Survey of Rational Homotopy Theory: Sullivan Models and Elliptic Spaces

Hicham YAMOUL

Department of Mathematics
Faculty Of Science Ain Chock
University Hassan II-Casablanca
Rational Homotopy Theory Moroccan Research Group
http://www.algtop.net/

Talk, CRMEF-Rabat School, June 3-5, 2015

0.1 Introduction

A minimal model is a particularly tractable kind of commutative differential graded algebra (cdga) that can be associated to any nice cdga or to any nice space. The word *minimal* emphasizes that, at least in many cases of interest, the model is calculable. The amazing feature of minimal models of spaces is their ability to algebraically encode all rational homotopy information about a space. This is, of course, why minimal models are important.

Among the cdga's, some have more interesting properties than others. This is the case for the so-called Sullivan cdga's and minimal cdga's.[3]

This lecture is based essentially on the section 12 of [2], the first chapters of [3]and the part concerned to Sullivan models, Elliptic Spaces of [1], we denote then by $C^*(X, \mathbb{K})$ the cochain algebra of normalized singular cochains on a topological space X. This algebra is almost never commutative although it is homotopy commutative.

We introduce a naturally defined commutative cochain algebra $A_{PL}(X; \mathbb{K})$, and natural cochain algebra quasi-isomorphisms

$$C^*(X, \mathbb{K}) \overset{\cong}{\to} D(X) \overset{\cong}{\leftarrow} A_{PL}(X; \mathbb{K}).$$

where $D(X)$ is a third natural cochain algebra. The construction of the functor $A_{PL}(X; \mathbb{K})$, due to Sullivan, is inspired from $\mathcal{C}^\infty$ differential forms and the functor $X \rightsquigarrow A_{PL}(X; \mathbb{K})$ equivalently

$$A_{PL} : \text{topological spaces} \rightsquigarrow \text{commutative cochain algebras}$$

is contravariant....

0.2 Sullivan models

Definition 0.2.1. *A Sullivan algebra is a commutative cochain algebra of the form $(\Lambda V, d)$, where*

- $V = \{V^p\}_{p \geq 1}$ *and, as usual, ΛV denotes the free graded commutative algebra on V;*
- $V = \bigcup_{k=0}^{\infty} V(k)$, *where $V(0) \subset V(1) \subset ...$ is an increasing sequence of graded subspaces such that*

$$d = 0 \text{ in } V(0) \text{ and } d : V(k) \to \Lambda V(k-1), k \geq 1.$$

The second condition is called the nilpotence condition on d. i.e, d preserves each $\Lambda V(k)$, and there exist graded subspaces $V_k \subset V(k)$ such that $\Lambda V(k) = \Lambda V(k-1) \otimes \Lambda V_k$, with $d : V_k \to \Lambda V(k-1)$.

The Sullivan algebra is completely described by the vector space V and the linear operator, d. Moreover, if a graded algebra A is connected, i.e. $H^0(A) = \mathbb{K}$, there always exists a quasi-isomorphism from a Sullivan algebra to (A, d). Another simple definition is the following

Definition 0.2.2. *A Sullivan cdga is a cdga $(\Lambda V, d)$ whose underlying algebra is free commutative, with $V = \{V^k\}_{k \geq 1}$ and such that V admits a basis x_i indexed by a well-ordered set such that $dx_i \in \Lambda(x_j)_{j<i}$.*

0.2.1 Topological spaces and the A_{PL} functor

Let Δ^n be the standard simplex in $\mathbb{R}^{n+1}$

$$\Delta^n := \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} | \forall i, 0 \leq i \leq n, \sum_{i=0}^{n} t_i = 1\}$$

and let ∂ denote the standard boundary operator. Consider the restriction to Δ^n of all the differential forms of $\mathbb{R}^{n+1}$ of type

$$\sum \phi_{i_1 \dots i_k} dt_{i_1} \wedge \dots \wedge dt_{i_k},$$

where $\phi_{i_1 \dots i_k}$ are polynomials in the variables $t_0, \dots, t_n$ over the rational numbers.

Denote all such forms by $A^*(\Delta^n)$; note that this algebra contains the two relations

$$\sum_{i=0}^{n} t_i = 1 \text{ and } \sum dt_i = 0.$$

Now let K be a simplicial complex and let $\sigma \in K$ a simplex. Define

$$A_{PL}(K) := \{(\omega_\sigma)_{\sigma \in K} | (\omega_\sigma) \in A^*(\sigma) \text{ and } \omega_\sigma|_\tau = \omega_\tau \text{ if } \tau \subset \partial\sigma\}.$$

Note that $A_{PL}(K)$ is a differential graded commutative algebra. If L L is another simplicial complex and $f : K \to L$ is a simplicial map, then there exists a dga morphism $A_{PL}(f) : A_{PL}(L) \to A_{PL}(K)$ given by

$$A_{PL}(f)(\omega_\sigma) = \omega_{f(\sigma)}.$$

Suppose that X is a topological space homeomorphic to a simplicial complex K; thus K is a simplicial complex whose spatial realization $|K|$ is homeomorphic to X. Then the functor A_{PL} associates to every such topological space X a differential graded commutative cochains algebra $A_{PL}(X)$ and to every continuous map $f : X \to Y$, $Y = |L|$ for some simplicial complex L, a dga morphism $A_{PL}(f) : A_{PL}(Y) \to A_{PL}(X)$. Notice that it is a contravariant functor.

Remark 1. *Any continuous function $f : X \to X$ induces a continuous map $F : K \to K$ using the simplicial approximation theorem, we can find a simplicial representative $\widetilde{F}$ of F. This means that, for every $n \in \mathbb{N}$, $\widetilde{F}$ sends n-simplices to n-simplices and that $\widetilde{F}$ is homotopic to F.*

Definition 0.2.3. *Let X be a topological space homeomorphic to a simplicial complex. Then the minimal model of X is defined to be the minimal model of the dga $A_{PL}(X)$.*

Now, coming back to Sullivan model

Definition 0.2.4. *1. A Sullivan model for a commutative cochain algebra (A, d) is a quasi-isomorphism*

$$m : (\Lambda V, d) \to (A, d)$$

from a Sullivan algebra $(\Lambda V, d)$.

2. If X is a path connected topological space then a Sullivan model for $A_{PL}(X)$,

$$m : (\Lambda V, d) \xrightarrow{\cong} A_{PL}(X),$$

is called a Sullivan model forX.

3. A Sullivan algebra (or model), $(\Lambda V, d)$ is called minimal if we can choose a basis $\{v_i\}$ for V whose enumeration agrees with degree so that the differential is decomposable for each v_i : $dv_i \in \Lambda V_{<i}$, where $V_{<i}$ is the space spanned by the generators $v_1, \dots, v_{i-1}$.

Equivalently, a Sullivan algebra (or model), $(\Lambda V, d)$ is called minimal if

$$Im\ d \subset \Lambda^+ V . \Lambda^+ V$$

Definition 0.2.5. *A (Sullivan) minimal cdga is a Sullivan cdga $(\Lambda V, d)$ satisfying the additional property that $dV \subset \Lambda^{\geq 2} V$*

If $(\Lambda V, d)$ is a minimal cdga and $a \in (\Lambda V)^k$ is a cocycle and a decomposable element, then we construct a new minimal cdga by introducing a new generator x in degree $k-1$ and putting $dx = a$. This gives the minimal cdga $(\Lambda(V \oplus \mathbb{K}x, d)$. By iterating this process, we can easily construct a lot of minimal cdga's. For instance, $(\Lambda(x, y, z), d)$ with $|x| = |y| = 2, |z| = 3, dx = dy = 0$ and $dz = x^2 - y^2$ is automatically a Sullivan minimal model.

If (A, d) is connected algebra then, it has always a minimal Sullivan model, and this is uniquely determined up to isomorphism.

Sullivan models for topological spaces X are, among all the commutative models, the ones that provide the key to unlocking the rational homotopy properties of X. For example, if $(\Lambda V, d)$ is a Sullivan model for X then, as with any commutative model,

$$H^*(\Lambda V, d) \stackrel{\cong}{\to} H^*(X; \mathbb{K}).$$

However, if $(\Lambda V, d)$ is minimal there is also a natural isomorphism

$$V \stackrel{\cong}{\to} \mathrm{Hom}_{\mathbb{Z}}(\pi_*(X); \mathbb{K}),$$

provided that X is simply connected and has rational homology of finite type.

Proposition 0.2.1. *If a simply connected topological spaces X and Y have the same rational homotopy type, then $A_{PL}(X)$ and $A_{PL}(Y)$ are weakly equivalent.*

The minimal models are unique up to isomorphism, then $A_{PL}(X)$ and $A_{PL}(Y)$ have isomorphic minimal models : *the isomorphism class of a minimal model of X is an invariant of its rational homotpy type.*

We can summarize two interesting correspondences as follows :

$$\{\text{Rational homotopy types}\} \stackrel{\cong}{\to} \{\text{Isomorphism classes of minimal Sullivan algebras over}\mathbb{Q}\}$$

Sullivan models also provide good descriptions of continuous maps, and of the relatin of homotopy. Indeed, let $\Lambda(t, dt)$ be the free commutative graded algebra on the basis $\{t, dt\}$ with $\deg t = 0$, $\deg dt = 1$, and let d be the differential sending $t \mapsto dt$. Define augmentations

$$\varepsilon_0, \varepsilon_1 : \Lambda(t, dt) \to \mathbb{K} \text{ by } \varepsilon_0(t) = 0, \varepsilon_1(t) = 1$$

Definition 0.2.6. *Two morphisms $\varphi_0, \varphi_1 : (\Lambda V, d) \to (A, d)$ from a Sullivan algebra to an arbitrary comutative cochain algebra are homotopic if there is a morphism*

$$\Phi : (\Lambda V, d) \to (A, d) \otimes (\Lambda(t, dt), d)$$

such that $(id.\varepsilon_i)\Phi = \varepsilon_i$, $i = 0, 1$. Here Φ is called a homotopy from φ_0 to φ_1, and we write $\varphi_0 \sim \varphi_1$.

Suppose $m_X : (\Lambda V, d) \to A_{PL}(X)$ and $m_Y : (\Lambda W, d) \to A_{PL}(Y)$ are Sullivan models defined over $\mathbb{Q}$, and that $f : X \to Y$ is a continuous map. Then it turns out that there is a unique homotopy class of morphism $\varphi : (\Lambda W, d) \to (\Lambda V, d)$ such that $m_X\varphi = A_{PL}(f)m_Y : \varphi$ is called a *Sullivan representative for f*. Furthermore, the homotopy class of φ depends only on the homotopy class of f. It is shown that $f \mapsto \varphi$ define a bijection and there is the following correspondence

$$\{\text{homotopy class of maps}X \to Y\} \stackrel{\cong}{\to} \{\text{homotopy classes of morphisms}(\Lambda W, d) \to (\Lambda V, d)\}$$

0.2.2 Sullivan algebras and models : constructions and examples

Firstly, we recall some notations and basic facts associated with free commutative graded algebras ΛV;

• $\Lambda V =$ symmetric algebra$(V^{even})\otimes$ exterior algebra(V^{odd}). The subalgebras $\Lambda(V^{\leq p}), \Lambda(V^{>q}), ...$ are denored $\Lambda V^{\leq p}, \Lambda V^{>q}, ...$

• If $\{v_i\}$ is a basis for V we write $\Lambda(\{v_i\})$ or $\Lambda(v_1, ...)$ for ΛV.

• $\Lambda^q V$ is the linear span of elements of the form $v_1 \wedge ... \wedge v_q$, $v_i \in V$. Elements in $\Lambda^q V$ have wordlength q.

• $\Lambda V = \bigoplus_q \Lambda^q V$ and we write $\Lambda^{\geq q} V = \bigoplus_{i\geq q} \Lambda^i V$ and $\Lambda^+ V = \Lambda^{\geq 1} V$.

• If $V = \bigoplus_i V_i$ then $\Lambda V = \bigotimes_i \Lambda V_i$.

• Any linear map of degree zero from V to a commutative graded algebra A extends to a unique graded algebra morphism $\Lambda V \to A$.

• Any linear map of degree k ($k \in \mathbb{Z}$) from V to ΛV extends to a unique derivation of degree k in ΛV.

In particular the differential in a Sulivan algebra $(\Lambda V, d)$ decomposes uniquely as the sum $d = d_0 + d_1 + d_2 + ...$ of derivations d_i raising the wordlength by i The derivation d_0 is called the linear part of d.

Proposition 0.2.2 (The Existence of Sullivan Models). *Any commutative cochain algebra (A, d) satisfying $H^0(A) = \mathbb{K}$ has a Sullivan model*

$$m : (\Lambda V, d) \xrightarrow{\cong} (A, d)$$

Démonstration. By construction, since V is the direct sum of graded subspaces V_k, $k \geq 0$ with $d = 0$ in V_0 and $d : V_k \to \Lambda(\bigoplus_{i=0}^{k-1} V_i)$. Choose $m_0 : (\Lambda V_0, 0) \to (A, d)$ so that

$$H(m_0) : V_0 \xrightarrow{\cong} H^+(A).$$

Since $H^0(A) = \mathbb{K}$, $H(m_0)$ is surjective.

Suppose m_0 has been extended to $m_k : (\Lambda(\bigoplus_{i=0}^{k} V_i), d) \to (A, d)$. Let z_α be cocycles in $\Lambda(\bigoplus_{i=0}^{k} V_i)$ such that $[z_\alpha]$ is a basis for $\ker H(m_k)$. Let V_{k+1} be a graded space with basis $\{v_\alpha\}$ in 1-1 correspondence with the z_α, and with $\deg v_\alpha = \deg z_\alpha - 1$. Extend d to a derivation in $\Lambda(\bigoplus_{i=0}^{k} V_i)$ by setting $dv_\alpha = z_\alpha$. Since d has odd degree, d^2 is a derivation. Since $d^2 v_\alpha = dz_\alpha = 0$, $d^2 = 0$.

Since $H(m_k)[z_\alpha] = 0$, $m_k z_\alpha = da_\alpha$, $a_\alpha \in A$. Extend m_k to a graded algebra morphism $m_{k+1} : \Lambda(\bigoplus_{i=0}^{k+1} V_i) \to A$ by setting $m_{k+1} v_\alpha = a_\alpha$. Then $m_{k+1} dv_\alpha = dm_{k+1} v_\alpha$, and so $m_{k+1} d = dm_{k+1}$.

This completes the construction of $m : (\Lambda V, d) \to (A, d)$ with $V = \bigoplus_{i=0}^{\infty} V_i$ and $m_{|V_k} = m_k$. Since $m_{|\Lambda V_0} = m_0$, and $H(m_0)$ is surjective, $H(m)$ is surjective as well. If $H(m)[z] = 0$ then, since z is necessary in some $\Lambda(\bigoplus_{i=0}^{k} V_i$, $H(m_k)[z] = 0$. By construction, z is a boundary in $\bigoplus_{i=0}^{k+1} V_i$. Thus $H(m)$ is an isomorphism.

We show next by induction on k that V_k is concentrated in degree ≥ 1. This is certainly true for $k = 0$, because $V_0 \cong H^+(A)$. Assume it true for V_i, $i \leq k$. Any element in $\Lambda(\bigoplus_{i=0}^{k} V_i)$ of degree 1 then has the form $v = v_0 + ... + v_k$, $v_i \in V_i^1$.

Thus if $dv = 0$ then $dv_k \in d(\Lambda \bigoplus_{i=0}^{k-1} V_i)$. By construction, this implies $v_k = 0$. Repeating this argument we find $v = v_0$ and $H(m_k)[v_0] = H(m_0)[v_0] \neq 0$, unless $v_0 = 0$. Thus $\ker H(m_k)$ vanishes in degree 1 ; i.e., it is concentrated in degrees ≥ 2. It follows that V_{k+1} is concentrated in degrees ≥ 1.

Finally, the nilpotence condition on d is built into the construction. $\square$

Example 1. *The spheres, S^k.*

The fundamental class $[S^k] \in H_k(S^k;\mathbb{Z})$. that determines a unique class $\omega \in H^k(A_{PL}(S^k))$ such that $\langle \omega, [S^k]\rangle = 1$, and $\{1,\omega\}$ is a basis for $H(A_{PL}(S^k))$. Let Φ be a representing cocycle for ω.

We can distinguish two cases;

If k is odd, then a minimal Sullivan model for S^k is given by

$$m : (\Lambda(e), 0) \xrightarrow{\cong} A_{PL}(S^k),\ \ deg\ e = k, me = \Phi.$$

Indeed, since k is odd, 1 and e are a basis for the exterior algebra $\Lambda(e)$.

If k is even, consider also

$$m : (\Lambda(e), 0) \xrightarrow{\cong} A_{PL}(S^k),\ \ deg\ e = k, me = \Phi.$$

But now, because $deg\ e$ is even, $\Lambda(e)$ has as basis $\{1, e, e^2, e^3, ...\}$ and this morphism is not a quasi-isomorphism. However, Φ^2 is certaint a coboundary, written as $\Phi^2 = d\varphi$ and extend m to

$$m : (\Lambda(e, e'), d) \to A_{PL}(S^k)$$

by setting $deg\ e' = 2k - 1, de' = e^2$ and $me' = \varphi$. The elements $1, e$ represent a basis of $H(\Lambda(e, e'), d)$. Thus this is a minimal model for S^k. Finally, observe that quasi-isomorphisms $(\Lambda e, 0) \to (H^*(S^k), 0)$, k odd and

$(\Lambda(e, e'), d) \to (H^*(S^k), 0)$, k even are given by $e \mapsto \omega$, $e' \mapsto 0$.

Example 2. *Complex projective spaces $\mathbb{CP}^n$ We know that $H^*(\mathbb{CP}^n) = \Lambda a/(a^{n+1})$, $\ deg\ a = 2$*

We choose $z \in A^2_{PL}(\mathbb{CP}^n)$ representing the generator of the cohomology algebra. As above, we choose $\Omega \in A^{2n+1}_{PL}(\mathbb{CP}^n)$ such that $d\Omega = z^{n+1}$. We define

$$\begin{aligned} \varphi \ \ : \ \ & (\Lambda(a, u), da = 0, du = a^{n+1}) \to A_{PL}(\mathbb{CP}^n) \\ & a \mapsto z \\ & u \mapsto \Omega \end{aligned}$$

Again one easily checks that the map

$$\begin{aligned} \overline{\varphi} \ \ : \ \ & (\Lambda(a, u)) \to A_{PL}(\mathbb{CP}^n) \\ & a \mapsto a \\ & u \mapsto 0 \end{aligned}$$

is a quasi-isomorphism. Therefore $\pi_*(\mathbb{CP}^n) \otimes \mathbb{Q} = \mathbb{Q}a^* \oplus \mathbb{Q}u^*$.

Example 3. *Products of topological spaces*

Suppose $m_X : (\Lambda V, d) \to A_{PL}(X)$ and $m_Y : (\Lambda W, d) \to A_{PL}(Y)$ are Sullivan models for path connected topological spaces X and Y. Assume further that the rational homology of one of these spaces has finite type. Let $p^X : X \times Y \to X$ and $p^Y : X \times Y \to Y$ be the projections. Then $A_{PL}(p^X).A_{PL}(p^Y) : A_{PL}(X) \otimes A_{PL}(Y) \to A_{PL}(X \times Y)$ is a quasi-isomorphism of cochain algebras.

In fact, $A_{PL}(p^X).A_{PL}(p^Y)$ is clearly morphism of graded vector spaces commuting with the differentials. It is a morphism of algebras because $A_{PL}(X \times Y)$ is commutative. To see that it is a quasi-isomorphism we use Corollary 10.10 of [2] to identify the induced map of cohomology with the map

$$H^*(X;\mathbb{K}) \otimes H^*(Y;\mathbb{K}) \to H^*(X \times Y;\mathbb{K})$$

given by $\alpha \otimes \beta \mapsto H^*(p^X)\alpha \cup H^*(p^Y)\beta$. *that is an isomorphism (Prop 5.3 (ii)[2]).*

Since $A_{PL}(p^X).A_{PL}(p^Y)$ *is a quasi-isomorphism so is*

$$m_X.m_Y : (\Lambda V, d) \otimes (\Lambda W, d) \xrightarrow{\cong} A_{PL}(X \times Y),$$

where $(m_X.m_Y)(a \otimes b) = A_{PL}(p^X)m_X a.A_{PL}(p^Y)m_Y b$. *This exhibits* $(\Lambda V, d) \otimes (\Lambda W, d)$ *as a Sullivan model for* $X \times Y$.

Example 4. *H-spaces have minimal Sullivan models of the form* $(\Lambda V, 0)$.

An H-space is a based topological space $(X, *)$ *together with a continuous map* $\mu : X \times X \to X$ *such that the self maps* $x \mapsto \mu(x, *)$ *and* $x \mapsto \mu(*, x)$ *of* X *are homotopic to the identity*

Theorem 0.2.1 (Hopf)**.** *If X is a path connected H-space such that $H_*(X;\mathbb{K})$ has a finite type, $H^*(X;\mathbb{K})$ is a free commutative graded algebra.*

Consider the map $\varphi : \Lambda V \xrightarrow{\cong} H^*(X;\mathbb{K})$

and let $w \in A_{PL}(X)$ *be a cocycle representing the cohomology classe* v. *The correspondence* $v \mapsto w$ *defines a linear map* $V \to A_{PL}(X)$ *wich extends to a unique morphism* $m : \Lambda V \to A_{PL}(X)$. *Since* φ *is an isomorphism it follows that* m *is a quasi-isomorphism :*

$$m : (\Lambda V, 0) \xrightarrow{\cong} A_{PL}(X)$$

is a minimal Sullivan model for the H-space X.

Example 5. *A cochain algebra* $(\Lambda V, d)$ *that is not a Sullivan algebra. Consider the cochain algebra* $(A, d) = (\Lambda(v_1, v_2, v_3), d)$, $degv_i = 1$, *with* $dv_1 = v_2v_3, dv_2 = v_3v_1$, *and* $dv_3 = v_1v_2$. *Here* (A, d) *is not a Sullivan algebra. (If it were, it would have to have a cocycle of degree 1,* $w = v_1v_2v_3$ *represent a basis for* $H(A)$, *and so it has a minimal model* $m : (\Lambda(w), 0) \xrightarrow{\cong} (\Lambda V, d)$, $degv = 3$, $m(w) = v_1v_2v_3$.*)*

Example 6. *The minimal Sullivan algebra* $(\Lambda(a, b, x, y, z), d)$, *where*

$$da = db = 0, dx = a^2, dy = ab, dz = b^2$$

and $dega = degb = 2$ *and* $degx = degy = degz = 3$.

Here, the cohomology algebra H has a basis 1, $\alpha = [a]$, $\beta = [b]$, $\gamma = [ay - bx]$, $\delta = [by - az]$, $\varepsilon = [aby - b^2x]$.

Note that $\alpha\delta = \varepsilon = \beta\gamma$, *and that all other products of basis elements in* H^+ *are zero.*

To construct a minimal model for the cochain algebra $(H, 0)$, *we consider* $m : (\Lambda V, d) \xrightarrow{\cong} (H, 0)$, *beginning with*

$V^2 = \langle v_1, v_2 \rangle$ *with* $dv_1 = dv_2 = 0$, $mv_1 = \alpha, mv_2 = \beta$ *and* $V^3 = \langle u_1, u_2, u_3 \rangle$ *with* $du_1 = v_1^2, du_2 = v_1v_2, du_3 = v_2^2$ *and* $mu_1 = mu_2 = mu_3 = 0$.

Note that necessarily $m(v_1u_2 - v_2u_1) = 0 = m(v_2u_2 - v_1u_3)$.

Thus we need to add $V^4 = \langle x_1, x_2 \rangle$ *with* $dx_1 = v_1u_2 - v_2u_1$, $dx_2 = v_2u_2 - v_1u_3$, *and* $mx_1 = mx_2 = 0$,

and $V^4 = \langle y_1, y_2 \rangle$ *with* $dy_1 = dy_2 = 0$ *and* $my_1 = \gamma$, $my_2 = \delta$.

The process turns out (but we can not yet prove this) to continue without end. Observe that this provides two distinct minimal Sullivan algebras with the same cohomology algebra.

0.3 Elliptic Spaces

This section is destined to the study of Sullivan minimal models in the case of the finiteness of cohomological and homotopical dimension, that is ellipticity, let X be a 1-connected CW-complex with finitely many cells in each dimension. It is well-known that, for each i, its i-th homotopy groups $\pi_i(X)$ is a finitely generated abelian groups.

The question that one can state is how do the size of the space X influences the size of $\pi_*(X)$? The rough answer is :

If X is small, then either $\pi_*(X)/(\text{torsion})$ is very small or very large.

0.3.1 Finiteness of the formal dimension

We assume in the following that all the minimal models $(\Lambda V, d)$ satisfy $V^1 = 0$ and that V has finite type, i.e. the topological space X is 1-connected and $H^i(X;\mathbb{Q})$ is finite dimensional for every i.

Definition 0.3.1. *The formal dimension $fd(X)$ of a commutative differential graded algebra $(\Lambda V, d)$ is equal to the maximum of all k such that $H^k(\Lambda V) \neq 0$. If no such k exists we write $fd(X) = \infty$.*

To begin with, we describe the important process which consists in killing variables in Sullivan models.

Suppose we are given a minimal Sullivan model $(\Lambda V, d)$; there always exists a $v \in V$ such that $dv = 0$. Let us choose a $\mathbb{Q}$-vector space such that V decomposes as $V = \mathbb{Q}v \oplus W$ with $dv = 0$, and define $(\Lambda W, d) = \Lambda v \otimes \Lambda W/(v)$.

The algebraic structure is given by :

$$\Lambda V = \Lambda v \otimes \Lambda W = (\mathbb{Q}v \oplus \Lambda^+ v) \otimes \Lambda W = \Lambda W \oplus \Lambda^+ v \otimes \Lambda W,$$

where the second summand is precisely the ideal $v.\Lambda V = (v)$, the ideal generated by v in ΛV. Because $dv = 0$, $v\Lambda V$ is d-stable (i.e. $d(v.\Lambda V) \subset v.d(\Lambda V)$). Then $\Lambda V / v.\Lambda V$ is a CDA ; as a graded algebra it is isomorphic to ΛW. Using this isomorphism, we endow ΛW with a differential ; let us denote this differential by $\overline{d}$. We say that $(\Lambda W, \overline{d})$ is the minimal Sullivan model obtained by killing the variable v.

Proposition 0.3.1. *Suppose that ΛV decomposes as $\Lambda v \otimes \Lambda W$.*

i) If $degv = 2n$ and $fd(\Lambda V) = k < \infty$, then $fd(\Lambda W) = k + 2n - 1$.

ii) If $degv = 2n + 1$ and $fd(\Lambda V) = k < \infty$, then

$$fd(\Lambda W) = k - (2n+1) \text{ or } fd(\Lambda W) = \infty.$$

0.3.2 Elliptic Models

Definition 0.3.2. *An elliptic Sullivan minimal model is a minimal model $(\Lambda V, d)$ such that*

$$\dim V < \infty \text{ and } \dim H(\Lambda V) < \infty.$$

For such models we can choose a finite basis $v_1, ..., v_r$ such that dv_i is a polynomial in the variables $v_1, ..., v_{i-1}$.

For each i we have the quotient model

$$(\Lambda(v_i, ..., v_r), \overline{d}) = (\Lambda V, d)/(v_1, ..., v_{i-1}).$$

Proposition 0.3.2. *Suppose* $(\Lambda V, d)$, *is elliptic. Then*
i) For each i, $(\Lambda(v_i, ..., v_r)$ *is elliptic.*
ii) $fd(\Lambda V) = \sum_{|v_i| odd} |v_i| - \sum_{|v_i| even}(|v_i| - 1)$.

Démonstration. Let $W = (v_2, ..., v_r)$. If $|v_i|$ is even, then by Proposition 0.. ΛW is elliptic and $\mathrm{fd}(\Lambda W) = \mathrm{fd}(\Lambda V) + |v_1| - 1$.

If $|v_1|$ is odd, let us consider the derivation $\theta : \Lambda W \to \Lambda W$ and its "linear part"

$$\theta_1 : \Lambda^q W \to \Lambda^q W.$$

Since θ_1 decreases the degree, there exists an N such that one actually has $(\theta_{1|W})^N = 0$, $(N \leq r-1)$. By Proposition 0.., this implies $H(\Lambda W) < \infty$; i.e. ΛW is elliptic and by Proposition 0.. $\mathrm{fd}(\Lambda W) = \mathrm{fd}(\Lambda V) - |v_1|$.

Repeating the process we get the announced formula. □

Remark 2. *Since* $(\Lambda v_r, 0)$ *is elliptic,* $|v_r|$ *must be odd.*

We now proceed to a characterization of elliptic models. Let P *be the span of the odd generators* $x_1, x_2, ..., x_k$ *with* $x_i = v_{2i+1}$. *Let* Q *be the span of the even generators* $y_1, y_2, ..., y_k$ *with* $y_i = v_{2i}$. *We identify*

$$\begin{aligned} \Lambda V &= \Lambda P \otimes \Lambda Q \\ &= \mathbb{K}[y_1, y_2, ..., y_l] \otimes E(x_1, x_2, ..., x_k) \\ &= \mathbb{K}[y_1, y_2, ..., y_l] \otimes \Lambda V \otimes P. \end{aligned}$$

In particular, $dy_i \in \Lambda V \otimes P$ *and we can write*

$$dx_i = f_i(y) + \Omega_i.$$

where f_i *is a polynomial and* $\Omega_i \in \Lambda V \otimes P$.
Define $n = \sum |v_i| - \sum(|v_i| - l)$.

and finally, we give the following theorem

Theorem 0.3.1. *The following conditions are equivalent for a CDA* $(\Lambda V, d)$ *with* $\dim V < \infty$:
i) $\dim H(\Lambda V) < \infty$; *i.e.*$(\Lambda V, d)$ *is elliptic.*
ii) $\mathbb{K}[y_1, y_2, ..., y_l]/(f_1, ..., f_k)$ *is finite dimensional.*
iii) $H^n(\Lambda V)$ *and* $H^i(\Lambda V) = 0$, *for* $n < i < 3n$.

Démonstration. See [1] □

Example 7. *Let* $\Lambda(x, y, z, a, u, v)$ *be the free graded commutative algebra generated by* x, y, z, a, u, v *in respective degrees 3,3,3,8,13,16 Define a differential by*

$$\begin{aligned} dx &= dy = dz = 0 \\ da &= xyz \\ du &= xya \\ dv &= a^2 + 2zu. \end{aligned}$$

Define a new differential δ *such that* $\delta x = \delta y = \delta z = \delta a = \delta u = 0, \delta v = a^2$. *We obtain*

$$\begin{aligned} H(\Lambda V, \delta) &= \Lambda(x, y, z, u) \otimes H(\Lambda(a, v), \delta v = a^2) \\ &= \Lambda(x, y, z, u) \otimes \Lambda a/(a^2), \end{aligned}$$

which is of dimension $32 < \infty$. *Thus* $H(\Lambda V, d)$ *is finite dimensional and*

$$fd(\Lambda V, d) = 3 + 3 + 3 - 7 + 13 + 15 = 30.$$

Example 8. $\Lambda(a_2, x_3, u_3, b_4, v_5, w_7; da = dx = 0, du = a^2, db = ax, dv = ab - ux, dw = b^2 - vx)$.

Here subscripts denote degrees. The differential δ *is given by* $\delta a = \delta b = \delta x = 0, \delta u = a^2, \delta v = ab, \delta w = b^2$. *Thus in* $H(\Lambda V, \delta)$ *we have* $[a]^2 = [b]^2 = 0$ *and so* $(\Lambda V, d)$ *is elliptic.*

0.3.3 Some equalities and inequalities

Suppose $(\Lambda V, d)$ is elliptic and generated by odd-degree generators$x_1, ..., x_k$, and even-degree generators $y_1, ..., y_l$. (we preserve all notations of the previous paragraph). Set

$$|y_i| = 2a_i, |x_j| = 2b_j - 1, \text{ and } n = \text{fd}(\Lambda V, d).$$

Recall from Proposition 0.2.2 ii) :
Fact one. $n = \sum_{j=1}^{l}(2b_j - 1) - \sum_{i=1}^{k}(2a_i - 1)$.
Next write $dx_j = f(y_1, ..., y_l) + \Lambda V \otimes P$ and recall that

$$\dim \mathbb{K}[y_1, y_2, ..., y_l]/(f_1, ..., f_k) < \infty, \ \deg f_j = 2b_j.$$

This means $k \geq l$,i.e. :
Fact two.
The number of odd generators$\geq$ the number of even generators. Now we renumber the indices such that

$$a_1 \geq ... \geq a_l \text{ and } b_1 \geq ... \geq b_k$$

Define a map from $\mathbb{K}[y_1, y_2, ..., y_l]/(f_1, ..., f_k)$ to

$$\mathbb{K}[y_1, y_2, ..., y_r]/(f_1(y_1, y_2, ..., y_r, 0, ..., 0), ..., f_k(y_1, y_2, ..., y_r, 0, ..., 0))$$

by $y_i \mapsto 0$, $i > r$.
At least r of the $f_j(y_1, y_2, ..., y_r, 0, ..., 0)$ are non-zero (because the last quotient algebra is finite, there are more equations than variables) ; say $f_{j_1}, ..., f_{j_r}$ out of $f_1, ..., f_k$. Thus $j_r \geq r$ and so

$$2b_r = \ \deg f_r \geq \ \deg f_{j_r} = \ \deg y_1^{m_1} ... y_r^{m_r} = \sum_i^r m_i 2a_i \geq 4a_r$$

and

$$b_r \geq 2a_r, 1 \leq r \leq l.$$

Now facts one and two combined yield

$$n \geq \sum_{j=1}^{l}(2b_j - 1) - \sum_{i=1}^{l}(2a_i - 1) = \sum_{j=1}^{l} 2(b_j - a_j).$$

Substituting the inequality (*) we get :

$$n \geq \sum_{j=1}^{l} 2a_j.$$

That is
Fact three.
$n \geq \sum \deg_{i=1}^{l} y_i$
Finally, we obtain

$$\begin{aligned} n &= \sum_{j=1}^{k} b_j + \sum_{j=1}^{k}(b_j - 1) - \sum_{i=1}^{l}(2a_i - 1) \\ &\geq \sum_{j=1}^{k} b_j + \sum_{j=1}^{k}(b_i - 2a_i) \geq \sum_{j=1}^{k} b_j, \end{aligned}$$

by inequality (*).
Fact four. $2n - 1 \geq \sum \deg_{j=1}^{k} x_j$.

0.3.4 Euler-Poincaré characteristic

We recall the Euler-Poincaré characteristic χ_M of a finite dimensional vector space by $\chi_M = \sum(-1)^i \dim M^i = \dim M^{even} - \dim M^{odd}$. If M is equipped with a differential, d, then $\chi_M = \chi_{H(M,d)}$.

Proposition 0.3.3. *Let χ be the Euler-Poincaré characteristic of the cohomology of an elliptic Sullivan algebra $(\Lambda V, d)$. Then*

$$\chi \geq 0 \text{ and } \dim V^{odd} - \dim V^{even} \geq 0.$$

Moreover, the following conditions are equivalent :
(i) $\chi > 0$.
(ii) $H(\Lambda V, d)$ is concentrated in even degrees.
(iii) $H(\Lambda V, d)$ is the quotient $\Lambda(y_1, ..., y_q)/(u_1, ..., u_q)$ of a polynomial algebra in variables (y_j) of even degree by an ideal generated by a regular sequence (u_j).
(iv)) $(\Lambda V, d)$ is isomorphic to a pure Sullivan algebra $(\Lambda Q \otimes \Lambda P, d)$ in which $Q = Q^{even}$, $P = P^{odd}$ and d maps a basis of P to a regular sequence in ΛQ.
(v) $\dim V^{odd} - \dim V^{even} = 0$.

Démonstration. See [2]. □

0.3.5 Topological interpretation

Definition 0.3.3. *A simply connected topological space X is rationally elliptic if*

$$\dim H^*(X;\mathbb{Q}) < \infty \text{ and } \dim \pi_*(X) \otimes \mathbb{Q} < \infty.$$

Let us call $n := \max\{i; H^i(X;\mathbb{Q}) \neq 0\}$ the *formal dimension* of X.

We recall that the Sullivan minimal model of any rational I-connected space X is a free CDA ΛV which satisfies
i) $H^*(\Lambda V, d) \cong H^*(X;\mathbb{Q})$.
ii) $\Lambda^+ V/\Lambda^+ V.\Lambda^+ V = \pi_*(X) \otimes \mathbb{Q}^\sharp$
Now we can state the Friedlander-Halperin theorem (topological version)

Theorem 0.3.2. *Suppose X is rationally elliptic of formal dimension n. Then*
i) $\dim \pi_{odd}(X) \otimes \mathbb{Q} \geq \dim \pi_{even}(X) \otimes \mathbb{Q}$.
ii) If $\{x_j\}$ is a basis of $\dim \pi_{odd}(X) \otimes \mathbb{Q}$ and $\{y_i\}$ a basis of $\dim \pi_{even}(X) \otimes \mathbb{Q}$, then

$$n = \sum degx_j - \sum (degy_i - 1).$$

iii) $n \geq degy_i$ and $2n - 1 \geq degx_j$.
iv) $\pi_i(X) \otimes \mathbb{Q} = 0$, for $i \geq 2n$.

Démonstration. See [2] for the algebraic version. □

Corollary 0.3.1. *If $(\Lambda V, d)$ is elliptic and has formal dimension n, then*

$$V^q = 0, q \geq 2n.$$

Bibliographie

[1] Marc Aubry, *Homotopy Theory and Models* Based on lectures held at a DMV Seminar in Blaubeuren by H.J. Baues, S. Halperin and J.-M. Lemaire.. Birhauser, 1995.

[2] Félix, Yves, Halperin, Stephen, and Thomas, Jean-Claude (2001). *Rational Homotopy Theory*, Volume 205 of Graduate Texts in Mathematics. Springer-Verlag, New York.

[3] Y.Félix,J.Oprea, and D.Tanré *Algebraic Models in Geometry*, Oxford Graduate Texts in Mathematics, 17 2008.