On the Hilali Conjecture for Configuration Spaces

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The Hilali Conjecture (or H Conjecture) is well known in Rational Homotopy Theory, it still an open problem for some kind of spaces. In this talk we try to give a short survey on some new results concerning this Conjecture for the Configuration Spaces as well as some open questions in relation in this subject. We leave the discussion open for the case of open manifolds and unordered configuration spaces...

Hilali Conjecture

In 1990, M.R.Hilali conjectured that for any simply-connected elliptic Space, the total dimension of the rational homotopy does not exceed that of its rational cohomology. Let us first recall some basic notions in Rational Homotopy Theory. A topological space X is called rationally elliptic when both of $\pi_*(X) \otimes \mathbb{Q}$ and $H^*(X; \mathbb{Q})$ are of finite dimension, otherwise it is called hyperbolic. For that kind of spaces, the Hilali Conjecture predicts that :

Hilali Conjecture

Hilali Conjecture *Topological version*. If X is an elliptic and simply connected topological space, then

$$\dim \pi_*(X) \otimes \mathbb{Q} \leq \dim H^*(X;\mathbb{Q}).$$

Hilali Conjecture *Algebraic version*. If $(\Lambda V, d)$ is 1-connected an elliptic Sullivan model, then

$$\dim V \leq \dim H^*(\Lambda V, d).$$

where $V = \bigoplus_{k\geq 2} V^k = \bigoplus_{k\geq 2} \mathrm{Hom}(\pi_k(X)\otimes\mathbb{Q}, \mathbb{Q})$ and $H^*(\Lambda V, d) = H^*(X;\mathbb{Q})$.

Cases of Validation

Until now, this conjecture was resolved in many interesting cases : for pure spaces, these are spaces whose Euler-Poincaré characteristic is nonzero, for H-spaces, for nilmanifolds, for symplectic and cosymplectic manifolds, for coformal spaces whose rational homotopy is concentrated in odd degrees,for formal spaces, and for the hyperelliptic spaces, the conjecture has been checked also for elliptic spaces under some restrictive assumptions on the formal dimension. And more recently, the H conjecture has been proved for the case of coformal spaces.(See Badr's Talk)

Ordered and Unordered Configuration Space

Let X be a topological space, the space of ordered configuration of k points in X is defined to be the space

$$F(X,k) := \{(x_1,...,x_k) \in X^k \mid x_i \neq x_j \text{ if } i \neq j\} \subset X^k.$$

The quotient of the space $F(X,k)$ by the symmetric group Σ_k is called *The unordered Configuration Spaces* and it is denoted by $C(X,k)$ and we have $C(X,k) := F(X,k)/\Sigma_k$.
If the original space is a manifold, the configuration space of distinct, unordered points is also a manifold, while the configuration space of not necessarily distinct unordered points is instead an orbifold.

Ordered and Unordered Configuration Space

The homotopy type of configuration spaces is not homotopy invariant, for example, that the spaces $F(\mathbb{R}^n, k)$ are not homotopic for any two distinct values of n. For instance, $F(\mathbb{R}, k)$ is not connected, $F(\mathbb{R}^n, k)$ is a $K(\pi, 1)$, and $F(\mathbb{R}^n, k)$ is simply connected for $n \geq 3$.

The rational Homotopy of Configuration Spaces

Let M be a $n-$dimensional compact and simply connected manifold, and let $q_1, \ldots, q_k$ be k distinct fixed points in M. For $i \leq k$, let $Q_i = \{q_1, \ldots, q_i\}$.

Theorem

If the cohomology algebra $H^*(M;\mathbb{Q})$ requires at least two generators, then we have an isomorphism

$$\pi_* F(M,k) \otimes \mathbb{Q} \cong \bigoplus_{i=0}^{k-1} \pi_*(M \setminus Q_i) \otimes \mathbb{Q}.$$

Lambrechts-Stanley Results

We need to give some necessary results to prove our results

Theorem (Lambrechts-Stanley)

Let M be a connected closed manifold orientable of dimension m such that $H^1(M;\mathbb{Q}) = H^2(M;\mathbb{Q}) = 0$. Let (A, d) be a CDGA-model of M such that A is a connected Poincaré duality algebra of formal dimension m and let $\Delta \in (A \otimes A)^m$ be the diagonal class. Then the ideal $(\Delta) = \Delta.(A \otimes A)$ is a differential ideal in $A \otimes A$ and the quotient

$$\frac{A \otimes A}{(\Delta)}$$

is a CDGA model of $F(M, 2)$.

Lambrechts-Stanley Results

Corollary (Lambrechts-Stanley)

If M is a closed connected formal manifold of dimension m such that $H^1(M;\mathbb{Q}) = H^2(M;\mathbb{Q}) = 0$ then $F(M,2)$ is a formal space and admits as a CDGA-model its cohomology algebra

$$H^*(F(M,2);\mathbb{Q}) \cong (H^*(M,;\mathbb{Q}) \otimes H^*(M,;\mathbb{Q}))/(\Delta)$$

where $\Delta := \sum_{i=1}^{n}(-1)^{|a_i|}a_i \otimes a_i^* \in (A \otimes A)^m$ is called the diagonal class. $(a_i)_{1\leq i\leq n}$ denote the homogeneous basis of A and $(a_i^*)_{1\leq i\leq n}$ its dual.

Hilali Conjecture for Configuration Spaces

Recently we solved the conjecture for configuration spaces of a closed manifold, we exhibit the new results obtained as follows

Theorem

If M is rationally elliptic, and $X := M - \{pt\}$ has a non-trivial rational homotopy group in dimension > 1 then $F(X, k)$ and $F(M, k)$ for $k > 2$, are rationally hyperbolic.

Hilali Conjecture for Configuration Spaces

Theorem

If M is a simply connected manifold of dimension at least 3, and has at least two linearly independent elements in its rational cohomology, then $F(M,3)$ and in general $F(M,k), k \geq 3$ is rationally hyperbolic.

And finally,

Theorem

If M is a closed and simply connected manifold, then $F(M,k)$ verifies the Hilali conjecture provided that $F(M,k)$ is elliptic.

Hilali Conjecture for Configuration Spaces

Remark : There is one exception in the case of '3 configuration' where $M = \mathbb{S}^n$. In fact, '3 configuration' still elliptic. Indeed, $F(\mathbb{S}^n, 3)$ is homotopy equivalent to the Stiefel manifold of orthonormal two frames in $\mathbb{R}^{n+1}$ wich is elliptic

We suggest many other directions of research that can be explored. For example :

Question 1 : M verifies the Hilali conjecture $\Rightarrow C(M,k)$ verifies it also ?

Question 2 : M verifies the Hilali conjecture $\Rightarrow M^{\circ}$ verifies it also ?

Question 3 : Try looking at the case where M is a compact, but not necessary closed, and simply connected manifold, and ask if the main Theorem and precedent questions still true.

References

- J.F. De Bobadilla, J. Fresan, V. MunŸoz, A. Murillo *The Hilali conjecture for hyper elliptic spaces*, Math ArXiv : 1205.1455v1
- Y.Félix,S.Halperin and J.-C.Thomas *Rational Homotopy Theory*, Graduate Texts in Mathematics 205. Springer (2000).
- E.Fadell and S.Husseini, Geometry and Topology of Configuration Spaces.-Springer Monographs in Mathematics. 2001
- Y.Félix,J.Oprea, and D.Tanré *Algebraic Models in Geometry*, Oxford Graduate Texts in Mathematics, 17 2008.
- M.R.Hilali *Action du tore $\mathbb{T}^n$sur les espaces simplement connexes*,Ph. D. Thesis, Université catholique de Louvain,

References

M.R.Hilali, M.I.Mamouni *A conjectured lower bound for the cohomological dimension of elliptic spaces.*,J. Homotopy Relat. Struct. 3, 379Ű384 (2008).

M.R.Hilali, M.I.Mamouni *A lower bound of cohomologic dimension for an elliptic space elliptic spaces.*,Topology Appl. 156, 274Ű283 (2008).

M.R.Hilali, M.I.Mamouni and H.Yamoul *On the Hilali Conjecture for configuration spaces of closed manifolds.*,preprint(2015).

I.Kriz *The Rational Homotopy Type of Configuration Spaces,*Annals of Mathematics 139 (1994), 227-237.