The Homotopy Lie Algebra Of Configuration Spaces

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Introduction

The aim of this talk is to give a some known results about the homotopy Lie algebra of Configuration spaces of closed compact manifold, simply connected, we try to adapt the results known for the minimal Sullivan model to Kriz model by its own derivation, on the other hand, we try to apply the results known from the works of Félix, Halperin and Thomas about the homotopy Lie algebra for $F(M, k)$.

The Homotopy Lie algebra of a minimal Sullivan algebra

Let $(\Lambda V, d)$ be a minimal Sullivan algebra, then d may be written as an infinite sum $d = d_1 + d_2 + ...$ of derivations, with d_k raising wordlength by k. [GTM 205], we recall some facts on the Homotopy Lie algebra for the minimal Sullivan algebra as follows ; Define a graded vector spaces L by requiring that

$$sL = Hom(V, \mathbb{K}),$$

where as usual the suspension sL is defined by $(sL)_k = L_{k-1}$.

Definition

The graded Lie algebra $L_X = (\pi_*(\Omega X) \otimes \mathbb{K}, [,])$ is called the homotopy Lie algebra of X with coefficients in $\mathbb{K}$.

Thus a pairing $\langle ; \rangle : V \times sL \to \mathbb{K}$ is defined by $\langle v; sx \rangle = (-1)^{|v|} sx(v)$. Extend this to $(k+1)-$linear maps

$$\Lambda^k V \times sL \times ... \times sL \to \mathbb{K}$$

by setting

$$\langle v_1 \wedge ... \wedge v_k; sx_k, ..., sx_1 \rangle = \sum_{\sigma \in S_k} \varepsilon_\sigma \langle v_{\sigma(1)}; sx_1 \rangle ... \langle v_{\sigma(k)}; sx_k \rangle,$$

where as usual S_k is the permutation group on k symbols and $v_{\sigma(1)} \wedge ... \wedge v_{\sigma(k)} = \varepsilon(\sigma) v_1 \wedge ... \wedge v_k$.

Definition

A pair of dual bases for V and for L consists of a basis (v_i) for V and a basis (x_j) for L such that $\langle v_i; sx_j\rangle = \delta_{ij}$.

L inherits a Lie bracket $[,]$ from d_1. Indeed, a biliear map $[,] : L \times L \to L$ is uniquely determined by the formula

$$\langle v; s[x, y]\rangle = (-1)^{|v|+1}\langle d_1 v; sx, sy\rangle, x, y \in L, v \in V.$$

The relation $v \wedge w = (-1)^{|v||w|} w \wedge v$ leads at once to $[x, y] = (-1)^{|x||y|+1}[y, x]$.

Definition

The Lie algebra L is called the homotopy Lie algebra of the Sullivan algebra $(\Lambda V, d)$.

Suppose X is a simply connected with rational homology of finite type. In [GTM 205], it is defined the homotopy Lie algebra $L_X = \pi_*(\Omega X) \otimes \mathbb{K}$. On the other hand, X has a Sullivan minimal model

$$m : (\Lambda V, d) \to A_{PL}(X)$$

with its own homotopy Lie algebra L.
Identify $sL_X = \pi_*(\Omega X) \otimes \mathbb{K}$ by setting $s\alpha = (-1)^{|\alpha|+1}\partial_*^{-1}\alpha$, where $\partial_* : \pi_*(X) \xrightarrow{\cong} \pi_{*-1}(\Omega X)$ is the connecting homomorphism for the path space fibration. Define a linear map $\theta : \pi_*(X) \otimes \mathbb{K} \to Hom(V, \mathbb{K})$ by $(\theta\alpha)v = (-1)^{|\alpha|}\langle v; \alpha\rangle$

Theorem

The linear map $\sigma : L_X \to L$ defined by $\theta(s\alpha) = s\sigma\alpha, \alpha \in L_X$, is an isomorphism of graded Lie algebras.

Another interesting result giving relation between the homotopy Lie algebra of a minimal model $(\Lambda V, d)$ and its LS-category is the following theorem given by Félix and Al in []

Theorem

Let $(\Lambda V, d)$ be a minimal model with homotopy Lie algebra L and UL its envoloping algebra. Then either

$$depth UL < cat(\Lambda V, d) \leq gl.\dim UL,$$

or

$$depth UL = cat(\Lambda V, d) = gl.\dim UL$$

Kriz Model for $F(M,k)$

The Kriz model [K] is a rational model for the cohomology of configuration spaces coming from the Fulton-MacPherson compactification [FM] of configuration spaces.
For a smooth complex projective variety M (of complex dimension m) :
We denote the Kriz model by $E(M,n)$ and

$$E(M,n) \cong (H^*(M)^{\otimes n} \otimes \wedge(\alpha_{ij}))/I$$

where I is the ideal generated by

Kriz Model for $F(M,k)$

(i) $\alpha_{ij} = \alpha_{ji}$
(ii) $p_j^*(x)\alpha_{ij} = p_i^*(x)\alpha_{ij}$, $1 \leq i < j \leq n, x \in H^*(M)$
(iii) $\alpha_{ik}\alpha_{jk} = \alpha_{ij}\alpha_{jk} - \alpha_{ij}\alpha_{ik}$ $1 \leq i < j < k \leq n$, (Arnold Relations)
where $\wedge(\alpha_{ij})$ denotes the exterior algebra generated by α_{ij} for $1 \leq i < j \leq n-1$ and p_i^* denotes the pullback of the $i-th$ projection $p_i : M^n \to M$.
The differantial is defined by $d_{|(H^*(M)^{\otimes n}} = 0$ and $d(\alpha_{ij}) = p_{ij}^*(\Delta)$ where Δ is the diagonal class.
Our aim goal is to find the homotopy Lie algebra of Kriz model defined above and compatible with its differential. For this we consider the problem of determining the structure of the Lie algebra of the homotopy groups of $F(M,n)$.

If $q_1, q_2, \ldots, q_n$ are n distinct points in M, the rational homotopy type of $F(M,k)$ related to that of $M - Q_i$ where $Q_i = \{q_1, \ldots, q_i\}$ is given by the following :

Theorem

If the cohomology algebra $H^*(M;\mathbb{Q})$ requires at least two generators, then we have an isomorphism

$$\pi_*(F(M,k)) \otimes \mathbb{Q} \cong \bigoplus_{i\geq 1} \pi_*(M \setminus Q_i) \otimes \mathbb{Q}.$$

In the case $k = 2$, there is the following result :

Theorem

Under the same hypothesis, the Lie algebra $L_{M,2}$ is subscribed in the Lie algebra exact sequence $0 \to L_{F_2} \to L_{F(M,2)} \to L_{M\times M} \to 0$ in which L_{F_2} is a free Lie algebra.

Theorem

Put $L_{k-r} = \pi_*(F(M - Q_r, k - r)) \otimes \mathbb{Q}$. The Lie algebra L_k contains the descending chain

$$L_k = L_{k-1} \supset ... \supset L_{k-r} \supset ... \supset L_k$$

of subalgebras. Moreover, each L_{k-r} is an ideal in L_k.

Remark : Let $\rho := Cat(F(M, k))$, then the radical, R, of L_k is finite dimensional and satisfies

$$\dim R \leq \rho.$$

1-For any conditions, the Lie algebra L_k are nilpotent ?
2-For the exact sequence $0 \to L_{F_2} \to L_{F(M,2)} \to L_{M\times M} \to 0$, can we $i : L_{F(M,2)} \to L_{M\times M}$ If there exists a subalgebra $\mathfrak{L}$ of $L_{F(M,2)}$ supplement of $Ker(i)$, is the restriction of i to $\mathfrak{L}$ an isomorphism of $\mathfrak{L}$ on $L_{M\times M}$?

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