The Serre Spectral Sequences for Loop Space of Configuration Spaces

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Contents

In this talk, we give a brief survey on the Serre spectral sequence $\{E^r_{*,*}(p), d^r\}$ over $\mathbb{K} = \mathbb{Z}$, and $\mathbb{Z}_2$, of the path space fibration

$$p : \mathcal{P}(M) \to M,$$

with $M = F(\mathbb{R}^{n+1}, k)$ or $F(S^{n+1}, k+1)$. Here the paths are based at an appropriate basepoint.
In the case where $M = \mathbb{R}^{n+1}$, the spectral sequence stabilizes at the n^{th} term, in the sense that

$$E^{n+1}_{*,*}(p) \cong E^{\infty}_{*,*}(p) \cong \mathbb{K}.$$

Consequenctly, regarding $H_*(\Omega M; \mathbb{K})$ as chain algebra, with the trivial differential and $\mathbb{K}$ as a trivial chain module over it, we interpret the $E^n_{*,*}$ term of the spectral sequence as an acyclic, free resolution over $\mathbb{K}$ over $H_*(\Omega M; \mathbb{K})$. First remind the Serre spectral sequence as follows :

Serre spectral sequence

The Serre Spectral Sequence expresses, in the language of homological algebra the singular (co)homology of the total space X of a (Serre) fibration in terms of the (co)homology of the base space B and the fiber F.

• **Cohomology spectral sequence**

Let $f : X \to B$ be a Serre fibration of topological spaces, and let F be the fiber. The Serre cohomology spectral sequence is the following :

$$E^2_{p,q} = H^p(B, H^q(F)) \Rightarrow H^{p+q}(X).$$

This spectral sequence can be derived from an exact couple built out of the long exact sequences of the cohomology of the pair (X_p, X_{p-1}) where X_p is the restriction of the fibration over the $p-$skeleton of B.

Serre spectral sequence

There is There is a multiplicative structure

$$E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t},$$

coinciding on the E_2-term with $(-1)^{qs}$ times the cup product, and with respect to which the differentials d_r are (graded) derivations inducing the product on the $E_{r+1}-$page from the one on the E_r-page. And we have $E_{r+1} = H^*(E_r)$.

• **Homology spectral sequence** Similarly to the cohomology spectral sequence, there is one for homology :

$$E^2_{p,q} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(X).$$

• **Main Application to Algebraic Topology : Cohomology via Fibrations**

We begin with a (somewhat) concrete example. Consider a fibration

$$F \to X \xrightarrow{f} B$$

Serre spectral sequence

and set

$$E_2^{p,q} := \begin{cases} H^p(B, H^q(F)) & \text{if } p, q \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

In the case where B is a $CW-$complex, setting $X^p := f^{-1}(B^p)$, we get a filtration

$$X_0 \subset X_1 \subset ...X_i \subset ... \subset X$$

We set $C_q := H^q(X)$ and $C_{p,q} := H^q(X_p)$. The inclusions

$$X_{i-1} \hookrightarrow X_i \hookrightarrow X$$

induce maps

$$H^*(X) \to H^*(X_i) \to H^*(X_{i-1})$$

and thus, setting $F_i := \ker(H^*(X) \to H^*(X_i))$, a filtration
$0 \subset F_1(X) \subset F_2(X) \subset ... \subset F_i(X) \subset ... \subset H^*(X)$
The relative cohomology $E_1^{p,q}$ is the relative Cohomology

Serre spectral sequence

∘ *A Basic Pathspace Fibration*

Consider the path space fibration

$$\Omega\mathbb{S}^{n+1} \hookrightarrow \mathcal{P}\mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$$

This is an example of a case where we can study the homology of a fibration by using the E^{∞} page (the homology of the total space) to control what can happen on the E^2 page. So recall that

$$E^2_{p,q} = H_p(S^{n+1}, H_q(\Omega\mathbb{S}^{n+1})).$$

Serre spectral sequence of path fibration

We consider the cases where $M = \mathbb{R}^{n+1}$ or $\mathbb{S}^{n+1}$ and we fix a set Q_r of r different points of M, $Q_r := \{q_1, ..., q_r\}$ where $1 \leq r \leq k$.
Let ΩM the based loop space and $\mathcal{L}M$ the free loop space.
With these notations, put $F_{k-r,r} := F(\mathbb{R}^{n+1} - Q^{k-r}, r)$, and consider the path fibration $p_{k-r,r} : \mathcal{P}F_{k-r,r} \to F_{k-r,r}$ that sends a based Moore path (α, r) to its endpoint $\alpha(r)$.
In the case where $n > 1$, we have the following theorem

Theorem

The Serre spectral sequence $\{E^t_{*,*}(p_{k-r,r}), d^t\}$ has the following properties :

i) $E^2_{*,*}(p_{k-r,r}) \cong H_*(F_{k-r,r}) \otimes H_*(\Omega F_{k-r,r})$,
ii) $E^2_{*,*}(p_{k-r,r}) \cong E^r_{*,*}(p_{k-r,r})$ for $2 \leq r \leq n$, and
iii) $E^{n+1}_{*,*}(p_{k-r,r}) \cong E^{\infty}_{*,*}(p_{k-r,r}) = \mathbb{Z}$,
where homology is with integral coefficients.

Serre spectral sequence of path fibration

Corollary

The Serre spectral sequence of $\mathcal{L}F_{k-r,r} \to F_{k-r,r}$ *is such that* $E^t_{*,*} \cong E^\infty_{*,*}$, *for* $t \geq (n+1)$.

The case of $F(\mathbb{S}^{n+1}, k+1)$, $(n+1)$ odd

In the case of configuration space of an odd sphere, we have the following

Theorem

The Serre spectral sequence $\{E^r_{*,*}(p_E), d^r(p_E)\}$ *has the following properties :*

i) $E^2_{*,*}(p_E) \cong E^r_{*,*}(p_E)$, *for* $2 \leq r \leq n$;

ii) the projection $p \circ p_E : \mathcal{P}(E) \to B$, *induces an isomorphism* $E^r_{*,*}(p_E) \cong E^r_{*,*}(p_B)$ *of DG-module, for* $r \geq n+1$; *and,*

iii) $E^r_{*,*}(p_E) \cong E^\infty_{*,*} \cong \mathbb{K}$, *for* $r > n+1$.

Serre spectral sequence of path fibration

The case of $F(\mathbb{S}^{n+1}, k+1)$, $(n+1)$ even

Theorem

The Serre rational homology spectral sequence of p_E has the following properties :
i) $E^r_{*,*}(p_E) \cong H_*(E) \otimes H_*(\Omega E)$, *for* $2 \leq r \leq n$;
ii) the map $p : E \to B$, *induces an isomorphism* $E^r_{*,*}(p) : E^r_{*,*}(p_E) \to E^r_{*,*}(p_B)$, *for* $r \geq n+1$; *and, consequently,*
iii) $E^r_{*,*}(p_E) \cong \mathbb{Z}_2$, *for* $r > n+1$.

Corollary

The natural injection $\mathcal{L}F(\mathbb{R}^{n+1}, k) \to \mathcal{L}F(\mathbb{S}^{n+1}, k+1)$ *induces an isomorphism*

$$H_*(\mathcal{L}F(\mathbb{R}^{n+1}, k)) \otimes H_*(\mathcal{L}\mathbb{S}^{n+1}) \to H_*(\mathcal{L}F(\mathbb{S}^{n+1}, k+1)).$$

Serre spectral sequence of path fibration

In the case of rational homology, a similar result holds as follows :

Theorem

The Serre rational homology spectral sequence of p_E has the following properties :
i) $E^r_{*,*}(p_E) \cong H_*(E) \otimes H_*(\Omega E)$, *for* $2 \leq r \leq n$;
ii) the map $p : E \to B$, *induces an isomorphism* $E^r_{*,*}(p) : E^r_{*,*}(p_E) \to E^r_{*,*}(p_B)$ *of spectral sequence for* $r \geq n+1$; *and, consequently,*
iii) $E^r_{*,*}(p_E) \cong \mathbb{Q}$, *for* $r > n+1$.

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