Pb:
$$\begin{tikzcd} X\times\{0\} \arrow[r, "f"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[d, "p"] \\ X\times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists?\tilde{F}"] & B \end{tikzcd}$$

$$F_0 : X \longrightarrow B$$
$$x \longmapsto F(x,0)$$

Rappel

$$\begin{tikzcd} & (E,e_0) \arrow[d, "p"] \\ I \arrow[r, "\gamma"'] \arrow[ur, dashed, "\exists!\tilde{\gamma}"] & (B,b_0) \end{tikzcd}$$
$\exists!\tilde{\gamma} \mid \tilde{\gamma}(0)=e_0$

donc $\forall x\in X$, on considère $F_x : t\longmapsto F(x,t)$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ I \arrow[r, "F_x"'] \arrow[ur, dashed, "\exists!\tilde{F}_x"] & B \end{tikzcd}$$
$\exists!\ \tilde{F}_x \mid \tilde{F}_x(0)=f(x,0)$

D'où $\tilde{F} : X\times I \longrightarrow E$
$$(x,t)\longmapsto \tilde{F}_x(t)$$

* $p\circ\tilde{F}(x,t)=p\circ\tilde{F}_x(t)$
$=F_x(t)$
$=F(x,t)$

Par conséquent $p\circ\tilde{F}=F$.

* $\tilde{F}\circ i(x,0)=\tilde{F}(x,0)=\tilde{F}_x(0)=f(x,0)\Rightarrow \tilde{F}\circ i=f$.

Reste à démontrer la continuité

Soit $(x_0,t_0)\in X\times I$

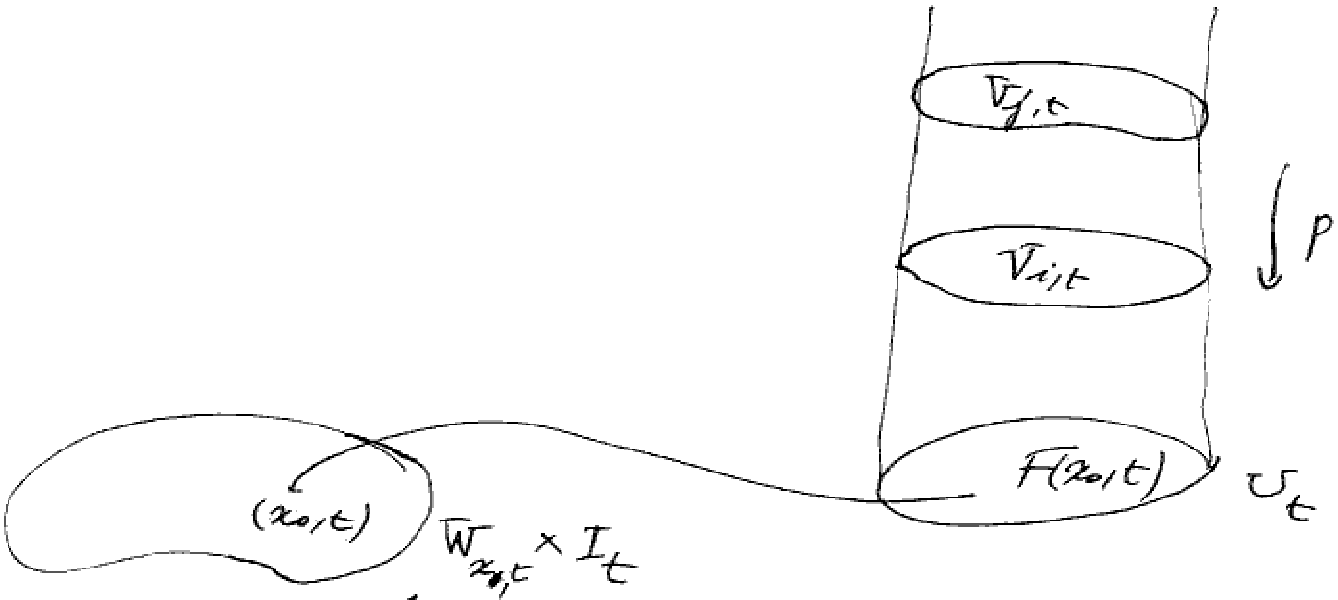
L'idée $\exists?\ W_{x_0}\in\mathcal{V}(x_0)$; $\tilde{F}_{|W_{x_0}\times I}=(p_{|V})^{-1}\circ F$

En effet $\forall (x_0,t)\in\{x_0\}\times I$

$\exists\ U_t\in\mathcal{V}(F(x_0,t))$ ouvert distingué (ou trivialisant) ; $p^{-1}(U_t)=\bigcup_{i\in\Lambda} V_{i,t}$ où $V_{i,t}\cap V_{j,t}=\emptyset$

$p_{|V_{i,t}} : V_{i,t}\xrightarrow{\simeq} U_t$

(Figure!)



donc $F^{-1}(U_t)$ voisinage de (x_0,t) donc

$\exists W_{x_0,t} \in \mathcal{V}(x_0)$ $\exists n \in \mathbb{N}$; $\forall k \in [\![0,n-1]\!]$, $\exists t_k$

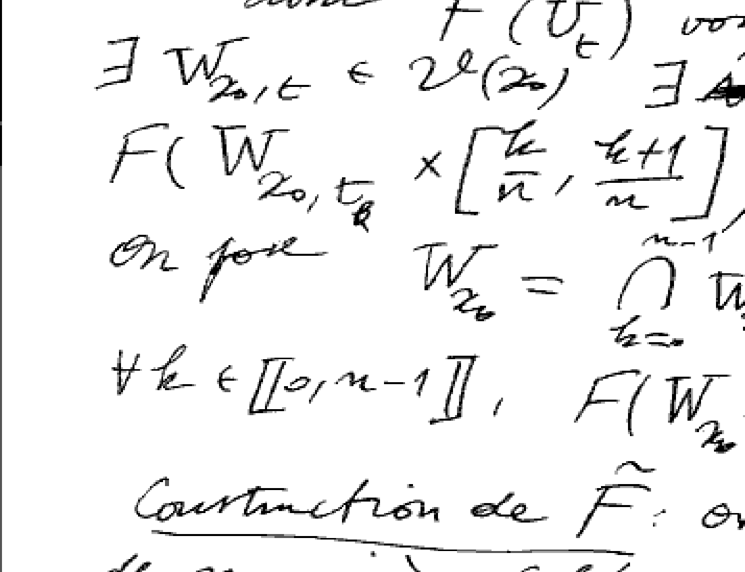
$F\left(W_{x_0,t_k} \times \left[\frac{k}{n},\frac{k+1}{n}\right]\right) \subset U_{t_k}$

On pose $W_{x_0} = \bigcap_{k=0}^{n-1} W_{x_0,t_k} \in \mathcal{V}(x_0)$ vérifiant

$\forall k \in [\![0,n-1]\!]$, $F\left(W_{x_0} \times \left[\frac{k}{n},\frac{k+1}{n}\right]\right) \subset U_{t_k}$

<u>Construction de $\tilde{F}$</u> : on va relever $F_{|W_{x_0}\times\left[\frac{k}{n},\frac{k+1}{n}\right]}$ de manière cohérente

* $k=0$



$\exists V_{i_0} \ni f(x_0,0)$, $p_{|V_{i_0}} : V_{i_0} \xrightarrow{\sim} U_{t_0}$

on définit $G_0 : W_{x_0}\times[0,\frac{1}{n}] \to V_{i_0}$

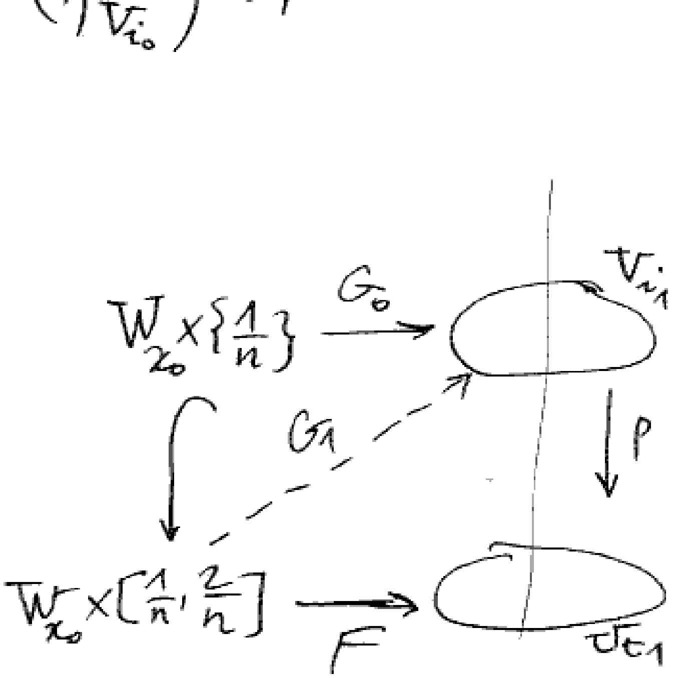
$G_0 = (p_{|V_{i_0}})^{-1} \circ F$

* <u>$k=1$</u> on va construire

$G_1 : W_{x_0}\times\left[\frac{1}{n},\frac{2}{n}\right] \longrightarrow U_{t_1}$

$p \circ G_1 = F$

et $G_1 = (p_{|V_{i_1}})^{-1} \circ F$



$$G_1|_{W_{x_0}\times\{\frac{1}{n}\}} = G_0|_{W_{x_0}\times\{\frac{1}{n}\}}$$

Et ainsi de suite on peut continuer

$$G_k : W_{x_0}\times\left[\frac{k}{n},\frac{k+1}{n}\right] \longrightarrow V_k$$

$G_k = (p|_{V_k})^{-1}\circ F$ alors $G : W_{x_0}\times I \longrightarrow E$

$(x,t)\longmapsto G_k(x,t)$ si $t\in\left[\frac{k}{n},\frac{k+1}{n}\right]$

G continue et vérifiant

$$\left\{\begin{array}{l} p\circ G = F\\ G(x,0)=f(x,0)\end{array}\right. \Rightarrow G = \tilde{F}|_{W_{x_0}\times I}$$

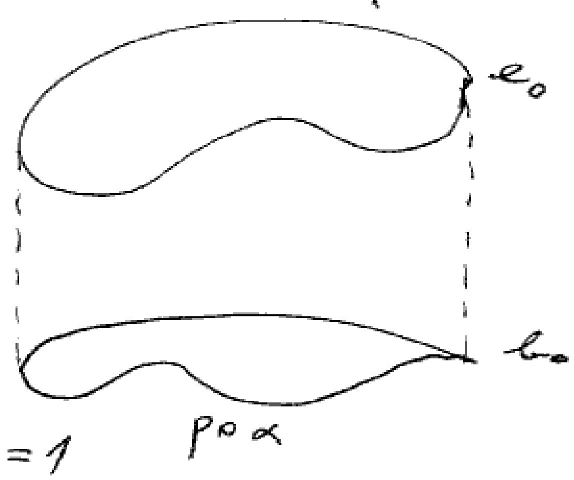
D'où la continuité de $\tilde{F}$.

<u>Remarque</u>. Les revêtements définissent le meilleur cadre pour l'étude de $\pi_1(B,b_0)$.

<u>Corollaire</u> Si $p:E\longrightarrow B$ revêtement alors $p_*:\pi_1(E,e_0)\longrightarrow\pi_1(B,b_0)$ est injective et par conséquent $p_*(\pi_1(E,e))$ s. groupe de $\pi_1(B,b_0)$

<u>En effet</u>.
$$p_* : \pi_1(E,e_0) \longrightarrow \pi_1(B,b_0)$$
$$[\alpha] \longmapsto [p\circ\alpha]$$

$$\begin{tikzcd} I \arrow[r, "\alpha"] \arrow[dr, "p\circ\alpha"'] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$



p_* injective car

soit $[\alpha]\in\pi_1(E,e_0)$; $p_*([\alpha])=1$

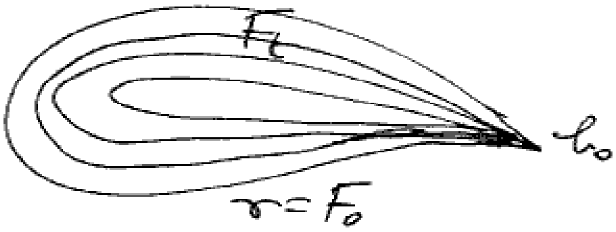
donc $\gamma = p\circ\alpha \sim c_{b_0} : I \longrightarrow B$
$t \longmapsto b_0$

<u>Pb</u>: $\alpha \sim c_{e_0}$?

Soit $F : I\times I \longrightarrow B$, $(s,t)\longmapsto F(s,t)$

une homotopie entre $\gamma = p\circ\alpha$ et c_{b_0}

ie



$$\begin{cases} F(s,0)=\gamma(s) \\ F(s,1)=b_0 \\ F(0,t)=b_0 \\ F(1,t)=b_0 \end{cases}$$

$$\begin{tikzcd}
I\times\{0\} \arrow[r, "\alpha"] \arrow[d, "i"'] & E \arrow[d, "p"] \\
I\times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists!\,\tilde F"] & B
\end{tikzcd}$$

$(s,0) \longmapsto \alpha(s)$

$\tilde F$ homotopie entre α et c_{e_0} ?

Car :

* $\tilde F(s,0) = (\tilde F\circ i)(s,0) = \alpha(s)$
* $\tilde F(s,1) = ?$

$p\circ\tilde F(s,1) = F(s,1) = b_0$

$\tilde F_1 : I \longrightarrow p^{-1}(b_0)$

$$\left\{\begin{array}{l} p^{-1}(b_0) \text{ discret} \\ \tilde F_1 \text{ continue} \end{array}\right. \Longrightarrow \tilde F_1 = \text{constante} = k_1$$

Calcul de la constante

$p\circ\tilde F(0,t) = F(0,t) = b_0$

$\tilde F_0 : I \xrightarrow{\text{continue}} p^{-1}(b_0)$

$t \longmapsto F(0,t)$

$\Rightarrow \tilde F_0 = k_2 = \tilde F(0,0) = \alpha(0) = e_0 = \tilde F(0,1) = k_1$

D'où l'injectivité de p_*

Corollaire 2 : $p^{-1}(b_0) \cong \pi_1(B,b_0)/p_*(\pi_1(E,e_0))$

Conséquences :

a) $B = S^1$ $\quad p : \mathbb{R} \longrightarrow S^1$, $\quad t \longmapsto e^{2i\pi t}$

$p^{-1}(1) = \mathbb{Z} \cong \pi_1(S^1,1)/p_*(\pi_1(\mathbb{R},e_0)) = \pi_1(S^1,1)$

b) $S^n \longrightarrow \mathbb{R}P^n = S^n/\mathbb{Z}_2$

$x \longmapsto [x]$

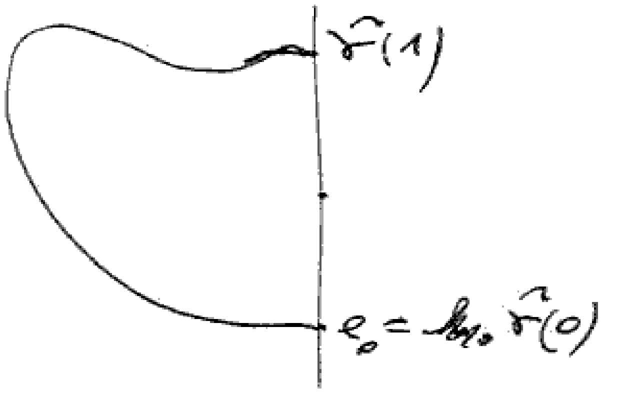
revêtement à 2 feuillets

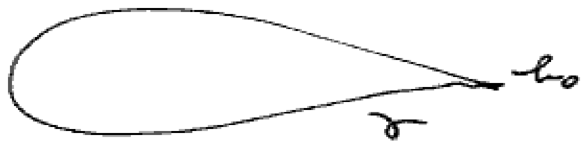
$$p^{-1}(b_0) \cong \mathbb{Z}_2$$
$$\cong \pi_1(\mathbb{R}P^n)/p_*(\pi_1(S^n,*)) \quad \forall n \geq 2$$

D'où $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2 \quad \forall n \geq 2$

<u>Esquisse de la preuve du corollaire 2</u>

$$\begin{aligned} \pi_1(B, b_0) &\xrightarrow{\varphi} p^{-1}(b_0) \\ [\gamma] &\longmapsto \tilde{\gamma}(1) \end{aligned}$$





φ passe au quotient

$$\begin{tikzcd} \pi_1(B, b_0) \arrow[r] \arrow[d] & p^{-1}(b_0) \\ \dfrac{\pi_1(B, b_0)}{p_*(\pi_1(E, e_0))} \arrow[ur, dashed, "\cong\ \tilde{\varphi}"'] & \end{tikzcd}$$

<u>Exemple</u> (lens space)

$$\begin{aligned} S^1 &\longrightarrow S^1 \\ z &\longmapsto z^n \end{aligned}$$

$$S^1 \longrightarrow S^1/\sim$$

$x \sim \lambda x, \ \lambda \in U_n$