induce maps from $\mathrm{Hom}_R(P_*, I^*)$ to $\mathrm{Hom}_R(A, I^*)$ and $\mathrm{Hom}_R(P_*, B)$

$$\begin{array}{ccccccc}
 & & \mathrm{Hom}_R(A, I^0) & \longrightarrow & \mathrm{Hom}_R(A, I^1) & \longrightarrow & \cdots \\
 & & \downarrow \varepsilon^* & & \downarrow \varepsilon^* & & \\
\mathrm{Hom}_R(P_0, B) & \xrightarrow{\eta_*} & \mathrm{Hom}_R(P_0, I^0) & \longrightarrow & \mathrm{Hom}_R(P_0, I^1) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_R(P_1, B) & \xrightarrow{\eta_*} & \mathrm{Hom}_R(P_1, I^0) & \longrightarrow & \mathrm{Hom}_R(P_1, I^1) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

<u>Def</u>. The cohomology groups $H^n(\mathrm{Hom}_R(A, I^*))$ and $H^n(\mathrm{Hom}_R(P_*, B))$ are called the Ext groups, and are denoted by $\mathrm{Ext}^n(A,B)$ and we have

$$\mathrm{Ext}^n(A,B) = H^n(\mathrm{Hom}_R(A, I^*)) \cong H^n(\mathrm{Hom}_R(P_*, B))$$

<u>Prop [2]</u> Let A be a dga s.t $\dim H^i(A) < \infty, \forall i$, if $H^*(A)$ is Gorenstein, then A is it also.

The converse is false in general (even if $H^*(A)$ is noetherian).

<u>Examples</u>. 1/ $(A,d) = (\Lambda(u, v, w, t), d)$, $du = dv = dw = 0$
$dt = uvw$.

$H^*(A) = \Lambda(u, v, w, t)/\langle uvw, aw + bv\rangle$, with $|u| = 2$, $|v| = |w| = 3$ and $|t| = 7$, $a = [vt]$ and $b = [wt]$

$H^*(A)$ is noetherian.

We know [1] that if $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$ then

$$fd(X; \mathbb{Q}) = \sum_{|x_i| \text{ odd}} |x_i| - \sum_{|x_i| \text{ even}} (|x_i| - 1)$$

Hence $fd(A) = 1 - |u| + |v| + |w| + |t| = 12$, then there exists an element f in $Ext_{H^*(A)}(\mathbb{Q}, H^*(A))$, $|f| = 12$

<u>The socle</u>

Let $A \xrightarrow{\varepsilon} \mathbb{K}$ an augmented cdga

the socle of A is defined by

$socle(A) = \{w \in A;\ w \cdot \ker \varepsilon = 0\}$

$Z_0 := \{\text{generators of } A\}$

$w \in socle(A) \Leftrightarrow w \cdot z_0 = 0,\ \forall z_0 \in Z_0$

<u>Prop</u> Let H be a graded alg. then

$Ext^{0,-}_H(\mathbb{Q}, H) = socle(H)$

But $Ext^{0,-}_{H^*(A)}(\mathbb{Q}, H^*(A)) = socle(H^*(A))$

and $[vw] \in socle(H^*(A))$, $|vw| = 6$

Then $Ext^{*,*}_{H^*(A)}(\mathbb{Q}, H^*(A))$ contains at least two elements of degrees 12 and 6 $\Rightarrow H^*(A)$ is not Gorenstein

<u>Example 2</u> Let $(A, d) = (\Lambda(a, x, y, z), d)$ with $dy = ax$ and $dz = x^2$, $|a|$ is even.

In the case $|a|$ even: $H(A) = \Lambda(x)/x^2 \oplus \Lambda(a, t)$ with $t = [xy + az]$ (Example 1)

Case $|a|$ odd: we have $H(A) = \bigoplus_{i \geq 0} (\mathbb{Q}u_i \oplus \mathbb{Q}v_i) = socle\, H(A)$ where $u_i = [xy^i]$ and $v_i = [ay^i]$

Then $H(A)$ is not Gorenstein alg. because $Ext^{0,-}_{H^*(A)}(\mathbb{Q}, H^*(A)) = socle(H^*(A))$ is of infinite dimension

<u>Formal Dimension</u>.

Def 1 [1] The formal dimension of a dga A denoted fd is: $fd(A;\mathbb{K}) := \sup\{r \in \mathbb{Z};\ [Ext_A(\mathbb{K},A)]^r \neq 0\}$

Prop. [1] If $\dim H^*(A;\mathbb{K}) < \infty$ then

$fd(A;\mathbb{K}) = \sup\{r \in \mathbb{N};\ H^r(A;\mathbb{K}) \neq 0\}$

Prop. Let (A,d) be a dga of Gorenstein, then $fd(A,\mathbb{K}) = |f|$ where $Ext_A(\mathbb{K},A)$ is generated by the class $[f]$.

Proof. $fd(A,\mathbb{K}) = \{r \in \mathbb{Z};\ [Ext_A(\mathbb{K},A)]^r \neq 0\} = |f|$.

proposition. Let X be 1-connected space s.t $\pi_*(X)\otimes\mathbb{Q}$ is of finite dimension, then X is a $\mathbb{Q}$-Gorenstein space

Rmk. Every elliptic space is a $\mathbb{Q}$-Gorenstein space in particular, if an elliptic space X verifies the H conjecture, then it is $\mathbb{Q}$-Gorenstein space.

III. Poincaré complexes

The Gorenstein spaces generalize the Poincaré Complexes.

The cap product $\cap$ is defined by:

$$(C^*(X;\mathbb{K}),d)\otimes(C_*(X;\mathbb{K}),d) \longrightarrow (C_*(X;\mathbb{K}),d)$$
$$x\otimes\alpha \longmapsto x\cap\alpha$$

We have the duality formula $\langle x\cup y,\alpha\rangle = \langle x, y\cap\alpha\rangle$ where $\cup$ denotes the cup product

$$\langle\, ,\rangle : H^*(X;\mathbb{K})\otimes H_*(X;\mathbb{K}) \longrightarrow \mathbb{K}$$
$$x\otimes\alpha \longmapsto \langle x,\alpha\rangle$$

Recall. Let X be a CW-complex of $\dim X = n$. X is said a Poincaré complex if there exists $u \in H_n(X,\mathbb{K})$ s.t: $\eta_u : H^*(X;\mathbb{K}) \longrightarrow H_*(X;\mathbb{K})$, $x \longmapsto x\cap u$ is an isomorphism

This means that $H^n(X;\mathbb{K}) = \mathbb{K}\omega$ and there exists for all i, $0\leq i\leq n$, an isom.

$$H^i(X,\mathbb{K}) \longrightarrow H^{n-i}(X;\mathbb{K})$$
$$x \longmapsto x^\vee \text{ (dual of } x)$$ s.t. $x\cup x^\vee = \omega$.

IV - Filtration

Theorem [2] - Let X be a Gorenstein space over $\mathbb{K}$.
$\dim H^*(X;\mathbb{K}) < \infty \Leftrightarrow H^*(X;\mathbb{K})$ satisfies the Poincaré duality

Theorem [1] - Let $F \to E \xrightarrow{p} B$ be a fibration of spaces simply connected s.t one of these hyp. holds:

(i) $\dim H^*(F;\mathbb{Q}) < \infty$

(ii) $\dim \pi_*(F)\otimes\mathbb{Q} < \infty$ and $\pi_*(p)\otimes\mathbb{Q}$ is surjective

Then there is the isom.

$$\varphi: Ext_{C^*(B;\mathbb{Q})}(\mathbb{Q}; C^*(B;\mathbb{Q})) \otimes Ext_{C^*(F;\mathbb{Q})}(\mathbb{Q}; E^*(F;\mathbb{Q})) \xrightarrow{\cong} Ext_{C^*(E;\mathbb{Q})}(\mathbb{Q}; C^*(E;\mathbb{Q}))$$

In particular E is a $\mathbb{Q}$-Gorenstein space $\Leftrightarrow$ B and F are $\mathbb{Q}$-Gorenstein space.

Theorem. [2] If $H^*(A;\mathbb{Q})$ is a Cohen-Macaulay alg and (A,d) is a cdga of Gorenstein then $H^*(A;\mathbb{Q})$ is a $\mathbb{Q}$-Gorenstein alg.

proof. See [2]

Lemma [2] Let (A,d) be a Gorenstein cdga and f be a regular element of $H^*(A;\mathbb{Q})$. Then there exists an extension

$$(A,d) \leftarrow (A\otimes\Lambda x, D) \to (\Lambda(x), 0) \quad \text{s.t}$$

$(A\otimes\Lambda x, D)$ be a Gorenstein over $\mathbb{Q}$ and

$$H^*(A\otimes\Lambda x;\mathbb{Q}) = H^*(A;\mathbb{Q})/(f)$$

V The Computation of Ext

In general, to calculate $Ext_H^{*,*}(\mathbb{Q},H)$, H is a grad. alg we use a projective resolution of $\mathbb{Q}$ in H-module

$$\cdots \xrightarrow{d} P_i \xrightarrow{d} \cdots \xrightarrow{d} P_1 \xrightarrow{d} H \xrightarrow{\varepsilon} \mathbb{Q} \to 0$$

$$\varphi \in Hom_H^i(P,H), \quad \varphi = \sum \lambda_j f_j, \quad f_j : P_j \to H$$

$$\lambda_j \in H; \quad |f_j| = i \quad \text{ie} \quad |f_j(p)| - |p| = i$$

The differential $D: Hom_H(P_j,H) \to Hom_H(P_{j+1},H)$ is defined by $Df_j = f_j\circ d$ and

$$Ext_H^{p,q}(\mathbb{Q},H) = H^p(Hom_H^{p+q}(P,H), D)$$

Example. $H = H(A)$ with $(A,d) = (\Lambda(u,v,w),d)$ with $du = dv = 0$, $dw = uv$, $|u| = |v| = 2$ and $|w| = 3$.

$H^*(A) = \Lambda u \oplus \Lambda v$ and the resolution is:

$$\cdots \xrightarrow{\tilde d} H\otimes(\Lambda \overline{Z})_i \xrightarrow{\tilde d} \cdots \xrightarrow{\tilde d} H\otimes(\Lambda\overline{Z})_2 \xrightarrow{\tilde d} H\otimes(\Lambda\overline{Z})_1 \xrightarrow{\tilde d} H \xrightarrow{\varepsilon} \mathbb{Q} \to 0$$

with $(\Lambda\overline{Z})_{2i} = \{\bar z^i, \bar z\bar y, \bar z^{i-1}\}$, $(\Lambda\overline{Z})_{2i+1} = \{\bar x\bar z^i; \bar y\bar z^i\}$

The diff. is defined by $\tilde d\bar u = u$, $\tilde d\bar v = v$, and $\tilde d\bar w = -y\bar x$.

$\mathbb{C}P^\infty$[2] $\dim \mathrm{Ext}_{C^*(\mathbb{C}P^\infty;\mathbb{Q})}(\mathbb{Q}, C^*(\mathbb{C}P^\infty;\mathbb{Q})) = 1$?

the Sullivan model of $\mathbb{C}P^\infty$: $(\Lambda x, 0) \xrightarrow{\simeq} A_{PL}(\mathbb{C}P^\infty)$
$|x| = 2$

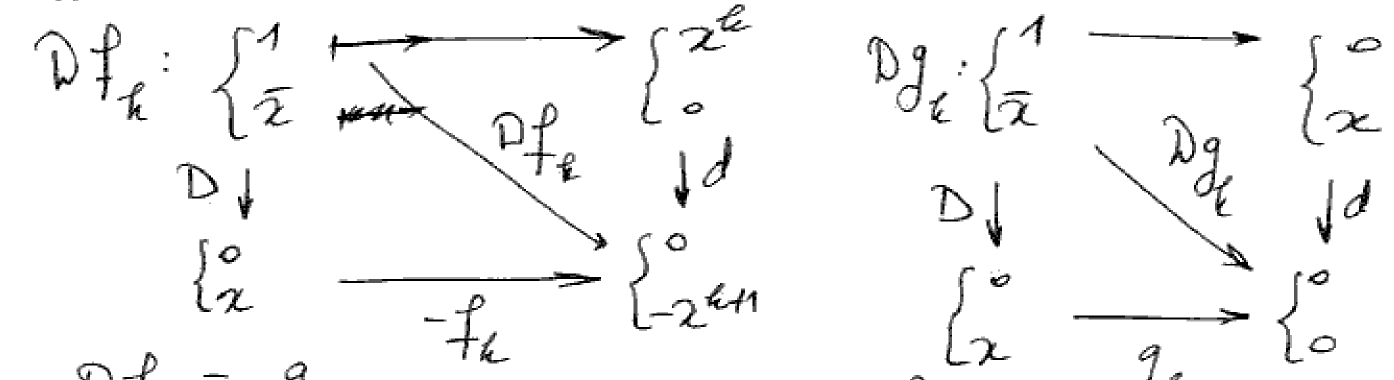
$$\mathrm{Ext}_{(\Lambda(x),0)}(\mathbb{Q}, (\Lambda(x),0)) = H^*(\mathrm{Hom}_{(\Lambda(x),0)}((\Lambda(x)\otimes\Lambda(\bar x), D), (\Lambda(x),0)), D)$$

is of dim 1. where $(\Lambda(x)\otimes\Lambda(\bar x), D) \xrightarrow{\simeq} \mathbb{Q}$ is the acyclic closure of $(\Lambda(x),0)$

S "$\mathrm{Hom}_{(\Lambda(x),0)}((\Lambda(x)\otimes\Lambda(\bar x), D), (\Lambda(x),0)), D) = \{\varphi : (\Lambda(x)\otimes\Lambda(\bar x), D) \to (\Lambda(x),0)$, $(\Lambda(x),0)$ - linear map$\}$

$f_k : 1 \longmapsto x^k$, $g_k : \bar x \longmapsto x^k$

f_k and g_k are generators of S.



$Df_k = -g_{k+1}$ $\qquad$ $Dg_k = 0$

Only g_0 ~~has~~ that is not a ~~bound~~ $g_0 : \bar x \longrightarrow 1$

$\Rightarrow \mathbb{C}P^\infty$ is a Gorenstein space.

References:

[1] Y. Félix, S. Halperin & J.-C. Thomas. Gorenstein spaces - Advances in Mathematics 71, 92-112 (1988)

[2] Gammelin, Hervé - Thèse 1997 - Espaces de Gorenstein & Applications d'Évaluation