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HOMOTOPICAL NILPOTENCY

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November 1, 2014

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1) Generalities

1.1) Grouplike

$\mathcal{G}$ associative H-groupe with :
- multiplication $\mu : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$
- strict unit $e \in \mathcal{G}$
- homotopy inverse

1.2) Commutator map

$$\boxed{c_2(a,b) = a b a^{-1} b^{-1}}$$ (defined up to homotopy)

1.3) higher commutator maps

$$\boxed{c_n(a_1 \ldots a_n) = c_2\left(a_1, c_{n-1}(a_1, \ldots, a_{n-1})\right)}$$

1.4) Homotopical nilpotency

Hnil(G) := least n st c_n null-homotopic
if no such integer exists we put
Hnil$(G) = \infty$

N.B Hnil$(G) = 1$ iff G homotopy abelian

- Examples
  - $Hnil(G) = 1$ iff G homotopy abelian
  - $Hnil(S^3) = 3$ (Porter, 1964)
  - $Hnil(SO(3)) = \infty$ (Rao, 1993)

- Basic results
  - Berstein - Ganea (1961)

    $$\boxed{Hnil(\Omega X) \geq WL(X)}$$

    when X path-connected
  - Salvatore (1996)

    $$\boxed{Hnil(\Omega X) = WL(X)}$$

    when X finite connected cw-complex

- Whitehead lenght

  $WL(X) :=$ greatest integer n st the n-iterated whitehead product in $\pi_*(X)$ do not vanish

1.5) Rational Homotopical Nilpotency

- MARUYAMA (1989)
  (G, μ) admits a rationalization grouplike $(G_{\mathbb{Q}}, \mu_{\mathbb{Q}})$, unique up to H-equivalence and yielding an H-space

- $$\boxed{\mathrm{Hnil}_{\mathbb{Q}}(G) := \mathrm{Hnil}(G_{\mathbb{Q}}) \leq \mathrm{Hnil}(G)}$$

1.6) Samelson bracket

[illegible]

- Whitehead product (1941)
  X pointed cw-complex
  $$[\ ,\ ] : \pi_n(X) \times \pi_m(X) \longrightarrow \pi_{n+m-1}(X)$$
  is defined as follows
  - $S^n \times S^m$ is obtained by attaching $(n+m)$-cell to $S^n \vee S^m$
  - the attaching map is $\varphi : S^{n+m-1} \to S^n \vee S^m$
  - if $f : S^n \to X$ and $g : S^m \to X$ then the homotopy class of
    $$(f \vee g) \circ \varphi : S^{n+m-1} \xrightarrow{\varphi} S^n \vee S^m \xrightarrow{f \vee g} X$$
    does not depend on the choice of representatives

we put $\boxed{[f, g] := [(f \vee g) \circ \varphi]}$

we define a bracket

$[\ ,\ ] : \pi_n(X) \times \pi_m(X) \longrightarrow \pi_{n+m-1}(X)$
called Whitehead product
(or bracket)

- Uehara et Massey (1957)
  the whitehead bracket ~~is~~ graded
  is bilinear symetric, and satisfies
  the Jacobi identity

$\boxed{(\pi_*(X), [\ ,\ ]) \text{ is graded Lie-algebra}}$

- Pontryagin product Bott-Samelson (1953)
  . X 1-connected top space
  . the composition of loops induced
    a natural multiplication
    on ΩX
  . It gives rise to a Pontryagin
    product $H_n(\Omega X) \times H_m(\Omega X) \longrightarrow H_{n+m}(\Omega X)$
    (similar than the concept
    introduced in 1939 by Pontryagin
    for groups)

- <u>Samelson product</u> (Samelson 1953)
is defined naturaly on $\pi_{*+1}(X) = \pi_*(\Omega X)$
as follows:

- $T : \pi_*(\Omega X) \longrightarrow \pi_{*+1}(X)$
natural isom

- $h : \pi_*(X) \longrightarrow H_*(X)$
hurewicz homorphism

- $$\begin{array}{ccc}
\pi_{n+1}(X) \times \pi_{m+1}(X) & \longrightarrow & \pi_{n+m+1}(X) \\
(a, b) & \longmapsto & \circledcirc \\
\downarrow {\scriptstyle T \times T} & & \uparrow {\scriptstyle h} \\
\pi_n(\Omega X) \times \pi_m(\Omega X) & & \\
\downarrow {\scriptstyle h \times h} & & \\
H_n(\Omega X) \times H_m(\Omega X) & \longrightarrow & H_{n+m}(\Omega X)
\end{array}$$

Pontryagin product

It's also the same that whitehead product

$[\ ,\ ] : \pi_{n+1}(X) \times \pi_{m+1}(X) \longrightarrow \pi_{n+m+1}(X)$

— General situation

- If (G,μ) connected grouplike space $\pi_*(G)$ admits similarly a natural Samelson product (see Whitehead, Elts of Homotopy)

- Scheerer (1989)
  The rational H-type of G is completely determined by the rational Samelson algebra $(\pi_*(G)\otimes\mathbb{Q}, [\,,\,])$

  i.e. $\boxed{G_1 \underset{H}{\overset{\mathbb{Q}}{\sim}} G_2 \text{ iff } \pi_*(G_1)\otimes\mathbb{Q}, [\,] \cong \pi_*(G_2)\otimes\mathbb{Q}, [\,]}$

- Felix, Lupton, Smith (2010)
  If $G=\Omega X$ connected CW gplike loop space where X simply connected space

  then

$$\boxed{\begin{aligned} \mathrm{Hnil}_{\mathbb{Q}}(G) &= \mathrm{Nil}(\pi_*(G)\otimes\mathbb{Q}) \\ &= \mathrm{WL}_{\mathbb{Q}}(X) \\ &\leq \mathrm{WL}(X) \\ &= \mathrm{Nil}(\pi_*(G)) \\ &\leq \mathrm{Hnil}(G) \end{aligned}}$$

2. Special case: Self homotopy equivalence

2.1 Introduction

X : CW-complex. aut(X) is known to be CW

Hnil(aut(X)) makes sense if aut(X) H-group

by thm 2.2, Ch X, ERfs of Homotopy whitehead

this holds when either

- X is finite (Milnor 1959)
- $\{k, \pi_k(X) \neq 0$ and finitely generated$\}$ is finite (Kahn 1984)

2.2. Known results (Salvatore 96)

. $\mathrm{Hnil}(\mathrm{aut}_1(S^1)) = 1$ since $\mathrm{aut}_1(S^1) \sim_H S^1$ ↑ H-abelian

. $\mathrm{Hnil}\ \mathrm{aut}_1(X) = 1$ if X Riemann surface of genus ≥ 2

(Hansen 1981)

. $\mathrm{Hnil}\ \mathrm{aut}_1(S^2) \geq 3$

. $\mathrm{Hnil}\ \mathrm{aut}_1(G) \geq \mathrm{Hnil}(G)$ if G. H-gpe

. $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(G/H)) = 1$ if G/H connected homogenous space with $\mathrm{rank}(G) = \mathrm{rank}(H)$

$\mathrm{aut}_1(X)$: path-component of aut(X) containing the identity

$\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(SU(n))) = n-1$, $\mathrm{Hnil}_{\mathbb{Q}}\ \mathrm{aut}_1(\vee S^{2n+1}) = +\infty$

Liu - Huang (2013)

$$\boxed{\mathrm{Hnil}_{\mathbb{Q}}\, \mathrm{aut}_1(\mathbb{C}P_n(\mathbb{C})) = n}$$

Felix - Lupton - Smith (2010)

- if X finite connected CW-complex
  then $\mathrm{Aut}(X) \sim$ CW-complex
  $\mathrm{aut}(X) \sim_H$ loop-space

with

$$\boxed{\begin{aligned} \mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}(X)) &= \mathrm{Nil}\, \pi_*(\mathrm{aut}(X) \otimes \mathbb{Q}) \\ &\leq \mathrm{Nil}\, \pi_*(\mathrm{aut}(X)) \\ &\leq \mathrm{Hnil}(\mathrm{aut}(X)) \end{aligned}}$$

$\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}(X)) \leq \mathrm{card}\,\{n \text{ st } \pi_n(X) \otimes \mathbb{Q} \neq 0\}$

replace
ind(?) by diff

Smith (2001)

- $\mathrm{Hnil}_{\mathbb{Q}}\, \mathrm{aut}_1\left(\prod_{\text{finite}} S^{2n} \times \prod_{\text{fini}} S^{2m+1}\right) \in \{1, c, c+1\}$

- $\mathrm{Hnil}_{\mathbb{Q}}\, \mathrm{aut}_1(G/H) = \max(n-m, 1)$

whenever

$G = U(n)$, $H = \prod U(m_i)$

or

$G = Sp(n)$, $H = \prod Sp(m_i)$

$m = \sum m_i$

- $\pi_*(\mathrm{aut}_1(X))$ is well described when X is two-stage space

recall: (formal $\Rightarrow$ two-stage) Felix - Halperin 82)

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