Topological Complexity and Motion Planning in Certain Real Grassmannians

Khalid BOUTAHIR

Faculté des Sciences de Meknès

11 Octobre 2014

INTRODUCTION: Topological Complexity of a space

A motion planner on a space X will usually have some amount of discontinuity.

A motion planner on a space X will usually have some amount of discontinuity.
An instrument to measure this discontinuity is the topological complexity of the space X .

INTRODUCTION: Topological Complexity of a space

A motion planner on a space X will usually have some amount of discontinuity.
An instrument to measure this discontinuity is the topological complexity of the space X .
A main reason of interest in this quantity is that it is a homotopy invariant of the configuration space.

INTRODUCTION: Topological Complexity of a space

Definition

The topological complexity of the space X , $TC(X)$, is the minimal number k such that there exists an open cover $X \times X = U_1 \cup ... \cup U_k$ with the property that each U_i admits a continuous motion planner $s_i : U_i \longrightarrow PX$.

INTRODUCTION: Topological Complexity of a space

Definition

The topological complexity of the space X , $TC(X)$, is the minimal number k such that there exists an open cover $X\times X = U_1\cup\ldots\cup U_k$ with the property that each U_i admits a continuous motion planner $s_i : U_i \longrightarrow PX$.

Definition

the Lusternik-Schnirelmann category of a topological space, cat(X), is the least integer n such that X can be covered by $(n + 1)$ open subsets contractible in X, and is infinite if no such n exists.

INTRODUCTION: Topological Complexity of a space

Definition

The topological complexity of the space X , TC(X), is the minimal number k such that there exists an open cover $X \times X = U_1 \cup ... \cup U_k$ with the property that each U_i admits a continuous motion planner $s_i : U_i \longrightarrow PX$.

Definition

the Lusternik-Schnirelmann category of a topological space, cat(X), is the least integer n such that X can be covered by $(n + 1)$ open subsets contractible in X, and is infinite if no such n exists.

Proposition 1 For a path-connected topological space X it holds

$$cat(X) \leq TC(X) \leq cat(X \times X) \leq 2cat(X)-1.$$

INTRODUCTION: Topological Complexity of a space

Definition

The topological complexity of the space X , TC(X), is the minimal number k such that there exists an open cover $X \times X = U_1 \cup \ldots \cup U_k$ with the property that each U_i admits a continuous motion planner $s_i : U_i \longrightarrow PX$.

Definition

the Lusternik-Schnirelmann category of a topological space, cat(X), is the least integer n such that X can be covered by $(n + 1)$ open subsets contractible in X, and is infinite if no such n exists.

Proposition 1 For a path-connected topological space X it holds

$$\mathrm{cat}(X) \leq TC(X) \leq \mathrm{cat}(X \times X) \leq 2\mathrm{cat}(X)\text{-}1.$$

Proposition 2 Given two polyhedra X and Y one has

$$TC(X \times Y) \leq TC(X) + TC(Y)\text{-}1.$$

TC of Spaces with Small Fundamental Group

Theorem (1)

If X is an r-connected simplicial polyhedron with covering dimension dimX, then

$$TC(X) < \frac{2\dim X + 1}{r+1} + 1.$$

In particular we have the general bound

$$TC(X) \leq 2dimX + 1.$$

Theorem (1)

If X is an r-connected simplicial polyhedron with covering dimension dimX, then

$$TC(X) < \frac{2\dim X + 1}{r+1} + 1.$$

In particular we have the general bound

$$TC(X) \leq 2dimX + 1.$$

Theorem (2)

Let X be a cell complex with $\pi_1(X) = \mathbb{Z}_2$. Then

$$TC(X) \leq 2dim(X).$$

Furthermore, for a closed manifold X with $\pi_1(X) = \mathbb{Z}_2$ it holds that

$$TC(X) \leq 2dim(X) - 1$$

TC of Spaces with Small Fundamental Group

Theorem (3)

For a closed connected n-dimensional manifold X with $\pi_1(X) = \mathbb{Z}_2$ one has cat(X) = dim(X) + 1 if and only if $\omega^n = 0 \in H^n(X; \mathbb{Z}_2)$ where $\omega \in H^1(X; \mathbb{Z}_2)$ is the generator.

We now have a clear picture of the case when the space X has fundamental group $\pi_1(X) = \mathbb{Z}_2$

TC of Spaces with Small Fundamental Group

Theorem (3)

For a closed connected n-dimensional manifold X with $\pi_1(X) = \mathbb{Z}_2$ one has $cat(X) = dim(X) + 1$ if and only if $\omega^n = 0 \in H^n(X; \mathbb{Z}_2)$ where $\omega \in H^1(X; \mathbb{Z}_2)$ is the generator.

We now have a clear picture of the case when the space X has fundamental group $\pi_1(X) = \mathbb{Z}_2$

Theorem (4)

Let X be a finite cell complex such that $\pi_1(X) = \mathbb{Z}_3$.

1. *Assume that either dim X is odd or dim $X = 2n$ is even and the 3-adic expansion of n contains at least one digit 2. Then, $TC(X) \leq 2\,dim(X)$.*
2. *For any integer $n \geq 1$ having only the digits 0 and 1 in its 3-adic expansion there exists a finite polyhedron X of dimension $2n$ with $\pi_1(X) = \mathbb{Z}_3$ and $TC(X) = 2\,dim(X) + 1$.*

Grassmannians

Definition

Let V be a finite-dimensional vector space over a field $\mathbb{k}$. The Grassmannian Gr(k,V) is the set of all k-dimensional linear subspaces of V. If V has dimension n, then the Grassmannian is also denoted Gr(k,n), and if the underlying field is $\mathbb{R}$ this Grassmannian is also denoted $G_k(\mathbb{R}^n)$ we can write, by identification, $G_k(\mathbb{R}^n) = O(n)/(O(k) \times O(n-k))$ where O(n) is the orthogonal group.

Grassmannians

Definition

Let V be a finite-dimensional vector space over a field $\mathbb{k}$. The Grassmannian Gr(k,V) is the set of all k-dimensional linear subspaces of V. If V has dimension n, then the Grassmannian is also denoted Gr(k,n), and if the underlying field is $\mathbb{R}$ this Grassmannian is also denoted $G_k(\mathbb{R}^n)$ we can write, by identification, $G_k(\mathbb{R}^n)$= O(n)/(O(k) $\times$O(n − k)) where O(n) is the orthogonal group.

Let $G_k(\mathbb{R}^{n+k})$ be the set of all real k-dimensional subspaces of $\mathbb{R}^{n+k}$. $G_k(\mathbb{R}^{n+k})$ is a compact connected differentiable manifold of real dimension nk with $\pi_1(G_k(\mathbb{R}^{n+k})) = \mathbb{Z}_2$. In particular, $G_1(\mathbb{R}^{n+1}) = \mathbb{R}P^n$.

From **theorem (2)** we have the upper bound $\mathrm{TC}(\mathrm{G}_k(\mathbb{R}^{n+k})) \leq$ 2kn.
Moreover, it follows from **theorem (2)** and **theorem (3)** that $\mathrm{TC}(\mathrm{G}_k(\mathbb{R}^{n+k})) \leq$ 2kn-1.
In [I. Berstein, **On the Lusternik-Schnirelmann category of Grassmannians**], the author shows that in some cases

$$\mathrm{cat}(\mathrm{G}_k(\mathbb{R}^{n+k})) = \dim(\mathrm{G}_k(\mathbb{R}^{n+k}))+1=\mathrm{nk}+1.$$

From **theorem (2)** we have the upper bound $\mathrm{TC}(G_k(\mathbb{R}^{n+k})) \leq 2\mathrm{kn}$.
Moreover, it follows from **theorem (2)** and **theorem (3)** that $\mathrm{TC}(G_k(\mathbb{R}^{n+k})) \leq 2\mathrm{kn}\text{-}1$.
In [I. Berstein, **On the Lusternik-Schnirelmann category of Grassmannians**], the author shows that in some cases

$$\mathrm{cat}(G_k(\mathbb{R}^{n+k})) = \dim(G_k(\mathbb{R}^{n+k}))+1=\mathrm{nk}+1.$$

Notice that by applying the upper bound given by **Proposition 1** only allows to establish the general dimensional upper bound $\mathrm{TC}(G_k(\mathbb{R}^{n})) \leq 2\mathrm{kn}+1$.

Definition

We define $\mathrm{ht}(\omega_1)$, the height of ω_1, to be $\mathrm{ht}(\omega_1) := \sup\{m;\ \omega_1^m \neq 0 \in H^*(G_k(\mathbb{R}^{n+k});\ \mathbb{Z}_2)\}$.

Definition

We define $\mathrm{ht}(\omega_1)$, the height of ω_1, to be $\mathrm{ht}(\omega_1) := \sup\{m;\ \omega_1^m \neq 0 \in H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2)\}$.

Proposition 3 (R. Stong, **Cup products in Grassmannians**, Topology Appl. 13, 103-113, (1982))
In $G_k(\mathbb{R}^{n+k})$, for $2 \leq k \leq n$ with $2^s < n + k \leq 2^{s+1}$, we have

$$\mathrm{ht}(\omega_1) = \begin{cases} 2^{s+1} - 2, & \text{if } k = 2 \text{ or if } k = 3 \text{ and } n + k = 2^s + 1, \\ 2^{s+1} - 1, & \text{otherwise}. \end{cases}$$

Theorem (5)

For $2 \leq k \leq n$ with $2^s < n + k \leq 2^{s+1}$, we have the following .

1. *If k=2, then $TC(G_2(\mathbb{R}^{n+2})) \geq n$.*
2. *If k=3 and if $n+3 = 2^s+1$, then $TC(G_3(\mathbb{R}^{n+3})) \geq 2n+2$.*
   If $k=3$ and if $nn+3 \neq 2^s+1$, then $TC(G_3(\mathbb{R}^{n+3})) \geq n+2$.
3. *If $4 \leq k \leq n$, then $TC(G_k(\mathbb{R}^{n+k})) \geq n+k-1$.*

Theorem (5)

For $2 \leq k \leq n$ with $2^s < n + k \leq 2^{s+1}$, we have the following .

1. *If k=2, then $TC(G_2(\mathbb{R}^{n+2})) \geq n$.*
2. *If k=3 and if $n+3 = 2^s + 1$, then $TC(G_3(\mathbb{R}^{n+3})) \geq 2n+2$.*
   If $k = 3$ and if $nn + 3 \neq 2^s + 1$, then $TC(G_3(\mathbb{R}^{n+3})) \geq n+2$.
3. *If $4 \leq k \leq n$, then $TC(G_k(\mathbb{R}^{n+k})) \geq n + k - 1$.*

Proof.

We apply **Proposition 1** to see that $\mathrm{TC}(\mathrm{G}_k(\mathbb{R}^{n+k})) \geq \mathrm{cat}(\mathrm{G}_k(\mathbb{R}^{n+k})) \geq \mathrm{ht}(\omega_1)$ and the assertions follow from **Proposition 3.** □

EXAMPLE

$\mathbf{TC}(\mathbf{G}_2(\mathbb{R}^4)$

It is a quick observation from **Proposition 1** and **Theorem (5)** that

$$2 \leq \mathrm{TC}(\mathrm{G}_2(\mathbb{R}^4)) \leq 9.$$

EXAMPLE

$\mathbf{TC}(\mathbf{G}_2(\mathbb{R}^4))$

It is a quick observation from **Proposition 1** and **Theorem (5)** that

$$2 \leq \mathrm{TC}(\mathrm{G}_2(\mathbb{R}^4)) \leq 9.$$

Lemma (1)

We have $3 \leq TC(G_2(\mathbb{R}^4)) \leq 5$.

EXAMPLE

$\mathbf{TC}(\mathbf{G}_2(\mathbb{R}^4)$

It is a quick observation from **Proposition 1** and **Theorem (5)** that

$$2 \leq \mathrm{TC}(\mathrm{G}_2(\mathbb{R}^4)) \leq 9.$$

Lemma (1)

We have $3 \leq TC(G_2(\mathbb{R}^4)) \leq 5$.

Proof.

In [H. Hiller, **On the cohomology of real Grassmannians**, p. 529], it is shown that $\mathrm{cat}(\mathrm{G}_2(\mathbb{R}^{2^s+2})) = 2^{\,s+1}$ - 1. Thus, $\mathrm{cat}(\mathrm{G}_2(\mathbb{R}^4)) = 3$. By **Theorem (5)**, we see that $3 \leq \mathrm{TC}(\mathrm{G}_2(\mathbb{R}^4)) \leq 5$. □

EXAMPLE

TC$(\mathbf{G}_2(\mathbb{R}^4)$

It is a quick observation from **Proposition 1** and **Theorem (5)** that

$$2 \leq \mathrm{TC}(\mathrm{G}_2(\mathbb{R}^4)) \leq 9.$$

Lemma (1)

We have $3 \leq TC(G_2(\mathbb{R}^4)) \leq 5$*.*

Proof.

In [H. Hiller, **On the cohomology of real Grassmannians**, p. 529], it is shown that $\mathrm{cat}(\mathrm{G}_2(\mathbb{R}^{2^s+2})) = 2^{\ s+1}$ - 1. Thus, $\mathrm{cat}(\mathrm{G}_2(\mathbb{R}^4)) = 3$. By **Theorem (5)**, we see that $3 \leq \mathrm{TC}(\mathrm{G}_2(\mathbb{R}^4)) \leq 5$. □

In [K.J.PEARSON AND TAN ZHANG, **Topological Complexity and Motion Planning in Certain Real Grassmannians]** there it was wrongly assumed that TC(X)=cat(X×X) (Theorem1.8). This compromises the result "$\mathrm{TC}(\mathrm{G}_2(\mathbb{R}^4))=5$"

MERCI