SPECTRAL SEQUENCE OF A FILTRED SPACE

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Summary

- Spectral Sequences and Convergence
- Construction of Spectral Sequences
- Some examples

Spectral Sequences and Convergence

Let C be an abelian category.

Definition (Spectral sequences): A (homologically graded) *spectral sequence* is a family of objects $\{E^r_{pq}\}$ of C, for all $p, q \in \mathbb{Z}$ and $r \geq a$ (with a fixed $a \in \mathbb{Z}$), together with differentials $d^r_{pq} : E^r_{pq} \longrightarrow E^r_{p-r,q+r-1}$ which satisfy $d^r \circ d^r = 0$. Furthermore we require that there are isomorphisms

$$E^{r+1}_{pq} \cong H(E^r_{pq}) = \ker(d^r_{pq}) \Big/ \operatorname{im}(d^r_{p+r,q-r+1}) .$$

By $n := p + q$ we denote the *total degree* of E^r_{pq}.

Spectral Sequences and Convergence

The collections $(E^r_{pq})_{p,q\in\mathbb{Z}}$ for fixed r are called the *sheets* or *pages* of the spectral sequence. By the isomorphisms required above, we imagine that we get from one sheet to the next one ("turing a page around") by taking homology.

Spectral Sequences and Convergence

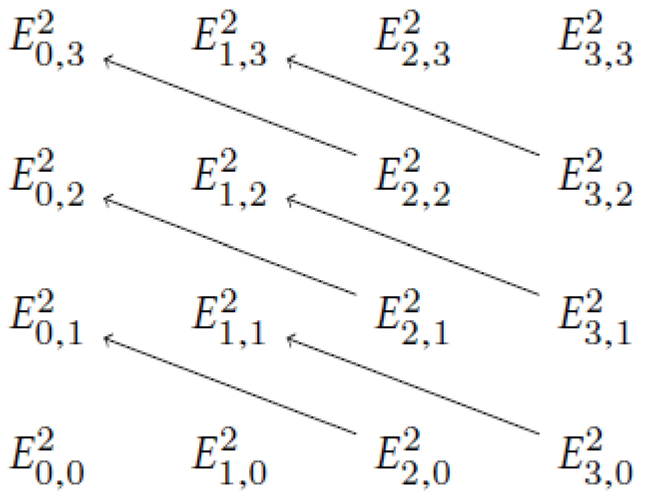
$$
r = 0. \qquad
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\downarrow & \downarrow & \downarrow & \downarrow \\
E^0_{0,3} & E^0_{1,3} & E^0_{2,3} & E^0_{3,3} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
E^0_{0,2} & E^0_{1,2} & E^0_{2,2} & E^0_{3,2} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
E^0_{0,1} & E^0_{1,1} & E^0_{2,1} & E^0_{3,1} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
E^0_{0,0} & E^0_{1,0} & E^0_{2,0} & E^0_{3,0} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\vdots & \vdots & \vdots & \vdots
\end{array}
$$

first sheet

$$
\begin{array}{c}
\cdots \longleftarrow E^1_{0,3} \longleftarrow E^1_{1,3} \longleftarrow E^1_{2,3} \longleftarrow E^1_{3,3} \longleftarrow \cdots \\
\cdots \longleftarrow E^1_{0,2} \longleftarrow E^1_{1,2} \longleftarrow E^1_{2,2} \longleftarrow E^1_{3,2} \longleftarrow \cdots \\
\cdots \longleftarrow E^1_{0,1} \longleftarrow E^1_{1,1} \longleftarrow E^1_{2,1} \longleftarrow E^1_{3,1} \longleftarrow \cdots \\
\cdots \longleftarrow E^1_{0,0} \longleftarrow E^1_{1,0} \longleftarrow E^1_{2,0} \longleftarrow E^1_{3,0} \longleftarrow \cdots
\end{array}
$$

Spectral Sequences and Convergence

the 2-sheet



Spectral Sequences and Convergence

Definition (Convergence): Let $\{H_n\}$ be a family of objects of $\mathcal{C}$.
We say a spectral sequence ...

(a) ... *weakly converges* to H_* if there exists a filtration

$$\ldots \subseteq F_{p-1}H_n \subseteq F_pH_n \subseteq F_{p+1}H_n \subseteq \ldots \subseteq H_n$$

for each $n \in \mathbb{Z}$ and furthermore isomorphisms

$$\beta_{pq} : E^{\infty}_{pq} \cong F_pH_{p+q} \big/ F_{p-1}H_{p+q} .$$

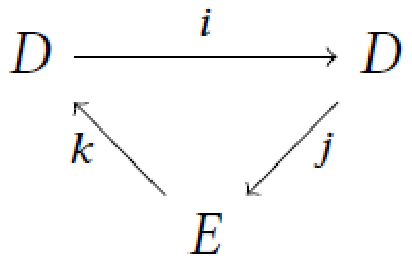
(b) ... *approaches* H_* if it weakly converges to H_* and[1]

$$H_n = \bigcup F_pH_n \quad \text{and} \quad \bigcap F_pH_n = 0.$$

(c) ... *converges* to H_* if it approaches H_* and $H_n = \varprojlim \left(H_n/F_pH_n\right)$.[2]
We denote convergence by $E^r_{pq} \Rightarrow H_{p+q}$.

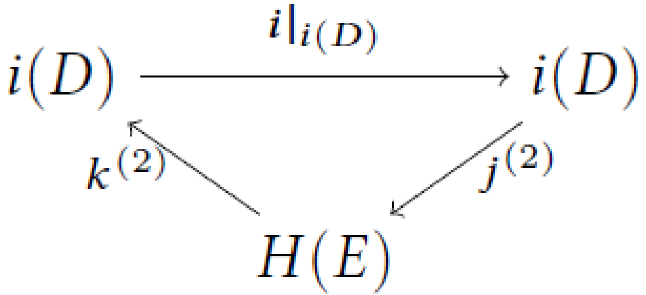
Construction of Spectral Sequences

Construction 2.1 (Exact couples and their derivations): An *exact couple* is a pair of objects D and E of C and morphisms i, j and k such that the diagram



is exact at each vertex. Since $d := jk$ complies with $d^2 = (jk)^2 = j(kj)k = j0k = 0$ we can apply homology by setting $H(E) = \ker(d)/\mathrm{im}(d)$ and get the *derived exact couple*

Construction of Spectral Sequences



with $j^{(2)}(i(x)) = [j(x)]$ and $k^{(2)}([e]) = k(e)$, $x \in D$ and $[e] \in H(E)$. It is an easy computation that $j^{(2)}$ and $k^{(2)}$ are well-defined and that the derived couple is exact. It suggests itself to iterate this process and set $E^1 := E$ and $E^r = H(E^{r-1})$, $d^1 := d = jk$ and $d^r := j^{(r)}k^{(r)}$.

Construction of Spectral Sequences

The $(r+1)$-th exact couple looks like

$$\begin{array}{ccc} i^r(D) & \xrightarrow{\; i|_{i^r(D)} \;} & i^r(D) \\ & {\scriptstyle k^{(r+1)}} \nwarrow \quad \swarrow {\scriptstyle j^{(r+1)}} & \\ & E^{r+1}. & \end{array}$$

Construction of Spectral Sequences

Let C_* be a complex with a filtration $\ldots F_pC_* \subseteq F_{p+1}C_* \subseteq \ldots \subseteq C_*$

such that there are integers $s < t$ for each n with $F_sC_n = 0$ and $F_tC_n = C_n$. (Such a filtration is called *bounded*. Particularly this means $F_kC_n = 0$ for all $k \leq s$ and $F_kC_n = C_n$ for all $k \geq t$. We will always consider *canonically bounded* filtrations, i.e. $s = -1$ and $F_nC_n = C_n$ for all n – what leads to a first quadrant spectral sequence.)

Construction of Spectral Sequences

We get short exact sequences

$$0 \longrightarrow F_{p-1}C_* \stackrel{i}{\longrightarrow} F_pC_* \stackrel{\pi_p}{\longrightarrow} F_pC_*/F_{p-1}C_* \longrightarrow 0$$

and, by applying homology, long exact secquences

$$\cdots \longrightarrow H_{p+q+1}(F_{p-1}C_*) \stackrel{i}{\longrightarrow} H_{p+q}(F_pC_*) \stackrel{j}{\longrightarrow} H_{p+q}\left(F_pC_*/F_{p-1}C_*\right) \stackrel{\delta}{\longrightarrow} H_{p-1+q}(F_pC_*) \longrightarrow \cdots$$

Construction of Spectral Sequences

Now we can roll up those long exact sequences into an exact triangle

$$\begin{array}{ccc} \bigoplus H_{p+q}(F_pC_*) & \xrightarrow{\quad i \quad} & \bigoplus H_{p+q}(F_pC_*) \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & \bigoplus H_{p+q}(F_pC_*/F_{p-1}C_*) & \end{array}$$

which gives us a spectral sequence E^r_{pq}.

SOME EXAMPLES

1. Leray-Serre spectral sequence. Given a fibration $F \hookrightarrow X \to B$ with trivial monodromy on the homology of the fibers, $E^2_{p,q} = H_p(B; H_q(F))$ and the spectral sequence abuts to $H_*(X)$. (If there is monodromy, this still works but replacing $E^2_{p,q} = H_p(B; H_q(F))$ with $E^2_{p,q} = H_p(B; \mathcal{H}_q(F))$, where $\mathcal{H}_q(F)$ is a local system on B). There is also a cohomology version.

Some examples

2. Atiyah-Hirzebruch spectral sequence. If F_* is a generalized homology theory (such as K-homology, bordism, etc.), then there is a spectral sequence with $E^2_{p,q} = H_p(X; F_q(pt))$ that abuts to $F_*(X)$. There is also a generalized cohomology version.

Some examples

Hodge-De Rham spectral sequence

Adams spectral sequence

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