Topological Complexity and Motion Planning in Certain Real Grassmannians

Khalid BOUTAHIR

Faculté des Sciences de Meknès

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A motion planner on a space $X$ will usually have some amount of discontinuity. An instrument to measure this discontinuity is the topological complexity of the space $X$. A main reason of interest in this quantity is that it is a homotopy invariant of the configuration space.
The topological complexity of the space $X$, $TC(X)$, is the minimal number $k$ such that there exists an open cover $X \times X = U_1 \cup \ldots \cup U_k$ with the property that each $U_i$ admits a continuous motion planner $s_i: U_i \rightarrow PX$. 

The Lusternik-Schnirelmann category of a topological space, $cat(X)$, is the least integer $n$ such that $X$ can be covered by $(n + 1)$ open subsets contractible in $X$, and is infinite if no such $n$ exists.

Proposition 1: For a path-connected topological space $X$ it holds $cat(X) \leq TC(X) \leq 2cat(X)-1$.

Proposition 2: Given two polyhedra $X$ and $Y$ one has $TC(X \sqcup Y) = TC(X) + TC(Y) - 1$. 

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INTRODUCTION: Topological Complexity of a space

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**Proposition 1** For a path-connected topological space $X$ it holds

$$\text{cat}(X) \leq \text{TC}(X) \leq \text{cat}(X \times X) \leq 2\text{cat}(X)-1.$$  

**Proposition 2** Given two polyhedra $X$ and $Y$ one has

$$\text{TC}(X \times Y) \leq \text{TC}(X) + \text{TC}(Y)-1.$$
Theorem (1)

If $X$ is an $r$-connected simplicial polyhedron with covering dimension $\dim X$, then

$$TC(X) < \frac{2\dim X + 1}{r + 1} + 1.$$ 

In particular we have the general bound

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TC of Spaces with Small Fundamental Group

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**Theorem (2)**

Let $X$ be a cell complex with $\pi_1(X) = \mathbb{Z}_2$. Then

\[ TC(X) \leq 2 \dim(X). \]

Furthermore, for a closed manifold $X$ with $\pi_1(X) = \mathbb{Z}_2$ it holds that

\[ TC(X) \leq 2 \dim(X) - 1 \]
Theorem (3)

For a closed connected n-dimensional manifold $X$ with $\pi_1(X) = \mathbb{Z}_2$ one has $\text{cat}(X) = \text{dim}(X) + 1$ if and only if $\omega_n = 0 \in H^n(X; \mathbb{Z}_2)$ where $\omega \in H^1(X; \mathbb{Z}_2)$ is the generator.

We now have a clear picture of the case when the space $X$ has fundamental group $\pi_1(X) = \mathbb{Z}_2$.
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Theorem (4)

Let $X$ be a finite cell complex such that $\pi_1(X) = \mathbb{Z}_3$.

1. Assume that either $\dim X$ is odd or $\dim X = 2n$ is even and the 3-adic expansion of $n$ contains at least one digit 2. Then, $\text{TC}(X) \leq 2 \dim(X)$.

2. For any integer $n \geq 1$ having only the digits 0 and 1 in its 3-adic expansion there exists a finite polyhedron $X$ of dimension $2n$ with $\pi_1(X) = \mathbb{Z}_3$ and $\text{TC}(X) = 2 \dim(X) + 1$. 
Grassmannians

Definition

Let $V$ be a finite-dimensional vector space over a field $\mathbb{k}$. The Grassmannian $\text{Gr}(k,V)$ is the set of all $k$-dimensional linear subspaces of $V$. If $V$ has dimension $n$, then the Grassmannian is also denoted $\text{Gr}(k,n)$, and if the underlying field is $\mathbb{R}$ this Grassmannian is also denoted $\text{G}_k(\mathbb{R}^n)$ we can write, by identification, $\text{G}_k(\mathbb{R}^n) = \text{O}(n)/(\text{O}(k) \times \text{O}(n-k))$ where $\text{O}(n)$ is the orthogonal group.
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Let $\text{Gr}_k(\mathbb{R}^{n+k})$ be the set of all real $k$-dimensional subspaces of $\mathbb{R}^{n+k}$. $\text{Gr}_k(\mathbb{R}^{n+k})$ is a compact connected differentiable manifold of real dimension $nk$ with $\pi_1(\text{Gr}_k(\mathbb{R}^{n+k})) = \mathbb{Z}_2$. In particular, $\text{Gr}_1(\mathbb{R}^{n+1}) = \mathbb{R}P^n$. 
From \textit{theorem (2)} we have the upper bound $\text{TC}(G_k(\mathbb{R}^{n+k})) \leq 2kn$. Moreover, it follows from \textit{theorem (2)} and \textit{theorem (3)} that $\text{TC}(G_k(\mathbb{R}^{n+k})) \leq 2kn-1$.

In [I. Berstein, \textit{On the Lusternik-Schnirelmann category of Grassmannians}], the author shows that in some cases

$$\text{cat}(G_k(\mathbb{R}^{n+k})) = \dim(G_k(\mathbb{R}^{n+k}))+1=nk+1.$$
Topological Complexity of Grassmannians

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Notice that by applying the upper bound given by **Proposition 1** only allows to establish the general dimensional upper bound \( \text{TC}(G_k(\mathbb{R}^n)) \leq 2kn+1 \).
Topological Complexity of Grassmannians

Definition

We define $ht(\omega_1)$, the height of $\omega_1$, to be $ht(\omega_1) := \sup\{m; \omega_1^m \neq 0 \in H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2)\}$. 
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In $G_k(\mathbb{R}^{n+k})$, for $2 \leq k \leq n$ with $2^s < n + k \leq 2^{s+1}$, we have

$$ht(\omega_1) = \begin{cases} 
2^{s+1} - 2, & \text{if } k = 2 \text{ or if } k = 3 \text{ and } n + k = 2^s + 1, \\
2^{s+1} - 1, & \text{otherwise}.
\end{cases}$$
Topological Complexity of Grassmannians

Theorem (5)

For $2 \leq k \leq n$ with $2^s < n + k \leq 2^{s+1}$, we have the following.

1. If $k=2$, then $TC(G_2(\mathbb{R}^{n+2})) \geq n$.
2. If $k=3$ and if $n + 3 = 2^s + 1$, then $TC(G_3(\mathbb{R}^{n+3})) \geq 2n + 2$.
   
   If $k = 3$ and if $n + 3 \neq 2^s + 1$, then $TC(G_3(\mathbb{R}^{n+3})) \geq n + 2$.
3. If $4 \leq k \leq n$, then $TC(G_k(\mathbb{R}^{n+k})) \geq n + k - 1$. 

Proof. We apply Proposition 1 to see that $TC(G_k(\mathbb{R}^{n+k})) \geq \text{cat}(G_k(\mathbb{R}^{n+k}))$ and the assertions follow from Proposition 3.
Theorem (5)

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Proof.

We apply Proposition 1 to see that $TC(G_k(\mathbb{R}^{n+k})) \geq cat(G_k(\mathbb{R}^{n+k})) \geq ht(\omega_1)$ and the assertions follow from Proposition 3.
It is a quick observation from **Proposition 1** and **Theorem (5)** that

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We have \(3 \leq \text{TC}(G_2(\mathbb{R}^4)) \leq 5.\)
**EXAMPLE**

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*We have* \[ 3 \leq \text{TC}(G_2(\mathbb{R}^4)) \leq 5. \]

**Proof.**

In [H. Hiller, *On the cohomology of real Grassmannians*, p. 529], it is shown that \( \text{cat}(G_2(\mathbb{R}^{2^s+2})) = 2^{s+1} - 1 \). Thus, \( \text{cat}(G_2(\mathbb{R}^4)) = 3 \). By **Theorem (5)**, we see that \( 3 \leq \text{TC}(G_2(\mathbb{R}^4)) \leq 5. \) □
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In [H. Hiller, On the cohomology of real Grassmannians, p. 529], it is shown that $\text{cat}(G_2(\mathbb{R}^{2s+2})) = 2^{s+1} - 1$. Thus, $\text{cat}(G_2(\mathbb{R}^4)) = 3$. By Theorem (5), we see that $3 \leq \text{TC}(G_2(\mathbb{R}^4)) \leq 5$.

In [K.J.PEARSON AND TAN ZHANG, Topological Complexity and Motion Planning in Certain Real Grassmannians] there it was wrongly assumed that $\text{TC}(X) = \text{cat}(X \times X)$ (Theorem 1.8). This compromises the result "$\text{TC}(G_2(\mathbb{R}^4)) = 5$"
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